Online Appendix to

"Pegging the Interest Rate on Bank Reserves: A Resolution of New Keynesian Puzzles and Paradoxes"

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Appendix A: Simple Model (Analytical Results)

A.1. Root Analysis

We first show that $0 < \rho < 1 < |\omega_1| \leq |\omega_2|$. Since $\mathcal{P}(0) = -1/\beta - \chi_y/(\beta\sigma\chi_i) < 0$ and $\mathcal{P}(1) = \kappa/(\beta\sigma\chi_i) > 0$, $\mathcal{P}(X)$ has either one or three real roots inside $(0, 1)$. Moreover, since $\mathcal{P}(X) < 0$ for all $X < 0$, $\mathcal{P}(X)$ has no negative real roots. Therefore, $\mathcal{P}(X)$ has at least one real root inside $(0, 1)$, which we denote by ρ, and its other two roots, which we denote by ω_1 and ω_2 with $|\omega_1| \leq |\omega_2|$, must be (i) both real and inside $(0, 1)$, or (ii) both real and larger than 1, or (iii) both complex and conjugates of each other. Now, given that $\mathcal{P}(X)$ is of type $X^3 - a_2X^2 + a_1X - a_0$, we have $\rho + \omega_1 + \omega_2 = a_2 \equiv 2 + 1/\beta + \kappa/(\beta\sigma) + \chi_y/(\sigma\chi_i) > 3$. Therefore, Case (i) is impossible, and in Case (iii) the common real part of ω_1 and ω_2 is larger than 1. As a consequence, in the remaining two possible cases, namely Cases (ii) and (iii), ω_1 and ω_2 lie outside the unit circle.

We now show that ω_1 and ω_2 can be real numbers, and that they can also be complex (non-real) numbers. Suppose, for instance, that χ_y and χ_i go to 0, with χ_y/χ_i constant. Then, $a_1 \equiv 1 + 2/\beta + (1 + 1/\chi_i)\kappa/(\beta\sigma) + (1 + 1/\beta)\chi_y/(\sigma\chi_i)$ goes to $+\infty$, while $a_2 \equiv 2 + 1/\beta + \kappa/(\beta\sigma) + \chi_y/(\sigma\chi_i)$ and $a_0 \equiv 1/\beta + \chi_y/(\beta\sigma\chi_i)$ remain constant. Therefore, for sufficiently small values of χ_y and χ_i, $\mathcal{P}(X) = X^3 - a_2X^2 + a_1X - a_0$ is positive for all $X \geq 1$, so that Case (ii) is impossible and ω_1 and ω_2 are complex numbers. By contrast, suppose now that χ_y and χ_i go to $+\infty$, with χ_y/χ_i constant. Then, $\mathcal{P}[1 + \chi_y/(\sigma\chi_i)]$ goes to $-[1 + \chi_y/(\sigma\chi_i)]\kappa\chi_y/(\beta\sigma^2\chi_i)$, which is negative. Therefore, for sufficiently large values of χ_y and χ_i, we have $\mathcal{P}[1 + \chi_y/(\sigma\chi_i)] < 0$, which, together with $\mathcal{P}(1) > 0$, implies that ω_1 and ω_2 are positive real numbers.

A.2. Resolution of the Paradox of Flexibility

Using the definition of Z_t, and after some simple algebra, we can rewrite (10) and (11) as

$$\begin{aligned}\pi_t &= -(1-\rho)\,p_{t-1} + \frac{\kappa}{\beta(\omega_2-\omega_1)}\mathbb{E}_t\left\{-\frac{1}{\sigma}\sum_{k=0}^{+\infty}\left(\omega_1^{-k-1}-\omega_2^{-k-1}\right)\left(i^*_{t+k}-r_{t+k}-\frac{M_{t+k}}{\chi_i}\right)\right.\\ &\left.-\sum_{k=0}^{+\infty}\left(\xi_1^g\omega_1^{-k-1}-\xi_2^g\omega_2^{-k-1}\right)g_{t+k}+\sum_{k=0}^{+\infty}\left(\xi_1^\varphi\omega_1^{-k-1}-\xi_2^\varphi\omega_2^{-k-1}\right)\delta_\varphi\varphi_{t+k}\right\}, \qquad (A.1)\end{aligned}$$

$$\begin{aligned}y_t &= -\vartheta p_{t-1} + g_t + \frac{\mathbb{E}_t}{\beta(\omega_2-\omega_1)}\left\{\frac{1}{\sigma}\sum_{k=0}^{+\infty}\left(\xi_1\omega_1^{-k-1}-\xi_2\omega_2^{-k-1}\right)\left(i^*_{t+k}-r_{t+k}-\frac{M_{t+k}}{\chi_i}\right)\right.\\ &\left.+\sum_{k=0}^{+\infty}\left(\xi_1\xi_1^g\omega_1^{-k-1}-\xi_2\xi_2^g\omega_2^{-k-1}\right)g_{t+k}-\sum_{k=0}^{+\infty}\left(\xi_1\xi_1^\varphi\omega_1^{-k-1}-\xi_2\xi_2^\varphi\omega_2^{-k-1}\right)\varphi_{t+k}\right\}, \qquad (A.2)\end{aligned}$$

where $\vartheta \equiv (1-\rho)(1-\beta\rho)/\kappa$ and

$$\begin{aligned}
\xi_j &\equiv \beta\left(\omega_j+\rho-1\right)-1, \\
\xi_j^g &\equiv \left(1-\delta_g\right)\left(\omega_j-1\right)+\frac{\delta_g \chi_y}{\sigma \chi_i}, \\
\xi_j^{\varphi} &\equiv \delta_{\varphi}\left(\omega_j-1\right)-\frac{\delta_{\varphi} \chi_y}{\sigma \chi_i}
\end{aligned}$$

for $j \in \{1,2\}$.

The only parameter that depends on the degree of price stickiness θ in the structural equations (1), (2), and (7) is the slope κ of the Phillips curve (2). We have $\lim_{\theta \to 0} \kappa = +\infty$ and therefore

$$-\beta\sigma \lim_{\theta \to 0}\left[\frac{\mathcal{P}(X)}{\kappa}\right] = X\left(X-\omega_1^n\right)$$

for any $X \in \mathbb{R}$, where $\omega_1^n \equiv (1+\chi_i)/\chi_i > 1$, which implies in turn that

$$\lim_{\theta \to 0} \rho = 0, \quad \lim_{\theta \to 0} \omega_1 = \omega_1^n, \quad \text{and} \quad \lim_{\theta \to 0} \omega_2 = +\infty. \tag{A.3}$$

Using (A.3) and

$$(1-\rho)\left(\omega_1-1\right)\left(\omega_2-1\right) = \mathcal{P}(1) = \frac{\kappa}{\beta\sigma\chi_i},$$

we also get that

$$\lim_{\theta \to 0} \frac{\kappa}{\omega_2} = \beta\sigma. \tag{A.4}$$

Using (A.3) and (A.4), we can easily determine the limits of (A.1) and (A.2) as $\theta \to 0$:

$$\begin{aligned}
\lim_{\theta \to 0} \pi_t &= -p_{t-1} - \mathbb{E}_t\left\{\sum_{k=0}^{+\infty}\left(\omega_1^n\right)^{-k-1}\left\{i_{t+k}^* - r_{t+k} - \frac{M_{t+k}}{\chi_i} + \left[\frac{\sigma\left(1-\delta_g\right)+\chi_y\delta_g}{\chi_i}\right] g_{t+k}\right.\right. \\
&\quad \left.\left. + \left[\frac{\left(\chi_y-\sigma\right)\delta_{\varphi}}{\chi_i}\right]\varphi_{t+k}\right\}\right\} + \sigma\left(1-\delta_g\right) g_t - \sigma\delta_{\varphi}\varphi_t, \qquad \text{(A.5)} \\
\lim_{\theta \to 0} y_t &= \delta_g g_t + \delta_{\varphi}\varphi_t. \qquad \text{(A.6)}
\end{aligned}$$

These limits are finite, unlike their counterparts in the basic NK model.

We now show that the right-hand sides of (A.5) and (A.6) coincide with the values taken by π_t and y_t when prices are perfectly flexible ($\theta = 0$). The flexible-price value of y_t is straightforwardly obtained by setting to zero the last term in the Phillips curve (2), which is proportional to (the log-deviation of) firms' marginal cost of production:

$$y_t = \delta_g g_t + \delta_{\varphi}\varphi_t. \tag{A.7}$$

This value is identical to the right-hand side of (A.6). Using the IS equation (1), the money-demand equation (7), the identity $m_t = M_t - p_t$, the exogenous policy-rate setting $i_t^m = i_t^*$, and the solution for flexible-price output (A.7), we get the following dynamic equation under flexible prices:

$$\begin{aligned}
p_t &= \left(\omega_1^n\right)^{-1}\mathbb{E}_t\left\{p_{t+1}\right\} - \left(\omega_1^n\right)^{-1}\left\{i_t^* - r_t - \frac{M_t}{\chi_i} - \left[\sigma\left(1-\delta_g\right) - \frac{\chi_y\delta_g}{\chi_i}\right] g_t\right. \\
&\quad \left. + \sigma\left(1-\delta_g\right)\mathbb{E}_t\left\{g_{t+1}\right\} + \left(\sigma + \frac{\chi_y}{\chi_i}\right)\delta_{\varphi}\varphi_t - \sigma\delta_{\varphi}\mathbb{E}_t\left\{\varphi_{t+1}\right\}\right\}.
\end{aligned}$$

Iterating this equation forward to $+\infty$ leads to the following value for the price level p_t in our simple model under flexible prices:

$$
\begin{aligned}
p_t &= -\mathbb{E}_t \left\{ \sum_{k=0}^{+\infty} (\omega_1^n)^{-k-1} \left\{ i_{t+k}^* - r_{t+k} - \frac{M_{t+k}}{\chi_i} + \left[\frac{\sigma(1-\delta_g) + \chi_y \delta_g}{\chi_i} \right] g_{t+k} \right. \right. \\
&\quad \left. \left. + \left[\frac{(\chi_y - \sigma)\delta_\varphi}{\chi_i} \right] \varphi_{t+k} \right\} \right\} + \sigma(1-\delta_g) g_t - \sigma \delta_\varphi \varphi_t,
\end{aligned}
$$

which implies in turn that the value of $\pi_t \equiv p_t - p_{t-1}$ in our simple model under flexible prices coincides with the right-hand side of (A.5). Thus, our simple model solves the paradox of flexibility: the limits of π_t and y_t as $\theta \to 0$ are finite and coincide with the values of π_t and y_t when $\theta = 0$.

A.3. Effects of Greater Price Flexibility

We measure the degree of price flexibility by the reduced-form parameter κ (which is inversely related to the degree of price stickiness θ). We show that: (i) $\partial^2\pi_t/\partial\kappa\partial r_{t+k} > 0$ for $k \geq 0$ in both our selected equilibrium and the standard equilibrium of the basic NK model; and (ii) $\partial^2 y_t/\partial\kappa\partial r_{t+k} < 0$ for $k \geq 0$ in our selected equilibrium, while $\partial^2 y_t/\partial\kappa\partial r_t = 0$ and $\partial^2 y_t/\partial\kappa\partial r_{t+k} > 0$ for $k \geq 1$ in the standard equilibrium.

We start with (i). In our selected equilibrium, we have $\partial\pi_t/\partial r_{t+k} = (1-\rho_b)(1-\omega_b^{-k-1})$. Using $\omega_b = [1+\beta+\kappa/\sigma+\sqrt{(1+\beta+\kappa/\sigma)^2 - 4\beta}]/(2\beta)$, we get $\partial\omega_b/\partial\kappa > 0$. In turn, using this result and $\rho_b\omega_b = 1/\beta$, we get $\partial\rho_b/\partial\kappa < 0$. We conclude that $\partial^2\pi_t/\partial\kappa\partial r_{t+k} > 0$. In the standard equilibrium, now, we have

$$
\frac{\partial \pi_t}{\partial r_{t+k}} = \frac{\kappa\left(\rho_b^{-k-1} - \omega_b^{-k-1}\right)}{\beta\sigma(\omega_b - \rho_b)} = \frac{\beta^k \kappa}{\sigma}\left(\frac{\omega_b^{k+1} - \rho_b^{k+1}}{\omega_b - \rho_b}\right) = \frac{\beta^k\kappa}{\sigma}\sum_{j=0}^{k} \omega_b^{k-j}\rho_b^j = \frac{\beta^k\kappa}{\sigma}\sum_{j=0}^{k}\beta^{-j}\omega_b^{k-2j},
$$

where the second and fourth equalities follow from $\rho_b\omega_b = 1/\beta$. For $k = 0$, thus, we straightforwardly get $\partial^2\pi_t/\partial\kappa\partial r_t = 1/\sigma > 0$. For $k \geq 1$, we introduce the function $x \mapsto f_k(x) \equiv \sum_{j=0}^{k}\beta^{-j}x^{k-2j}$, and we write $\partial\pi_t/\partial r_{t+k} = \beta^k\kappa f_k(\omega_b)/\sigma$. The first derivative of $f_k(x)$ is

$$
f_k'(x) = \frac{1}{x}\sum_{j=0}^{k}(k-2j)\beta^{-j}x^{k-2j} = \frac{1}{x}\sum_{j=0}^{\lfloor (k-1)/2 \rfloor}(k-2j)\beta^{-j}\left[x^{k-2j} - \frac{1}{(\beta x)^{k-2j}}\right],
$$

where $\lfloor . \rfloor$ denotes the floor operator. Using $\rho_b\omega_b = 1/\beta$, we then get

$$
f_k'(\omega_b) = \frac{1}{\omega_b}\sum_{j=0}^{\lfloor (k-1)/2 \rfloor}(k-2j)\beta^{-j}\left(\omega_b^{k-2j} - \rho_b^{k-2j}\right) > 0.
$$

So, we have

$$
\frac{\partial^2\pi_t}{\partial\kappa\partial r_{t+k}} = \frac{\beta^k}{\sigma}\left[f_k(\omega_b) + \kappa f_k'(\omega_b)\frac{\partial\omega_b}{\partial\kappa}\right] > 0.
$$

We now turn to (ii). In our selected equilibrium, we have

$$
\frac{\partial y_t}{\partial r_{t+k}} = \frac{\rho_b}{\sigma}\left[1 + \beta(1-\rho_b)\omega_b^{-k}\right] = \frac{1}{\sigma}\left(\rho_b + \beta^{k+1}\rho_b^{k+1} - \beta^{k+1}\rho_b^{k+2}\right),
$$

where the second equality follows from $\rho_b\omega_b = 1/\beta$. Therefore, we get

$$\begin{aligned}\frac{\partial^2 y_t}{\partial\kappa\partial r_{t+k}} &= \frac{1}{\sigma}\left[1+(k+1)\beta^{k+1}\rho_b^k-(k+2)\beta^{k+1}\rho_b^{k+1}\right]\frac{\partial\rho_b}{\partial\kappa} \\ &= \frac{1}{\sigma}\left[\left(1-\beta^{k+1}\rho_b^{k+1}\right)+(k+1)\beta^{k+1}\rho_b^k(1-\rho_b)\right]\frac{\partial\rho_b}{\partial\kappa} \\ &< 0.\end{aligned}$$

In the standard equilibrium, we have

$$\begin{aligned}\frac{\partial y_t}{\partial r_{t+k}} &= \frac{\kappa}{\beta\sigma^2(\omega_b-\rho_b)}\left(\frac{\rho_b^{-k}}{1-\rho_b}+\frac{\omega_b^{-k}}{\omega_b-1}\right)=\frac{1}{\sigma(\omega_b-\rho_b)}\left[(\omega_b-1)\rho_b^{-k}+(1-\rho_b)\omega_b^{-k}\right] \\ &= \frac{\beta^k}{\sigma}\left[(\omega_b-1)\left(\frac{\omega_b^k-\rho_b^k}{\omega_b-\rho_b}\right)+\rho_b^k\right],\end{aligned}$$

where the second equality follows from $\rho_b+\omega_b = 1+1/\beta+\kappa/(\beta\sigma)$ and $\rho_b\omega_b = 1/\beta$, and the third one from $\rho_b\omega_b = 1/\beta$. For $k=0$, we straightforwardly get $\partial y_t/\partial r_t = 1/\sigma$ and hence $\partial^2 y_t/\partial\kappa\partial r_t = 0$. For $k=1$, we get $\partial y_t/\partial r_{t+1} = (\beta/\sigma)(\rho_b+\omega_b-1) = (1/\sigma)(1+\kappa/\sigma)$ and hence $\partial^2 y_t/\partial\kappa\partial r_{t+1} = \sigma^{-2} > 0$. Finally, for $k\geq 2$, we get

$$\frac{\partial y_t}{\partial r_{t+k}} = \frac{\beta^k}{\sigma}\left[(\omega_b-1)f_{k-1}(\omega_b)+\rho_b^k\right] = \frac{\beta^k}{\sigma}\left[(\omega_b-1)f_{k-1}(\omega_b)+\beta^{-k}\omega_b^{-k}\right],$$

where the function $f_{k-1}(.)$ is defined above and where the second equality follows from $\rho_b\omega_b = 1/\beta$. So,

$$\begin{aligned}\frac{\partial^2 y_t}{\partial\kappa\partial r_{t+k}} &= \frac{\beta^k}{\sigma}\left[f_{k-1}(\omega_b)+(\omega_b-1)f'_{k-1}(\omega_b)-k\beta^{-k}\omega_b^{-k-1}\right]\frac{\partial\omega_b}{\partial\kappa} \\ &= \frac{\beta^k}{\sigma}\left[(\omega_b-1)f'_{k-1}(\omega_b)+\frac{1}{\omega_b}\left(\sum_{j=0}^{k-1}\omega_b^{k-j}\rho_b^j-k\rho_b^k\right)\right]\frac{\partial\omega_b}{\partial\kappa} \\ &> 0,\end{aligned}$$

where the second equality follows from $\rho_b\omega_b = 1/\beta$.

Appendix B: MIU Model (Presentation and Log-Linearization)

In this appendix (and the following ones), to lighten up the notation, we sometimes omit function arguments when no ambiguity results.

B.1. Households

Households get utility from consumption (c_t) and real money (m_t), and disutility from labor (h_t). Their intertemporal utility is

$$\mathcal{U}_t = \mathbb{E}_t\left\{\sum_{k=0}^{\infty}\beta^k\zeta_{t+k}\left[u(c_{t+k},m_{t+k})-\frac{v(h_{t+k})}{\varphi_{1,t+k}}\right]\right\},$$

where $\beta\in(0,1)$. The utility function u, defined over the set of pairs of positive real numbers $\mathbb{R}^2_{>0}$, is twice differentiable, strictly increasing ($u_c>0$, $u_m>0$), strictly concave ($u_{cc}<0$, $u_{mm}<0$, $u_{cc}u_{mm}-(u_{cm})^2>0$), with $u_{cm}\geq 0$, and it satisfies the standard Inada conditions

$$\begin{aligned}\lim_{c_t\to 0}u_c(c_t,m_t) &= +\infty, & \lim_{c_t\to+\infty}u_c(c_t,m_t) &= 0, \\ \lim_{m_t\to 0}u_m(c_t,m_t) &= +\infty, & \lim_{m_t\to+\infty}u_m(c_t,m_t) &= 0.\end{aligned}$$

The labor-disutility function v, defined over the set of non-negative real numbers $\mathbb{R}_{\geq 0}$, is twice differentiable, strictly increasing ($v' > 0$), and weakly convex ($v'' \geq 0$). The intertemporal utility $\mathcal{U}_t$ is affected by two stochastic exogenous shocks of mean one: the discount-factor shock ζ_t, and the labor-disutility shock $\varphi_{1,t}$. The latter shock is the first of the four alternative supply shocks that we consider.

Households choose c_t, h_t, m_t, and real bonds b_t to maximize their utility function subject to their budget constraint

$$c_t + b_t + m_t \leq \frac{I_{t-1}}{\Pi_t} b_{t-1} + \frac{I^m_{t-1}}{\Pi_t} m_{t-1} + w_t h_t + \tau_t, \tag{B.1}$$

where I_t denotes the gross nominal interest rate on bonds, I^m_t the gross nominal interest rate on money, $\Pi_t \equiv P_t/P_{t-1}$ the gross inflation rate (with P_t the price level), w_t the real wage, and τ_t captures firm profits and the government's lump-sum taxes or transfers. Let λ_t denote the Lagrange multiplier on the period-t budget constraint. The first-order conditions of this maximization problem are

$$\lambda_t = \zeta_t u_c(c_t, m_t), \tag{B.2}$$

$$\frac{1}{I_t} = \beta \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\lambda_t \Pi_{t+1}} \right\}, \tag{B.3}$$

$$\lambda_t w_t = \frac{\zeta_t v'(h_t)}{\varphi_{1,t}}, \tag{B.4}$$

$$\zeta_t u_m(c_t, m_t) = \lambda_t - \beta I^m_t \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\Pi_{t+1}} \right\}.$$

Using (B.2) and (B.3), we can rewrite the last condition as

$$\frac{I^m_t}{I_t} = 1 - \frac{u_m(c_t, m_t)}{u_c(c_t, m_t)}. \tag{B.5}$$

B.2. Firms

There is a continuum of monopolistically competitive firms owned by households and indexed by $i \in [0,1]$. Each firm i uses $h_t(i)$ units of labor to produce

$$y_t(i) = \varphi_{2,t} f[h_t(i)] \tag{B.6}$$

units of output. The production function f, defined over $\mathbb{R}_{\geq 0}$, is twice differentiable, strictly increasing ($f' > 0$), weakly concave ($f'' \leq 0$), and such that $f(0) = 0$. The stochastic exogenous technology shock $\varphi_{2,t}$, of mean one, is the second of the four alternative supply shocks that we consider. The third supply shock that we consider, $\varphi_{3,t}$, also of mean one, captures a labor subsidy received by firms (when $\varphi_{3,t} > 1$) or labor tax paid by firms (when $\varphi_{3,t} < 1$): if W_t denotes the pre-subsidy or pre-tax nominal wage, then the after-subsidy or after-tax nominal wage paid by firms is $W_t/\varphi_{3,t}$.

Following Calvo (1983), we assume that at any date, each firm, whatever its history, has the probability $\theta \in [0,1)$ not to be allowed to reset its price. If allowed to reset its price at date t, firm i chooses its new price $P^*_t(i)$ to maximize the present value of the profits that this price will generate:

$$\mathbb{E}_t \left\{ \sum_{k=0}^{+\infty} (\beta\theta)^k \frac{\lambda_{t+k}}{\lambda_t \Pi_{t,t+k}} \left[P^*_t(i) y_{t+k}(i) - \frac{W_{t+k} h_{t+k}(i)}{\varphi_{3,t+k}} \right] \right\},$$

subject to the production function (B.6) and the demand schedule

$$y_{t+k}(i) = \left[\frac{P^*_t(i)}{P_{t+k}} \right]^{-\varepsilon \varphi_{4,t+k}} y_{t+k}, \tag{B.7}$$

where $\Pi_{t,t+k} \equiv P_{t+k}/P_t$ for any $k \in \mathbb{N}$, $\varepsilon > 0$ denotes the steady-state elasticity of substitution between differentiated goods, and $y_t \equiv [\int_0^1 y_t(i)^{(\varepsilon\varphi_{4,t}-1)/(\varepsilon\varphi_{4,t})} di]^{\varepsilon\varphi_{4,t}/(\varepsilon\varphi_{4,t}-1)}$. The stochastic exogenous shock $\varphi_{4,t}$, of mean one, affecting the elasticity of substitution between differentiated goods, is the last of the four alternative supply shocks that we consider.

Using (B.6), we can rewrite the present value of the profits generated by $P_t^*(i)$ as

$$\mathbb{E}_t \left\{ \sum_{k=0}^{+\infty} (\beta\theta)^k \frac{\lambda_{t+k}}{\lambda_t \Pi_{t,t+k}} \left[P_t^*(i)\, y_{t+k}(i) - \frac{W_{t+k}}{\varphi_{3,t+k}} f^{-1} \left[\frac{y_{t+k}(i)}{\varphi_{2,t+k}} \right] \right] \right\}.$$

Choosing $P_t^*(i)$ to maximize this present value subject to (B.7) leads to the first-order condition

$$\mathbb{E}_t \left\{ \sum_{k=0}^{+\infty} (\beta\theta)^k \frac{\lambda_{t+k}(\varepsilon\varphi_{4,t+k}-1)}{\lambda_t \Pi_{t,t+k}} \left[P_t^*(i) - \left(\frac{\varepsilon\varphi_{4,t+k}}{\varepsilon\varphi_{4,t+k}-1} \right) \frac{W_{t+k}}{\varphi_{2,t+k}\varphi_{3,t+k} f'[h_{t+k}(i)]} \right] y_{t+k}(i) \right\} = 0. \quad \text{(B.8)}$$

In the limit case of perfectly flexible prices ($\theta = 0$), and in a symmetric equilibrium ($P_t^*(i) = P_t$ and $h_t(i) = h_t$), this first-order condition becomes

$$P_t = \left(\frac{\varepsilon\varphi_{4,t}}{\varepsilon\varphi_{4,t}-1} \right) \frac{W_t}{\varphi_{2,t}\varphi_{3,t} f'(h_t)}. \quad \text{(B.9)}$$

B.3. Government

The government consists of a fiscal authority and a monetary authority. The fiscal authority consumes an exogenous quantity $g_t \geq 0$ of goods, does not issue bonds, and sets lump-sum taxes on households so as to balance its budget (making fiscal policy Ricardian). We assume for simplicity that government purchases g_t are wasted, but the results would be unchanged if they entered households' utility function in a separable way.

The monetary authority – i.e., the central bank – has two independent instruments: the nominal stock of money $M_t > 0$, or equivalently its (gross) growth rate $\mu_t \equiv M_t/M_{t-1} > 0$, and the (gross) nominal interest rate on money $I_t^m \geq 0$. We assume that the central bank injects reserves via lump-sum transfers.[1] The consolidated budget constraint of the government is thus

$$M_t = I_{t-1}^m M_{t-1} + P_t g_t - T_t, \quad \text{(B.10)}$$

where T_t denotes the net lump-sum tax imposed by the government (the fiscal authority's tax minus the monetary authority's transfer).

To capture a lower bound on I_t^m in a simple and stark way, we assume that cash (with no interest payments) is a perfect substitute for deposits at the central bank in terms of providing liquidity services to households. This introduces a zero lower bound (ZLB) for the net nominal IOR rate $I_t^m - 1$ in our model. In an equilibrium with $I_t^m > 1$, households will hold no cash. In an equilibrium with $I_t^m = 1$, the decomposition of money into reserves and cash will be indeterminate, but also inconsequential.

B.4. Market-Clearing Conditions

The bond-market-clearing condition is

$$b_t = 0,$$

the money-market-clearing condition is

$$m_t = \frac{M_t}{P_t}, \quad \text{(B.11)}$$

and the goods-market-clearing condition is

$$c_t + g_t = y_t. \quad \text{(B.12)}$$

[1] It would be straightforward to modify our model and allow changes in money balances to be matched by changes in the monetary authority's holdings of bonds issued by households or the fiscal authority; such features, however, would not play a role in our analysis.

B.5. Steady-State Existence and Uniqueness

We consider steady-state values of policy-instruments such that $I^m \geq 1$, $\mu = 1$, and $g \geq 0$. Since $\mu = 1$, the set of steady states is the same under sticky prices ($\theta > 0$) as under flexible prices ($\theta = 0$), so that we can use the first-order condition of firms' optimization problem under flexible prices (B.9) to characterize this set. We first use (B.2), (B.4), (B.6), (B.9), and (B.12) to get

$$u_c [f(h) - g, m] = \left(\frac{\varepsilon}{\varepsilon - 1}\right) \frac{v'(h)}{f'(h)}. \tag{B.13}$$

We then consider two alternative cases in turn, separable utility ($u_{cm} = 0$) and non-separable utility ($u_{cm} > 0$). We show that in both cases, the necessary and sufficient condition for steady-state existence and uniqueness is $I^m < 1/\beta$.

In the separable-utility case, the left-hand side of (B.13) does not depend on m and decreases from $+\infty$ to 0 as h increases from $\underline{h} \equiv f^{-1}(g)$ to $+\infty$. The right-hand side of (B.13) increases as h increases from $\underline{h}$ to $+\infty$. Therefore, there is a unique value of h in $(\underline{h}, +\infty)$ that satisfies (B.13). Moreover, (B.3), (B.5), and (B.13) imply

$$\left(\frac{\varepsilon - 1}{\varepsilon}\right) \frac{f'(h)}{v'(h)} u_m [f(h) - g, m] = 1 - \beta I^m. \tag{B.14}$$

The left-hand side of (B.14) decreases from $+\infty$ to 0 as m increases from 0 to $+\infty$. Therefore, there is a unique value of m that satisfies (B.14) if and only if the right-hand side of (B.14) is positive. In other words, there exists a unique steady state if and only if $I^m < 1/\beta$.

In the non-separable-utility case, (B.13) implicitly and uniquely defines a function $\mathcal{M}$ such that

$$m = \mathcal{M}(h). \tag{B.15}$$

This function is defined over $(\underline{h}, +\infty)$, and it is strictly increasing ($\mathcal{M}' > 0$). We then use (B.3), (B.5), (B.13), and (B.15) to get

$$\left(\frac{\varepsilon - 1}{\varepsilon}\right) \frac{f'(h)}{v'(h)} u_m [f(h) - g, \mathcal{M}(h)] = 1 - \beta I^m. \tag{B.16}$$

The function $z(h) \equiv u_m [f(h) - g, \mathcal{M}(h)]$ is strictly decreasing in h. The reason is that (B.13) implies that $u_c [f(h) - g, \mathcal{M}(h)]$ is strictly increasing in h, i.e. that

$$u_{cc} [f(h) - g, \mathcal{M}(h)] f'(h) + u_{cm} [f(h) - g, \mathcal{M}(h)] \mathcal{M}'(h) > 0,$$

which implies in turn that

$$z'(h) = u_{cm} f'(h) + u_{mm} \mathcal{M}'(h) < \frac{-f'(h)}{u_{cm}} \left(u_{cc} u_{mm} - u_{cm}^2\right) \leq 0,$$

where the functions u_{cc}, u_{mm}, and u_{cm} are evaluated at $[f(h) - g, \mathcal{M}(h)]$. Since $z'(h) < 0$, the left-hand side of (B.16) decreases from $+\infty$ to 0 as h increases from $\underline{h}$ to $+\infty$. Therefore, there is a unique value of h that satisfies this equation if and only if its right-hand side is positive. In other words, there exists a unique steady state if and only if $I^m < 1/\beta$.

B.6. Log-Linearization

We assume that $I^m < 1/\beta$ and log-linearize the equilibrium conditions of the model around its unique steady state. To derive the Phillips curve (C.2), we log-linearize firms' first-order condition (B.8), and use the definition of the real wage $w_t \equiv W_t/P_t$, to get

$$\widehat{P}_t^* = (1 - \beta\theta) \mathbb{E}_t \left\{ \sum_{k=0}^{+\infty} (\beta\theta)^k \left[\widehat{w}_{t+k} + \widehat{P}_{t+k} - \widehat{mp}_{t+k|t} - \widehat{\varphi}_{3,t+k} - \frac{\widehat{\varphi}_{4,t+k}}{\varepsilon - 1} \right] \right\}, \tag{B.17}$$

where variables with hats denote log-deviations from steady-state values, $i_t \equiv \widehat{I}_t$, and $mp_{t+k|t}$ denotes the marginal productivity in period $t+k$ for a firm whose price was last set in period t. Log-linearizing the production function (B.6) gives

$$\widehat{h}_t = \frac{f}{f'h}\left(\widehat{y}_t - \widehat{\varphi}_{2,t}\right), \tag{B.18}$$

so that we can rewrite $\widehat{mp}_{t+k|t}$ as

$$\begin{aligned}\widehat{mp}_{t+k|t} &= \widehat{\varphi}_{2,t} + \frac{f''h}{f'}\widehat{h}_{t+k|t} = \widehat{mp}_{t+k} + \frac{f''h}{f'}\left(\widehat{h}_{t+k|t} - \widehat{h}_{t+k}\right) \\ &= \widehat{mp}_{t+k} + \frac{ff''}{(f')^2}\left(\widehat{y}_{t+k|t} - \widehat{y}_{t+k}\right) = \widehat{mp}_{t+k} - \frac{\varepsilon ff''}{(f')^2}\left(\widehat{P}_t^* - \widehat{P}_{t+k}\right),\end{aligned} \tag{B.19}$$

where mp_{t+k} denotes the average marginal productivity in period $t+k$. Using this result and

$$\pi_t \equiv \log(\Pi_t) = (1-\theta)\left(\widehat{P}_t^* - \widehat{P}_{t-1}\right),$$

and following the same steps as in, e.g., Galí (2008, Chapter 3), we can rewrite (B.17) as

$$\pi_t = \beta\mathbb{E}_t\{\pi_{t+1}\} + \frac{(1-\theta)(1-\beta\theta)}{\theta\left[1 - \frac{\varepsilon ff''}{(f')^2}\right]}\left(\widehat{w}_t - \widehat{mp}_t - \widehat{\varphi}_{3,t} - \frac{\widehat{\varphi}_{4,t}}{\varepsilon - 1}\right). \tag{B.20}$$

Now, log-linearizing the goods-market-clearing condition (B.12) gives

$$\widetilde{c}_t + \widetilde{g}_t = \widehat{y}_t, \tag{B.21}$$

where $\widetilde{c}_t \equiv (c/y)\widehat{c}_t$ and $\widetilde{g}_t \equiv (g/y)\widehat{g}_t$. Log-linearizing the first-order condition (B.4), and using (B.18) and (B.21), gives

$$\widehat{w}_t = \left(-\frac{u_{cc}y}{u_c} + \frac{v''h}{v'}\frac{f}{f'h}\right)\widehat{y}_t - \frac{u_{cm}m}{u_c}\widehat{m}_t + \frac{u_{cc}y}{u_c}\widetilde{g}_t - \widehat{\varphi}_{1,t} - \frac{v''h}{v'}\frac{f}{f'h}\widehat{\varphi}_{2,t}. \tag{B.22}$$

Moreover, we have

$$\widehat{mp}_t = \widehat{\varphi}_{2,t} + \frac{ff''}{(f')^2}\left(\widehat{y}_t - \widehat{\varphi}_{2,t}\right). \tag{B.23}$$

Using (B.22) and (B.23), we can then rewrite (B.20) as the Phillips curve

$$\pi_t = \beta\mathbb{E}_t\{\pi_{t+1}\} + \kappa\left(\widehat{y}_t - \delta_m\widehat{m}_t - \delta_g\widetilde{g}_t - \delta_\varphi\widehat{\varphi}_t\right) \tag{B.24}$$

with

$$\begin{aligned}\kappa &\equiv \frac{(1-\theta)(1-\beta\theta)}{\theta\left[1 - \frac{\varepsilon ff''}{(f')^2}\right]}\psi > 0, \\ \delta_m &\equiv \left(\frac{u_{cm}m}{u_c}\right)\psi^{-1} \geq 0, \\ \delta_g &\equiv \left(\frac{-u_{cc}y}{u_c}\right)\psi^{-1} \in (0,1), \\ \delta_\varphi &\equiv \left\{\mathbb{1}_{\varphi_t=\varphi_{1,t}} + \left[1 + \frac{v''h}{v'}\frac{f}{f'h} - \frac{ff''}{(f')^2}\right]\mathbb{1}_{\varphi_t=\varphi_{2,t}} + \mathbb{1}_{\varphi_t=\varphi_{3,t}} + \left(\frac{1}{\varepsilon-1}\right)\mathbb{1}_{\varphi_t=\varphi_{4,t}}\right\}\psi^{-1} > 0,\end{aligned}$$

where

$$\psi \equiv \frac{-u_{cc} y}{u_c} + \frac{v'' h}{v'} \frac{f}{f' h} - \frac{f f''}{(f')^2} > 0.$$

Note that, to write this Phillips curve in a compact way, we have considered a single supply shock $\varphi_t \in \{\varphi_{1,t}, \varphi_{2,t}, \varphi_{3,t}, \varphi_{4,t}\}$ and used indicator functions in the definition of δ_φ: for any $k \in \{1,2,3,4\}$, $\mathbb{1}_{\varphi_t = \varphi_{k,t}}$ takes the value one if $\varphi_t = \varphi_{k,t}$ and the value zero otherwise.

Log-linearizing the first-order condition (B.5) and using (B.21) gives the money-demand equation

$$\widehat{m}_t = \chi_y \left(\widehat{y}_t - \widetilde{g}_t \right) - \chi_i \left(i_t - i_t^m \right), \tag{B.25}$$

where $i_t^m \equiv \widehat{I}_t^m$ and

$$\chi_y \equiv \left(\frac{u_{cm} m}{u_c} - \frac{u_{mm} m}{u_m} \right)^{-1} \left(\frac{u_{cm} y}{u_m} - \frac{u_{cc} y}{u_c} \right) > 0,$$

$$\chi_i \equiv \left(\frac{u_{cm} m}{u_c} - \frac{u_{mm} m}{u_m} \right)^{-1} \left(\frac{\beta I^m}{1 - \beta I^m} \right) > 0.$$

Finally, log-linearizing the first-order condition (B.3) and using (B.21) gives the IS equation

$$\widehat{y}_t = \mathbb{E}_t \left\{ \widehat{y}_{t+1} \right\} - \frac{1}{\sigma} \left(i_t - \mathbb{E}_t \left\{ \pi_{t+1} \right\} - r_t \right) - \eta \mathbb{E}_t \left\{ \Delta \widehat{m}_{t+1} \right\} - \mathbb{E}_t \left\{ \Delta \widetilde{g}_{t+1} \right\}, \tag{B.26}$$

where $\Delta \equiv 1 - L$ denotes the first-difference operator, $r_t \equiv -\mathbb{E}_t\{\Delta \widehat{\zeta}_{t+1}\}$, and

$$\sigma \equiv \frac{-u_{cc} y}{u_c} > 0,$$

$$\eta \equiv \left(\frac{-u_{cc} y}{u_c} \right)^{-1} \frac{u_{cm} m}{u_c} \geq 0.$$

Appendix C: MIU Model (Log-Linearized Version)

This appendix proves Proposition 6 (stated in the main text), which essentially says that our MIU model delivers the same results as our simple model. The first subsection provides an outline of the proof, following the same steps as in Section 3 for our simple model. The following subsections prove some specific claims made in the first subsection.

For convenience, we keep the same notations as in our simple model in Section 3 for the reduced-form parameters (σ, κ, δ_g, δ_φ, χ_y, χ_i), the characteristic polynomial ($\mathcal{P}(X)$), the roots of this polynomial (ρ, ω_1, ω_2), and the exogenous driving term in the dynamic equation (Z_t), although all of them are in fact model-specific.

C.1. Outline of the Proof of Proposition 6

We start from the log-linearized reduced form of our MIU model, made of the IS equation (B.26), the Phillips curve (B.24), and the money-demand equation (B.25) derived in Appendix B.6. For simplicity, we replace the notations $\widehat{y}_t$, $\widehat{m}_t$, $\widetilde{g}_t$, and $\widehat{\varphi}_t$ with the notations y_t, m_t, g_t, and φ_t (as everywhere in the main text), and we thus write these equations as

$$y_t = \mathbb{E}_t \left\{ y_{t+1} \right\} - \frac{1}{\sigma} \left(i_t - \mathbb{E}_t \left\{ \pi_{t+1} \right\} - r_t \right) + \eta \left(m_t - \mathbb{E}_t \left\{ m_{t+1} \right\} \right) + g_t - \mathbb{E}_t \left\{ g_{t+1} \right\}, \tag{C.1}$$

$$\pi_t = \beta \mathbb{E}_t \left\{ \pi_{t+1} \right\} + \kappa \left(y_t - \delta_m m_t - \delta_g g_t - \delta_\varphi \varphi_t \right), \tag{C.2}$$

$$m_t = \chi_y \left(y_t - g_t \right) - \chi_i \left(i_t - i_t^m \right), \tag{C.3}$$

where $\beta \in (0,1)$, $\eta \geq 0$, $\delta_m \geq 0$, $\delta_g \in (0,1)$, and all the other parameters are positive.

In the case in which the utility function is not separable in consumption and money, we have $\eta > 0$ and $\delta_m > 0$. In this case, the IS equation (C.1) involves real-money terms (in m_t and $\mathbb{E}_t\{m_{t+1}\}$) because the marginal utility of consumption in the consumption Euler equation depends on real money. Similarly, the Phillips curve (C.2) involves a real-money term (in m_t) because real money increases the marginal utility of consumption, which in turn decreases the real wage and hence the marginal cost of production of firms. In the alternative case in which the utility function is separable in consumption and money, we have $\eta = \delta_m = 0$, and these two equations become identical to the IS equation (1) and the Phillips curve (2) of the two models considered so far (the basic NK model in Section 2 and our simple model in Section 3). The money-demand equation (C.3) is isomorphic to its counterpart (7) in our simple model, except for the presence of the government-purchases shock g_t. This shock appears in (C.3) because money demand now depends on consumption, which we have eliminated using the goods-market-clearing condition.

Our MIU model implies, in particular, the following two restrictions on the reduced-form parameters:

$$\eta = \frac{\delta_m}{\delta_g}, \tag{C.4}$$

$$\delta_m \chi_y < 1, \tag{C.5}$$

as we show in Appendix C.2. These restrictions will play an important role in our determinacy result below (as we will see). The equality (C.4) says that the weight of m_t relative to g_t (and $\mathbb{E}_t\{m_{t+1}\}$ relative to $\mathbb{E}_t\{g_{t+1}\}$) in the IS equation, η, is identical to the weight of m_t relative to g_t in the Phillips curve, δ_m/δ_g. The reason is that m_t and g_t come exclusively from the marginal utility of consumption in both equations. The marginal utility of consumption depends negatively on consumption, and therefore positively on g_t for a given y_t (through the goods-market-clearing condition); and it depends non-negatively on m_t, with a weight of m_t relative to g_t equal to $\eta = \delta_m/\delta_g$. The inequality (C.5) reflects how holding money mitigates changes in firms' marginal cost of production (through the real wage). For a given spread $i_t - i_t^m$, a rise in output y_t has two opposite effects on firms' marginal cost of production (i.e., on the term in factor of κ in the Phillips curve): a standard positive direct effect (with elasticity 1), and a negative indirect effect via the implied rise in money m_t (with elasticity $\delta_m \chi_y$). The inequality states that the direct effect dominates the indirect one (i.e., $\delta_m \chi_y < 1$).

Under permanently exogenous monetary-policy instruments i_t^m and M_t (in particular $i_t^m = i_t^*$ exogenous for all $t \in \mathbb{Z}$), the IS equation (C.1), the Phillips curve (C.2), the money-demand equation (C.3), and the identities $m_t = M_t - p_t$ and $\pi_t = p_t - p_{t-1}$ lead to the following dynamic equation relating p_t to $\mathbb{E}_t\{p_{t+2}\}$, $\mathbb{E}_t\{p_{t+1}\}$, p_{t-1}, and exogenous terms:

$$\mathbb{E}_t\left\{L\mathcal{P}\left(L^{-1}\right)p_t\right\} = Z_t$$

$$\begin{aligned}
\text{with} \quad \mathcal{P}(X) \equiv\ & X^3 - \left[2 + \frac{1}{\beta} + \frac{\kappa}{\beta\sigma} + \frac{(1-\delta_g)\,\delta_m\kappa}{\beta\delta_g} + \frac{\chi_y}{\sigma\chi_i}\right]X^2 + \left[1 + \frac{2}{\beta} + \frac{\kappa}{\beta\sigma}\right. \\
& \left. + \frac{(1-\delta_g)\,\delta_m\kappa}{\beta\delta_g} + \frac{(1+\beta)\,\chi_y}{\beta\sigma\chi_i} + \frac{(1-\delta_m\chi_y)\,\kappa}{\beta\sigma\chi_i}\right]X - \left(\frac{1}{\beta} + \frac{\chi_y}{\beta\sigma\chi_i}\right),
\end{aligned}$$

$$\begin{aligned}
Z_t \equiv\ & \frac{-\kappa}{\beta\sigma}\left(i_t^* - r_t\right) + \left[\frac{(1-\delta_g)\,\delta_m}{\delta_g} + \frac{1-\delta_m\chi_y}{\sigma\chi_i}\right]\frac{\kappa}{\beta}M_t - \frac{(1-\delta_g)\,\delta_m\kappa}{\beta\delta_g}\mathbb{E}_t\left\{M_{t+1}\right\} \\
& + \left(1 + \frac{\chi_y}{\sigma\chi_i}\right)\frac{(1-\delta_g)\,\kappa}{\beta}g_t - \frac{(1-\delta_g)\,\kappa}{\beta}\mathbb{E}_t\left\{g_{t+1}\right\} - \left(1 + \frac{\chi_y}{\sigma\chi_i}\right)\frac{\delta_\varphi\kappa}{\beta}\varphi_t + \frac{\delta_\varphi\kappa}{\beta}\mathbb{E}_t\left\{\varphi_{t+1}\right\},
\end{aligned}$$

where we have used the equality (C.4) to replace η by δ_m/δ_g. Using the inequality (C.5), we show in Appendix C.3 that, as in our simple model of Section 3, the characteristic polynomial $\mathcal{P}(X)$ has one root inside the unit circle ($\rho \in (0,1)$) and two roots outside the unit circle (ω_1 and ω_2 with $|\omega_1| \leq |\omega_2|$). With one eigenvalue inside the unit circle (ρ) for one predetermined variable (p_{t-1}), thus, our MIU model satisfies

Blanchard and Kahn's (1980) conditions and has a unique bounded solution under permanently exogenous monetary-policy instruments.

In the MIU model, as in the simple model of Section 3, setting exogenously i_t^m and M_t amounts to following a "shadow Wicksellian rule" for i_t. Indeed, if the price level rises (making real money fall, given that nominal money is fixed), or if output rises, then the marginal utility of real money increases. Since the IOR rate is fixed, the interest rate on bonds has then to increase for private agents to remain indifferent between holding money and holding bonds. What is different from Section 3, however, is that existing results for Wicksellian rules in the basic NK model (e.g., Woodford, 2003, Chapter 4) do not apply to the MIU model with non-separable utility (i.e. with $\eta = \delta_m/\delta_g > 0$). In fact, not all Wicksellian rules would ensure determinacy in this model. What our determinacy result says, thus, is that the specific shadow Wicksellian rule that arises under permanently exogenous monetary-policy instruments, given the restriction (C.5) that the model imposes on its coefficients, always delivers determinacy.

We solve the dynamic equation forward in the same way as in Section 3, and obtain that inflation in the unique bounded solution is again characterized by (10) − keeping in mind, though, that the roots ρ, ω_1, ω_2, and the exogenous driving term Z_t have changed. Using the solution for inflation (10), the Phillips curve (C.2), and the identities $m_t = M_t - p_t$ and $\pi_t = p_t - p_{t-1}$, we then get the solution for output:

$$y_t = -\vartheta p_{t-1} + \delta_m M_t + \delta_g g_t + \delta_\varphi \varphi_t - \frac{\mathbb{E}_t}{(\omega_2 - \omega_1)\,\kappa} \left\{ \sum_{k=0}^{+\infty} \left(\xi_1 \omega_1^{-k-1} - \xi_2 \omega_2^{-k-1} \right) Z_{t+k} \right\}, \tag{C.6}$$

where now $\vartheta \equiv (1-\rho)(1-\beta\rho)/\kappa + \delta_m \rho$ and $\xi_j \equiv \beta(\omega_j + \rho - 1) + \kappa\delta_m - 1$ for $j \in \{1, 2\}$. Like our simple model's equilibrium (10)-(11), and unlike the basic NK model's standard equilibrium (5)-(6), the MIU model's equilibrium (10) and (C.6) involves only ω_1^{-k} and ω_2^{-k} terms with $\omega_1 > 1$ and $\omega_2 > 1$. Therefore, the longer the horizon k, the smaller the effects of shocks occurring at date $t+k$ on inflation and output at date t in the MIU model, regardless of which type of shock (preference, monetary, fiscal, or supply) we consider. In particular, neither the forward-guidance puzzle nor the fiscal-multiplier puzzle can arise in the MIU model. Moreover, because determinacy obtains for any degree of price stickiness $\theta \in (0, 1)$ and in particular as $\theta \to 0$, the paradox of flexibility does not arise either. In Appendix C.4, we show that the limits of π_t and y_t as $\theta \to 0$ take finite values, and that these values coincide with the values that π_t and y_t take under perfectly flexible prices.

In Appendix C.5, we show that we can asymptotically remove the monetary friction from our MIU model in (at least) two cases: the case with separable utility, and the case of utility over a constant-elasticity-of-substitution (CES) aggregator of money and consumption. In either case, as we remove the monetary friction at the same speed as we shrink the steady-state spread between the interest rate on bonds and the IOR rate, the steady state and reduced form of our MIU model converge to the steady state and reduced form of the basic NK model, with real money balances bounded away from zero and infinity along the way. In particular, the reduced-form parameters σ, κ, δ_g, and δ_φ converge to their counterparts in the basic NK model, while η, δ_m, $1/\chi_i$, and χ_y/χ_i converge to zero. As a result, the characteristic polynomial $\mathcal{P}(X)$ goes to $(X-1)\mathcal{P}_b(X)$; its roots ρ, ω_1, and ω_2 go respectively to ρ_b, 1, and ω_b; and the exogenous driving term Z_t goes to Z_t^b. Using these limit results, we get that the unique local equilibrium of our MIU model (10) and (C.6) converges to (12)-(13). Thus, our MIU model serves to select the same equilibrium of the basic NK model under a permanently exogenous policy rate as our simple model in the previous section.

Proposition 6 follows.

C.2. Restrictions on the Reduced-Form Parameters

The reduced-form parameters η, δ_m, and δ_g, which are defined in Appendix B.6, are straightforwardly linked to each other through the equality (C.4). The reduced-form parameters δ_m and χ_y, which are also defined in Appendix B.6, satisfy the inequality (C.5) because

$$
\begin{aligned}
1-\delta_m\chi_y &= 1-\left[\frac{-u_{cc}y}{u_c}+\frac{v''h}{v'}\frac{f}{f'h}-\frac{ff''}{(f')^2}\right]^{-1}\left(\frac{u_{cm}m}{u_c}-\frac{u_{mm}m}{u_m}\right)^{-1}\left(\frac{u_{cm}c}{u_m}-\frac{u_{cc}y}{u_c}\right)\frac{u_{cm}m}{u_c}\\
&= \left[\frac{-u_{cc}y}{u_c}+\frac{v''h}{v'}\frac{f}{f'h}-\frac{ff''}{(f')^2}\right]^{-1}\left(\frac{u_{cm}m}{u_c}-\frac{u_{mm}m}{u_m}\right)^{-1}\left\{\left[\frac{v''h}{v'}\frac{f}{f'h}-\frac{ff''}{(f')^2}\right]\right.\\
&\quad \left.\left(\frac{u_{cm}m}{u_c}-\frac{u_{mm}m}{u_m}\right)+\frac{(y-c)\,mu_{cc}u_{mm}}{u_cu_m}+\frac{cm}{u_cu_m}\left(u_{cc}u_{mm}-u_{cm}^2\right)\right\}\\
&> 0.
\end{aligned}
$$

C.3. Root Analysis

We first show that $0<\rho<1<|\omega_1|\leq|\omega_2|$. To that aim, we write the polynomial $\mathcal{P}(X)$ as

$$\mathcal{P}(X)=X^3-a_2X^2+a_1X-a_0$$

with

$$
\begin{aligned}
a_2 &\equiv 2+\frac{1}{\beta}+\frac{\kappa}{\beta\sigma}+\frac{(1-\delta_g)\,\delta_m\kappa}{\beta\delta_g}+\frac{\chi_y}{\sigma\chi_i}>3,\\
a_1 &\equiv 1+\frac{2}{\beta}+\frac{\kappa}{\beta\sigma}+\frac{(1-\delta_g)\,\delta_m\kappa}{\beta\delta_g}+\frac{(1+\beta)\,\chi_y}{\beta\sigma\chi_i}+\frac{(1-\delta_m\chi_y)\,\kappa}{\beta\sigma\chi_i}>0,\\
a_0 &\equiv \frac{1}{\beta}+\frac{\chi_y}{\beta\sigma\chi_i}>0,
\end{aligned}
$$

where the inequality $a_2>3$ comes from $\beta\in(0,1)$ and $\delta_g\in(0,1)$, and the inequality $a_1>0$ from $\delta_g\in(0,1)$ and (C.5).

We have $\mathcal{P}(0)=-a_0<0$ and $\mathcal{P}(1)=(1-\delta_m\chi_y)\kappa/(\beta\sigma\chi_i)>0$, where the last inequality comes from (C.5). Therefore, $\mathcal{P}(X)$ has either one or three real roots inside $(0,1)$. Moreover, the inequalities $a_2>0$, $a_1>0$, and $a_0>0$ imply that $\mathcal{P}(X)<0$ for all $X<0$, so that $\mathcal{P}(X)$ has no negative real roots. Therefore, $\mathcal{P}(X)$ has at least one real root inside $(0,1)$, which we denote by ρ, and its other two roots, which we denote by ω_1 and ω_2 with $|\omega_1|\leq|\omega_2|$, must be (i) both real and inside $(0,1)$, or (ii) both real and larger than 1, or (iii) both complex and conjugates of each other. Now, we have $\rho+\omega_1+\omega_2=a_2>3$. Therefore, Case (i) is impossible, and in Case (iii) the common real part of ω_1 and ω_2 is larger than 1. As a consequence, in the remaining two possible cases, namely Cases (ii) and (iii), ω_1 and ω_2 lie outside the unit circle.

We now show that ω_1 and ω_2 can be real numbers, and that they can also be complex (non-real) numbers. Consider, for example, the separable and iso-elastic specification

$$u(c_t,m_t)=\frac{c_t^{1-\sigma_c}-1}{1-\sigma_c}+\frac{m_t^{1-\sigma_m}-1}{1-\sigma_m},$$

where $\sigma_c>0$ and $\sigma_m>0$. Under this specification, σ, κ, δ_m, δ_g, and χ_y/χ_i do not depend on σ_m, but χ_i and χ_y do. Therefore, a_2 and a_0 do not depend on σ_m, but a_1 does. Since $\lim_{\sigma_m\to+\infty}\chi_i=0$, we have $\lim_{\sigma_m\to+\infty}a_1=+\infty$. As a consequence, for sufficiently large values of σ_m, $\mathcal{P}(X)=X^3-a_2X^2+a_1X-a_0$ is positive for all $X\geq1$, so that Case (ii) is impossible and ω_1 and ω_2 are complex numbers. Moreover, since $\lim_{\sigma_m\to0}\chi_i=+\infty$, we have

$$\lim_{\sigma_m\to0}\mathcal{P}\left(1+\frac{\chi_y}{\sigma\chi_i}\right)=-\left(1+\frac{\chi_y}{\sigma\chi_i}\right)\frac{\chi_y\kappa}{\beta\sigma^2\chi_i}<0.$$

Therefore, for sufficiently small values of σ_m, we have $\mathcal{P}[1+\chi_y/(\sigma\chi_i)]<0$, which, together with $\mathcal{P}(1)>0$, implies that ω_1 and ω_2 are positive real numbers.

By continuity, there also exist non-separable specifications of u that can make ω_1 and ω_2 real or complex depending on the calibration. Consider, for instance, the iso-elastic specification

$$u(c_t, m_t) = \frac{c_t^{1-\sigma_c} - 1}{1-\sigma_c} + \frac{m_t^{1-\sigma_m} - 1}{1-\sigma_m} + \epsilon c_t^{\nu} m_t^{1-\nu},$$

where $\nu \in (0,1)$ and $\epsilon > 0$. If ϵ is sufficiently small, then, as above, ω_1 and ω_2 will be real for sufficiently small values of σ_m and complex for sufficiently large values of σ_m.

C.4. Resolution of the Paradox of Flexibility

Using the definition of Z_t, and after some simple algebra, we can rewrite (10) and (C.6) as

$$\begin{aligned}
\pi_t \quad = \quad & -(1-\rho)\, p_{t-1} + \frac{\kappa}{\beta(\omega_2 - \omega_1)} \mathbb{E}_t \left\{ -\frac{1}{\sigma} \sum_{k=0}^{+\infty} \left(\omega_1^{-k-1} - \omega_2^{-k-1}\right)\left(i_{t+k}^* - r_{t+k}\right) \right. \\
& + \sum_{k=0}^{+\infty} \left(\xi_1^M \omega_1^{-k-1} - \xi_2^M \omega_2^{-k-1}\right) M_{t+k} - \sum_{k=0}^{+\infty} \left(\xi_1^g \omega_1^{-k-1} - \xi_2^g \omega_2^{-k-1}\right) g_{t+k} \\
& \left. + \sum_{k=0}^{+\infty} \left(\xi_1^\varphi \omega_1^{-k-1} - \xi_2^\varphi \omega_2^{-k-1}\right) \varphi_{t+k} \right\}, \qquad \text{(C.7)} \\
y_t \quad = \quad & -\vartheta p_{t-1} + \frac{\delta_m}{\delta_g} M_t + g_t + \frac{\mathbb{E}_t}{\beta(\omega_2 - \omega_1)} \left\{ \frac{1}{\sigma} \sum_{k=0}^{+\infty} \left(\xi_1 \omega_1^{-k-1} - \xi_2 \omega_2^{-k-1}\right)\left(i_{t+k}^* - r_{t+k}\right) \right. \\
& - \sum_{k=0}^{+\infty} \left(\xi_1 \xi_1^M \omega_1^{-k-1} - \xi_2 \xi_2^M \omega_2^{-k-1}\right) M_{t+k} + \sum_{k=0}^{+\infty} \left(\xi_1 \xi_1^g \omega_1^{-k-1} - \xi_2 \xi_2^g \omega_2^{-k-1}\right) g_{t+k} \\
& \left. - \sum_{k=0}^{+\infty} \left(\xi_1 \xi_1^\varphi \omega_1^{-k-1} - \xi_2 \xi_2^\varphi \omega_2^{-k-1}\right) \varphi_{t+k} \right\}, \qquad \text{(C.8)}
\end{aligned}$$

where $\vartheta \equiv (1-\rho)(1-\beta\rho)/\kappa + \delta_m \rho$ and

$$\begin{aligned}
\xi_j \quad & \equiv \quad \beta(\omega_j + \rho - 1) + \kappa\delta_m - 1, \\
\xi_j^M \quad & \equiv \quad \frac{1 - \delta_m \chi_y}{\sigma \chi_i} - \frac{(1-\delta_g)(\omega_j - 1)\,\delta_m}{\delta_g}, \\
\xi_j^g \quad & \equiv \quad \left(\omega_j - 1 - \frac{\chi_y}{\sigma\chi_i}\right)(1-\delta_g), \\
\xi_j^\varphi \quad & \equiv \quad \left(\omega_j - 1 - \frac{\chi_y}{\sigma\chi_i}\right)\delta_\varphi
\end{aligned}$$

for $j \in \{1, 2\}$.

The only parameter that depends on the degree of price stickiness θ in the structural equations (C.1), (C.2), and (C.3) is the slope κ of the Phillips curve (C.2). We have $\lim_{\theta \to 0} \kappa = +\infty$ and hence

$$\left[\frac{-\beta\sigma\delta_g}{\delta_g + (1-\delta_g)\,\sigma\delta_m}\right] \lim_{\theta\to 0} \left[\frac{\mathcal{P}(X)}{\kappa}\right] = X\left(X - \omega_1^n\right)$$

for any $X \in \mathbb{R}$, where

$$\omega_1^n \equiv 1 + \left[\frac{1 - \delta_m \chi_y}{\delta_g + (1-\delta_g)\,\sigma\delta_m}\right] \frac{\delta_g}{\chi_i} > 1,$$

where in turn the inequality follows from $\delta_g \in (0,1)$ and (C.5). Therefore, we get

$$\lim_{\theta\to 0} \rho = 0, \quad \lim_{\theta\to 0} \omega_1 = \omega_1^n, \quad \text{and} \quad \lim_{\theta\to 0} \omega_2 = +\infty. \tag{C.9}$$

Using (C.9) and

$$(1-\rho)(\omega_1 - 1)(\omega_2 - 1) = \mathcal{P}(1) = \frac{(1-\delta_m\chi_y)\kappa}{\beta\sigma\chi_i},$$

we also get that

$$\lim_{\theta\to 0} \frac{\kappa}{\omega_2} = \frac{\beta\sigma\delta_g}{\delta_g + (1-\delta_g)\sigma\delta_m}. \tag{C.10}$$

Using (C.9) and (C.10), we can easily determine the limits of (C.7) and (C.8) as $\theta \to 0$:

$$\begin{aligned}
\lim_{\theta\to 0} \pi_t \quad = \quad & -p_{t-1} + \frac{\delta_g}{\delta_g + (1-\delta_g)\sigma\delta_m}\mathbb{E}_t\left\{\sum_{k=0}^{+\infty} (\omega_1^n)^{-k-1}\left\{-\left(i^*_{t+k} - r_{t+k}\right)\right.\right. \\
& \left.\left. + (\omega_1^n - 1) M_{t+k} + \left[\frac{\chi_y}{\chi_i} - \sigma(\omega_1^n - 1)\right]\left[(1-\delta_g) g_{t+k} - \delta_\varphi \varphi_{t+k}\right]\right\}\right\} \\
& + \frac{\sigma\delta_g}{\delta_g + (1-\delta_g)\sigma\delta_m}\left[\frac{(1-\delta_g)\delta_m}{\delta_g} M_t + (1-\delta_g) g_t - \delta_\varphi\varphi_t\right],
\end{aligned} \tag{C.11}$$

$$\begin{aligned}
\lim_{\theta\to 0} y_t \quad = \quad & \frac{\delta_g\delta_m}{\delta_g + (1-\delta_g)\sigma\delta_m}\mathbb{E}_t\left\{\sum_{k=0}^{+\infty} (\omega_1^n)^{-k-1}\left\{i^*_{t+k} - r_{t+k} - (\omega_1^n - 1) M_{t+k}\right.\right. \\
& \left.\left. + \left[\sigma(\omega_1^n - 1) - \frac{\chi_y}{\chi_i}\right]\left[(1-\delta_g) g_{t+k} - \delta_\varphi \varphi_{t+k}\right]\right\}\right\} \\
& + \frac{\delta_g}{\delta_g + (1-\delta_g)\sigma\delta_m}\left[\delta_m M_t + \delta_g g_t + \left(1 + \frac{\sigma\delta_m}{\delta_g}\right)\delta_\varphi\varphi_t\right].
\end{aligned} \tag{C.12}$$

These limits are finite, unlike their counterparts in the basic NK model.

We now show that the right-hand sides of (C.11) and (C.12) coincide with the values taken by π_t and y_t when prices are perfectly flexible ($\theta = 0$). To determine these values, we first log-linearize the first-order condition of firms' optimization problem under flexible prices (B.9), and use (B.18), to get

$$\widehat{w}_t = \frac{ff''}{(f')^2}\widehat{y}_t + \left[1 - \frac{ff''}{(f')^2}\right]\widehat{\varphi}_{2,t} + \widehat{\varphi}_{3,t} + \frac{\widehat{\varphi}_{4,t}}{\varepsilon - 1}. \tag{C.13}$$

Using (B.22) and (C.13), considering a single supply shock $\varphi_t \in \{\varphi_{1,t}, \varphi_{2,t}, \varphi_{3,t}, \varphi_{4,t}\}$, and replacing the notations $\widehat{y}_t$, $\widehat{m}_t$, $\widetilde{g}_t$, and $\widehat{\varphi}_t$ by the notations y_t, m_t, g_t, and φ_t (for simplicity and consistency with the main text), we then get

$$y_t = \delta_m m_t + \delta_g g_t + \delta_\varphi \varphi_t. \tag{C.14}$$

Finally, using the IS equation (C.1), the money-demand equation (C.3), the identity $m_t = M_t - p_t$, the exogenous policy-rate setting $i_t^m = i_t^*$, and the solution for flexible-price output (C.14), we get the following dynamic equation under flexible prices:

$$\begin{aligned}
p_t \quad = \quad & (\omega_1^n)^{-1}\mathbb{E}_t\{p_{t+1}\} + \frac{\delta_g(\omega_1^n)^{-1}}{\delta_g + (1-\delta_g)\sigma\delta_m}\left\{-(i_t^* - r_t) + \left[\frac{1-\delta_m\chi_y}{\chi_i} + \frac{(1-\delta_g)\sigma\delta_m}{\delta_g}\right] M_t\right. \\
& - \frac{(1-\delta_g)\sigma\delta_m}{\delta_g}\mathbb{E}_t\{M_{t+1}\} + \left(\sigma + \frac{\chi_y}{\chi_i}\right)(1-\delta_g) g_t - \sigma(1-\delta_g)\mathbb{E}_t\{g_{t+1}\} \\
& \left. - \left(\sigma + \frac{\chi_y}{\chi_i}\right)\delta_\varphi\varphi_t + \sigma\delta_\varphi\mathbb{E}_t\{\varphi_{t+1}\}\right\},
\end{aligned}$$

where we have used the equality (C.4) to replace η by δ_m/δ_g. Iterating this equation forward to $+\infty$ leads to the following value for the price level p_t in our MIU model under flexible prices:

$$\begin{aligned} p_t \quad = \quad & \frac{\delta_g}{\delta_g + (1-\delta_g)\,\sigma\delta_m}\mathbb{E}_t\left\{\sum_{k=0}^{+\infty}(\omega_1^n)^{-k-1}\left\{-\left(i_{t+k}^* - r_{t+k}\right) + (\omega_1^n - 1)\,M_{t+k}\right.\right. \\ & \left.\left. + \left[\frac{\chi_y}{\chi_i} - \sigma\,(\omega_1^n - 1)\right]\left[(1-\delta_g)\,g_{t+k} - \delta_\varphi\varphi_{t+k}\right]\right\}\right\} \\ & + \frac{\sigma\delta_g}{\delta_g + (1-\delta_g)\,\sigma\delta_m}\left[\frac{(1-\delta_g)\,\delta_m}{\delta_g}M_t + (1-\delta_g)\,g_t - \delta_\varphi\varphi_t\right], \end{aligned} \tag{C.15}$$

which implies in turn that the value of $\pi_t \equiv p_t - p_{t-1}$ in our MIU model under flexible prices coincides with the right-hand side of (C.11). In turn, using (C.14), (C.15), and the identity $m_t = M_t - p_t$, we get that the value of y_t in our MIU model under flexible prices coincides with the right-hand side of (C.12). Thus, our MIU model solves the paradox of flexibility: the limits of π_t and y_t as $\theta \to 0$ are finite and coincide with the values of π_t and y_t when $\theta = 0$.

C.5. Convergence to the Basic NK Model

We start with the separable specification

$$u(c_t, m_t) = u_1\,(c_t) + \gamma u_2\,(m_t)\,,$$

where $\gamma > 0$ is a scale parameter. Under this specification, the steady-state value h of h_t, given by (B.13) with $u_c[f(h) - g, m] = u_1'[f(h) - g]$, is identical to the steady-state value of h_t in the basic NK model (with consumption-utility function u_1). The IS equation (C.1) and the Phillips curve (C.2) are also identical to the IS equation (1) and the Phillips curve (2) of the basic NK model, in the sense that their reduced-form parameters take the same values (in particular $\eta = 0$ and $\delta_m = 0$). The steady-state value m of m_t is given by (B.14), which can be rewritten as

$$u_2'\,(m) = \left(\frac{1-\beta I^m}{\gamma}\right)\left(\frac{\varepsilon}{\varepsilon - 1}\right)\frac{v'\,(h)}{f'\,(h)}. \tag{C.16}$$

If (I^m, γ) goes to $(1/\beta, 0)$ with $(1-\beta I^m)/\gamma$ bounded away from zero and infinity, as in the thought experiment of Subsection 4.1, then m is bounded away from zero and infinity. In this case, $\chi_y = (-u_2''m/u_2')^{-1}(-u_1''y/u_1')$ is also bounded away from zero and infinity, while $\chi_i = (-u_2''m/u_2')^{-1}\beta I^m/(1-\beta I^m)$ goes to infinity. Therefore, $1/\chi_i$ and χ_y/χ_i converge to zero, and the money-demand equation (C.3) converges to $i_t = i_t^m$. Alternatively, if I^m goes to $1/\beta$ holding γ constant, as in the policy experiment of Section 5, then m goes to infinity (asymptotic satiation). In that case, we still have $1/\chi_i$ and χ_y/χ_i converging to zero, and (C.3) converging to $i_t = i_t^m$, if the elasticity $-u_2''m/u_2'$ is bounded from above – a condition that is met, in particular, for isoelastic u_2 functions.

We now turn to the CES-based specification

$$u(c_t, m_t) = U\left\{\left[(1-\gamma)\,c_t^\alpha + \gamma m_t^\alpha\right]^{1/\alpha}\right\},$$

where $\alpha \in (-\infty, 1)$, $\gamma \in (0, 1)$, and the function U, defined over the set of positive real numbers $\mathbb{R}_{>0}$, is twice differentiable, strictly increasing ($U' > 0$), and strictly concave ($U'' < 0$). In addition, we impose that $U''(x)x/U'(x) \leq 1 - \alpha$ for any $x > 0$, which is the necessary and sufficient condition for $u_{cm} \geq 0$. Under this specification, (B.5) at the steady state can be rewritten as $m = \phi c = \phi[f(h) - g]$, where

$$\phi \equiv \left[\frac{\gamma}{(1-\gamma)\,(1-\beta I^m)}\right]^{\frac{1}{1-\alpha}},$$

which implies that (B.13) can in turn be rewritten as

$$(1-\gamma)\left[(1-\gamma)+\gamma\phi^{\alpha}\right]^{\frac{1-\alpha}{\alpha}} U'\left\{\left[(1-\gamma)+\gamma\phi^{\alpha}\right]^{\frac{1}{\alpha}}\left[f(h)-g\right]\right\} = \left(\frac{\varepsilon}{\varepsilon-1}\right)\frac{v'(h)}{f'(h)}. \tag{C.17}$$

If (I^m, γ) goes to $(1/\beta, 0)$ with $(1-\beta I^m)/\gamma$ bounded away from zero and infinity, as in the thought experiment of Subsection 4.1, then ϕ is bounded away from zero and infinity, and (C.17) converges to

$$U'\left[f(h)-g\right] = \left(\frac{\varepsilon}{\varepsilon-1}\right)\frac{v'(h)}{f'(h)}.$$

This last equation, which characterizes the limit value of h, is the same as the equation implicitly and uniquely defining the steady-state value $h^\star$ of h_t in the basic NK model (with consumption-utility function U). Therefore, h, y, and c converge respectively to $h^\star$, $y^\star \equiv f(h^\star)$, and $c^\star \equiv f(h^\star) - g$, while $m = \phi[f(h) - g]$ is bounded away from zero and infinity. As a consequence, we have $C \equiv [(1-\gamma)c^\alpha + \gamma m^\alpha]^{1/\alpha} \to c^\star$ and

$$\begin{aligned}
\frac{-u_{cc}y}{u_c} &= \left(\frac{c}{C}\right)^{\alpha}\frac{y}{c}\left\{(1-\alpha)\gamma\phi^{\alpha} + (1-\gamma)\left[\frac{-U''(C)C}{U'(C)}\right]\right\} &&\longrightarrow \frac{-U''(c^\star)y^\star}{U'(c^\star)},\\
\frac{u_{cm}m}{u_c} &= \gamma\phi^{\alpha}\left(\frac{c}{C}\right)^{\alpha}\left\{(1-\alpha) - \left[\frac{-U''(C)C}{U'(C)}\right]\right\} &&\longrightarrow 0,\\
\frac{-u_{mm}m}{u_m} &= \left(\frac{c}{C}\right)^{\alpha}\left\{(1-\alpha)(1-\gamma) + \gamma\phi^{\alpha}\left[\frac{-U''(C)C}{U'(C)}\right]\right\} &&\longrightarrow 1-\alpha,\\
\frac{u_{cm}y}{u_m} &= (1-\gamma)\left(\frac{c}{C}\right)^{\alpha}\frac{y}{c}\left\{(1-\alpha) - \left[\frac{-U''(C)C}{U'(C)}\right]\right\} &&\longrightarrow (1-\alpha)\frac{y^\star}{c^\star} - \left[\frac{-U''(c^\star)y^\star}{U'(c^\star)}\right].
\end{aligned}$$

Using these limit results, we get that the reduced-form parameters σ, κ, δ_g, and δ_φ converge to their counterparts in the basic NK model, while η, δ_m, $1/\chi_i$, and χ_y/χ_i converge to zero. We conclude that the steady state and reduced form of our MIU model, under the CES-based specification, converge to the steady state and reduced form of the basic NK model.

Appendix D: Model With Banks

This appendix proves Proposition 7 (stated in the main text), which essentially says that our model with banks delivers the same results as our simple model and our MIU model. The first subsection provides an outline of the proof, following the same steps as in Section 3 for our simple model and Appendix C.1 for our MIU model. The following subsections prove some specific claims made in the first subsection.

D.1. Outline of the Proof of Proposition 7

As we show in Diba and Loisel (2020), the IS equation of our model with banks is the same as the IS equation (1) of the basic NK model, while the Phillips curve and the money-demand equation of our model with banks are

$$\pi_t = \beta\mathbb{E}_t\{\pi_{t+1}\} + \kappa\left(y_t - \delta_m m_t - \delta_g g_t - \delta_\varphi \varphi_t\right), \tag{D.1}$$

$$m_t = \chi_y y_t - \chi_i\left(i_t - i_t^m\right) - \chi_g g_t - \chi_\varphi \varphi_t, \tag{D.2}$$

where $\beta \in (0,1)$, $\delta_g \in (0,1)$, $\chi_\varphi \geq 0$, and all the other parameters are positive. The Phillips curve (D.1) is isomorphic to its counterpart (C.2) in the MIU model – but not identical to it, since the reduced-form parameters κ, δ_m, δ_g, and δ_φ have changed (even though we keep, for convenience, the same notation). The reason why real reserves m_t appear in the Phillips curve (D.1) is that they reduce banking costs, which in turn lowers the borrowing costs of firms and hence their marginal cost of production. Like its counterpart (C.3) in the MIU model, the money-demand equation (D.2) involves the government-purchases shock g_t

because money demand depends on consumption, which we have eliminated using the goods-market-clearing condition.[2] Unlike (C.3), however, it also involves the supply shock φ_t, because the demand for reserves now depends on the volume of loans, which in turn depends on firms' wage bill, which in turn depends on the supply shock for a given output level.[3]

Our model with banks implies, in particular, the following restriction on the reduced-form parameters:

$$\sigma < \chi_y < \frac{1}{\delta_m}, \tag{D.3}$$

as we show in Diba and Loisel (2020). This double inequality will play a key role in our results below (as we will see). The first inequality in (D.3) arises from the fact that bank loans serve to finance the wage bill (or some fraction of it). If output y_t increases by 1% for given government purchases g_t, the marginal utility of consumption decreases by σ%; so, the wage, the wage bill, and loans all increase by more than σ%; and, in turn, so does the demand for reserves m_t for a given spread $i_t - i_t^m$ (i.e., $\chi_y > \sigma$). The second inequality in (D.3) is similar to the inequality (C.5) in the MIU model. Here, it reflects how holding reserves mitigates changes in banking costs. For a given spread $i_t - i_t^m$, a rise in output y_t has two opposite effects on firms' marginal cost of production (i.e., on the term in factor of κ in the Phillips curve): a standard positive direct effect (with elasticity 1), and a negative indirect effect via the implied rise in reserves m_t (with elasticity $\delta_m\chi_y$). The inequality states that the direct effect dominates the indirect one (i.e., $\delta_m\chi_y < 1$).

Under permanently exogenous monetary-policy instruments i_t^m and M_t (in particular $i_t^m = i_t^*$ exogenous for all $t \in \mathbb{Z}$), the IS equation (1), the Phillips curve (D.1), the money-demand equation (D.2), and the identities $m_t = M_t - p_t$ and $\pi_t = p_t - p_{t-1}$ lead to the following dynamic equation relating p_t to $\mathbb{E}_t\{p_{t+2}\}$, $\mathbb{E}_t\{p_{t+1}\}$, p_{t-1}, and exogenous terms:

$$\mathbb{E}_t\left\{L\mathcal{P}\left(L^{-1}\right)p_t\right\} = Z_t,$$

$$\begin{aligned}
\text{where}\quad \mathcal{P}(X) &\equiv X^3 - \left[2 + \frac{1}{\beta} + \frac{\chi_y}{\sigma\chi_i} + \left(\frac{1}{\sigma} - \delta_m\right)\frac{\kappa}{\beta}\right]X^2 + \left[1 + \frac{2}{\beta} + \left(1 + \frac{1}{\beta}\right)\frac{\chi_y}{\sigma\chi_i}\right.\\
&\quad \left. + \left(\frac{1}{\sigma} - \delta_m\right)\frac{\kappa}{\beta} + (1 - \delta_m\chi_y)\frac{\kappa}{\beta\sigma\chi_i}\right]X - \left(\frac{1}{\beta} + \frac{\chi_y}{\beta\sigma\chi_i}\right),\\
Z_t &\equiv \frac{-\kappa}{\beta\sigma}\left(i_t^* - r_t\right) + \left[\frac{1}{\sigma\chi_i} - \left(1 + \frac{\chi_y}{\sigma\chi_i}\right)\delta_m\right]\frac{\kappa}{\beta}M_t + \frac{\delta_m\kappa}{\beta}\mathbb{E}_t\left\{M_{t+1}\right\}\\
&\quad + \left[\left(1 + \frac{\chi_g}{\sigma\chi_i}\right) - \left(1 + \frac{\chi_y}{\sigma\chi_i}\right)\delta_g\right]\frac{\kappa}{\beta}g_t - \frac{(1-\delta_g)\kappa}{\beta}\mathbb{E}_t\left\{g_{t+1}\right\}\\
&\quad + \left[\frac{\chi_\varphi}{\sigma\chi_i} - \left(1 + \frac{\chi_y}{\sigma\chi_i}\right)\delta_\varphi\right]\frac{\kappa}{\beta}\varphi_t + \frac{\delta_\varphi\kappa}{\beta}\mathbb{E}_t\left\{\varphi_{t+1}\right\}.
\end{aligned}$$

Using the double inequality (D.3), we show in Appendix D.2 that the roots of the characteristic polynomial $\mathcal{P}(X)$ are three real numbers ρ, ω_1, and ω_2 such that $0 < \rho < 1 < \omega_1 < \omega_2$. With one eigenvalue inside the unit circle (ρ) for one predetermined variable (p_{t-1}), thus, our model with banks satisfies Blanchard and Kahn's (1980) conditions and has a unique bounded solution under permanently exogenous monetary-policy instruments.

In our model with banks, setting exogenously i_t^m and M_t also amounts to following a "shadow Wicksellian rule" for i_t, as in the previous two models (the simple model of Section 3 and the MIU model of Subsection 4.1). Existing results for Wicksellian rules in the basic NK model do not apply to our model with banks, and not all Wicksellian rules would ensure determinacy in this model. What our determinacy result says is that

[2] We no longer have $\chi_g = \chi_y$, though, because money demand now depends on y_t not only through consumption (via the goods-market-clearing condition), but also through loans (which are proportional to the wage bill).

[3] The only exception is when the supply shock is a markup shock – in which case $\chi_\varphi = 0$.

the specific shadow Wicksellian rule that arises under permanently exogenous monetary-policy instruments, given the restriction (D.3) that the model imposes on its coefficients, always delivers determinacy.

We determine the unique local equilibrium of our model with banks in the same way as the unique local equilibrium of our MIU model in Appendix C.1. We obtain that inflation and output in this equilibrium are again characterized by (10) and (C.6) – keeping in mind, though, that the roots ρ, ω_1, ω_2, the reduced-form parameters κ, δ_m, δ_g, δ_φ, and the exogenous driving term Z_t have changed. Since (10) and (C.6) involve only ω_1^{-k} and ω_2^{-k} terms with $\omega_1 > 1$ and $\omega_2 > 1$, neither the forward-guidance puzzle nor the fiscal-multiplier puzzle can arise in our model with banks. Moreover, because determinacy obtains for any degree of price stickiness $\theta \in (0,1)$ and in particular as $\theta \to 0$, the paradox of flexibility does not arise either in this model, as we formally show in Appendix D.3.

As we elaborate in Appendix D.4, as the scale parameter of banking costs and the steady-state interest-rate spread are shrunk to zero (at suitable rates, to keep a positive and finite level of steady-state real reserve balances in the limit), the steady state and reduced form of our model with banks converge to the steady state and reduced form of the basic NK model. Therefore, as previously, the characteristic polynomial $\mathcal{P}(X)$ goes to $(X-1)\mathcal{P}_b(X)$; its roots ρ, ω_1, and ω_2 go respectively to ρ_b, 1, and ω_b; and the exogenous driving term Z_t goes to Z_t^b. Using these limit results, we get again that the unique local equilibrium of our model with banks (10) and (C.6) converges to (12)-(13). Thus, our model with banks serves to select the same equilibrium of the basic NK model under a permanently exogenous policy rate as our previous two models. Proposition 7 follows.

D.2. Root Analysis

We show that $0 < \rho < 1 < \omega_1 < \omega_2$. The polynomial $\mathcal{P}(X)$ can be rewritten as

$$\begin{aligned}\mathcal{P}(X) &= X^3 - \left(\frac{1+2\beta+\beta\Theta_1+\Theta_2}{\beta}\right)X^2 + \left[\frac{2+\beta+(1+\beta)\Theta_1+\Theta_2+\Theta_3}{\beta}\right]X - \left(\frac{1+\Theta_1}{\beta}\right)\\ &= (X-1-\Theta_1)\left[X^2 - \left(\frac{1+\beta+\Theta_2}{\beta}\right)X + \frac{1}{\beta}\right] - \left(\frac{\Theta_1\Theta_2-\Theta_3}{\beta}\right)X,\end{aligned}$$

where $\Theta_1 \equiv \chi_y/(\sigma\chi_i) > 0$, $\Theta_2 \equiv (1/\sigma - \delta_m)\kappa$, and $\Theta_3 \equiv (1-\delta_m\chi_y)\kappa/(\sigma\chi_i)$. The double inequality (D.3) implies $\Theta_2 > 0$, $\Theta_3 > 0$, and $\Theta_1\Theta_2 - \Theta_3 = (\chi_y - \sigma)\kappa/(\sigma^2\chi_i) > 0$. Therefore, we get $\mathcal{P}(0) = -(1+\Theta_1)/\beta < 0$, $\mathcal{P}(1) = \Theta_3/\beta > 0$, $\mathcal{P}(1+\Theta_1) = -(\Theta_1\Theta_2 - \Theta_3)(1+\Theta_1)/\beta < 0$, and $\lim_{X\to+\infty}\mathcal{P}(X) = +\infty > 0$. As a consequence, the roots of $\mathcal{P}(X)$ are three real numbers ρ, ω_1, and ω_2 such that $0 < \rho < 1 < \omega_1 < 1+\Theta_1 < \omega_2$.

D.3. Resolution of the Paradox of Flexibility

Using the definition of Z_t, and after some simple algebra, we can rewrite (10) and (C.6) as

$$\begin{aligned}\pi_t = & -(1-\rho)p_{t-1} + \frac{\kappa}{\beta(\omega_2-\omega_1)}\mathbb{E}_t\left\{-\frac{1}{\sigma}\sum_{k=0}^{+\infty}\left(\omega_1^{-k-1} - \omega_2^{-k-1}\right)\left(i_{t+k}^* - r_{t+k}\right)\right.\\ & + \sum_{k=0}^{+\infty}\left(\xi_1^M\omega_1^{-k-1} - \xi_2^M\omega_2^{-k-1}\right)M_{t+k} - \sum_{k=0}^{+\infty}\left(\xi_1^g\omega_1^{-k-1} - \xi_2^g\omega_2^{-k-1}\right)g_{t+k}\\ & \left.+ \sum_{k=0}^{+\infty}\left(\xi_1^\varphi\omega_1^{-k-1} - \xi_2^\varphi\omega_2^{-k-1}\right)\varphi_{t+k}\right\},\end{aligned} \tag{D.4}$$

$$
\begin{aligned}
y_t &= -\vartheta p_{t-1} + g_t + \frac{\mathbb{E}_t}{\beta(\omega_2-\omega_1)} \left\{ \frac{1}{\sigma} \sum_{k=0}^{+\infty} \left(\xi_1 \omega_1^{-k-1} - \xi_2 \omega_2^{-k-1}\right) \left(i_{t+k}^* - r_{t+k}\right) \right. \\
&\quad - \sum_{k=0}^{+\infty} \left(\xi_1 \xi_1^M \omega_1^{-k-1} - \xi_2 \xi_2^M \omega_2^{-k-1}\right) M_{t+k} + \sum_{k=0}^{+\infty} \left(\xi_1 \xi_1^g \omega_1^{-k-1} - \xi_2 \xi_2^g \omega_2^{-k-1}\right) g_{t+k} \\
&\quad \left. - \sum_{k=0}^{+\infty} \left(\xi_1 \xi_1^\varphi \omega_1^{-k-1} - \xi_2 \xi_2^\varphi \omega_2^{-k-1}\right) \varphi_{t+k} \right\}, \qquad \text{(D.5)}
\end{aligned}
$$

where $\vartheta \equiv (1-\rho)(1-\beta\rho)/\kappa + \delta_m \rho$ and

$$
\begin{aligned}
\xi_j &\equiv \beta(\omega_j + \rho - 1) + \kappa \delta_m - 1, \\
\xi_j^M &\equiv \delta_m(\omega_j - 1) + \frac{1-\delta_m \chi_y}{\sigma \chi_i}, \\
\xi_j^g &\equiv (1-\delta_g)(\omega_j - 1) + \frac{\delta_g \chi_y - \chi_g}{\sigma \chi_i}, \\
\xi_j^\varphi &\equiv \delta_\varphi(\omega_j - 1) + \frac{\chi_\varphi - \delta_\varphi \chi_y}{\sigma \chi_i}
\end{aligned}
$$

for $j \in \{1, 2\}$.

The only parameter that depends on the degree of price stickiness θ in the structural equations (1), (D.1), and (D.2) is the slope κ of the Phillips curve (D.1). We have $\lim_{\theta \to 0} \kappa = +\infty$ and hence

$$
\left(\frac{-\beta\sigma}{1-\sigma\delta_m}\right) \lim_{\theta \to 0} \left[\frac{\mathcal{P}(X)}{\kappa}\right] = X(X - \omega_1^n)
$$

for any $X \in \mathbb{R}$, where

$$
\omega_1^n \equiv 1 + \frac{1-\delta_m \chi_y}{(1-\sigma\delta_m)\chi_i} > 1,
$$

where in turn the inequality follows from (D.3). Therefore, we get

$$
\lim_{\theta \to 0} \rho = 0, \quad \lim_{\theta \to 0} \omega_1 = \omega_1^n, \quad \text{and} \quad \lim_{\theta \to 0} \omega_2 = +\infty. \qquad \text{(D.6)}
$$

Using (D.6) and

$$
(1-\rho)(\omega_1 - 1)(\omega_2 - 1) = \mathcal{P}(1) = \frac{(1-\delta_m \chi_y)\kappa}{\beta\sigma\chi_i},
$$

we also get that

$$
\lim_{\theta \to 0} \frac{\kappa}{\omega_2} = \frac{\beta\sigma}{1-\sigma\delta_m}. \qquad \text{(D.7)}
$$

Using (D.6) and (D.7), we can easily determine the limits of (D.4) and (D.5) as $\theta \to 0$:

$$
\begin{aligned}
\lim_{\theta \to 0} \pi_t &= -p_{t-1} + \frac{1}{1-\sigma\delta_m} \mathbb{E}_t \left\{ \sum_{k=0}^{+\infty} (\omega_1^n)^{-k-1} \left\{ -\left(i_{t+k}^* - r_{t+k}\right) + (\omega_1^n - 1) M_{t+k} \right. \right. \\
&\quad \left. \left. - \left[\sigma(1-\delta_g)(\omega_1^n - 1) + \frac{\delta_g \chi_y - \chi_g}{\chi_i}\right] g_{t+k} + \left[\sigma\delta_\varphi(\omega_1^n - 1) + \frac{\chi_\varphi - \delta_\varphi \chi_y}{\chi_i}\right] \varphi_{t+k} \right\} \right\} \\
&\quad + \frac{\sigma}{1-\sigma\delta_m} \left[-\delta_m M_t + (1-\delta_g) g_t - \delta_\varphi \varphi_t\right], \qquad \text{(D.8)}
\end{aligned}
$$

$$
\begin{aligned}
\lim_{\theta\to 0} y_t &= \frac{\delta_m}{1-\sigma\delta_m}\mathbb{E}_t\left\{\sum_{k=0}^{+\infty}(\omega_1^n)^{-k-1}\left\{i_{t+k}^* - r_{t+k} - (\omega_1^n - 1)M_{t+k}\right.\right.\\
&\left.\left. + \left[\sigma(1-\delta_g)(\omega_1^n-1) + \frac{\delta_g\chi_y - \chi_g}{\chi_i}\right]g_{t+k} - \left[\sigma\delta_\varphi(\omega_1^n - 1) + \frac{\chi_\varphi - \delta_\varphi\chi_y}{\chi_i}\right]\varphi_{t+k}\right\}\right\}\\
&+\frac{1}{1-\sigma\delta_m}\left[\delta_m M_t + (\delta_g - \sigma\delta_m)g_t + \delta_\varphi\varphi_t\right]. \qquad \text{(D.9)}
\end{aligned}
$$

These limits are finite, unlike their counterparts in the basic NK model.

We now show that the right-hand sides of (D.8) and (D.9) coincide with the values taken by π_t and y_t when prices are perfectly flexible ($\theta = 0$). The flexible-price value of y_t is straightforwardly obtained by setting to zero the last term in the Phillips curve (D.1), which is proportional to (the log-deviation of) firms' marginal cost of production:

$$
y_t = \delta_m m_t + \delta_g g_t + \delta_\varphi\varphi_t. \qquad \text{(D.10)}
$$

Using the IS equation (1), the money-demand equation (D.2), the identity $m_t = M_t - p_t$, the exogenous policy-rate setting $i_t^m = i_t^*$, and the solution for flexible-price output (D.10), we get the following dynamic equation under flexible prices:

$$
\begin{aligned}
p_t &= (\omega_1^n)^{-1}\mathbb{E}_t\{p_{t+1}\} + \frac{(\omega_1^n)^{-1}}{1-\sigma\delta_m}\left\{-(i_t^* - r_t) + \left(\frac{1-\delta_m\chi_y}{\chi_i} - \sigma\delta_m\right)M_t + \sigma\delta_m\mathbb{E}_t\{M_{t+1}\}\right.\\
&+\left[\frac{\chi_g - \delta_g\chi_y}{\chi_i} + \sigma(1-\delta_g)\right]g_t - \sigma(1-\delta_g)\mathbb{E}_t\{g_{t+1}\}\\
&\left.+\left(\frac{\chi_\varphi - \delta_\varphi\chi_y}{\chi_i} - \sigma\delta_\varphi\right)\varphi_t + \sigma\delta_\varphi\mathbb{E}_t\{\varphi_{t+1}\}\right\}.
\end{aligned}
$$

Iterating this equation forward to $+\infty$ leads to the following value for the price level p_t in our model with banks under flexible prices:

$$
\begin{aligned}
p_t &= \frac{1}{1-\sigma\delta_m}\mathbb{E}_t\left\{\sum_{k=0}^{+\infty}(\omega_1^n)^{-k-1}\left\{-\left(i_{t+k}^* - r_{t+k}\right) + (\omega_1^n - 1)M_{t+k}\right.\right.\\
&\left.\left. - \left[\sigma(1-\delta_g)(\omega_1^n-1) + \frac{\delta_g\chi_y - \chi_g}{\chi_i}\right]g_{t+k} + \left[\sigma\delta_\varphi(\omega_1^n - 1) + \frac{\chi_\varphi - \delta_\varphi\chi_y}{\chi_i}\right]\varphi_{t+k}\right\}\right\}\\
&+\frac{\sigma}{1-\sigma\delta_m}\left[-\delta_m M_t + (1-\delta_g)g_t - \delta_\varphi\varphi_t\right], \qquad \text{(D.11)}
\end{aligned}
$$

which implies in turn that the value of $\pi_t \equiv p_t - p_{t-1}$ in our model with banks under flexible prices coincides with the right-hand side of (D.8). In turn, using (D.10), (D.11), and the identity $m_t = M_t - p_t$, we get that the value of y_t in our model with banks under flexible prices coincides with the right-hand side of (D.9). Thus, our model with banks solves the paradox of flexibility: the limits of π_t and y_t as $\theta \to 0$ are finite and coincide with the values of π_t and y_t when $\theta = 0$.

D.4. Convergence to the Basic NK Model

In a previous version of this paper (Diba and Loisel, 2019), we show that the steady state and reduced form of our model with banks converge to the steady state and reduced form of the basic NK model, with the steady-state stock of real reserves m bounded away from zero and infinity, as the scale parameter of banking costs γ and the steady-state interest-rate spread $1/\beta - I^m$ are shrunk to zero at the same speed (as in the thought experiment of Subsection 4.2).

More specifically, we prove this result in three steps. First, we show that the steady-state values of all endogenous variables in our model with banks converge, as $(I^m, \gamma) \to (1/\beta, 0)$, to their counterparts in the corresponding basic NK model – with the exception of the steady-state value of real reserves m, which does not exist in the basic NK model. Second, we show that m remains bounded away from zero and infinity along the way, provided that $(1/\beta - I^m)/\gamma$ is itself bounded away from zero and infinity, i.e. provided that $1/\beta - I^m$ and γ are shrunk to zero at the same speed. Third, we build on the first two steps to show that the reduced form of our model with banks converges to the reduced form of the basic NK model as $(I^m, \gamma) \to (1/\beta, 0)$ with $(1/\beta - I^m)/\gamma$ bounded away from zero and infinity.

In essence, making the steady-state IOR rate I^m go to the steady-state interest rate on bonds $I = 1/\beta$ asymptotically removes the steady-state opportunity cost of holding reserves. Making the banking-cost-scale parameter γ go to zero asymptotically removes the steady-state marginal banking cost (provided that m is bounded away from zero) and the steady-state marginal benefit of holding reserves (even when m is bounded from above). Imposing that $(1/\beta - I^m)/\gamma$ be bounded away from zero and infinity ensures that the steady-state opportunity cost and marginal benefit of holding reserves go hand in hand to zero, so that m is itself bounded away from zero and infinity. Asymptotically, given that all steady-state costs related to banking and reserve holding are removed, the steady state and reduced form of the model converge to the steady state and reduced form of the basic NK model.

It is straightforward to check that we still get this steady-state and reduced-form convergence, this time with m going to infinity (asymptotic satiation), as we hold γ constant and shrink only $1/\beta - I^m$ to zero (as in the policy experiment of Section 5), provided that two conditions are met. The first condition is that the marginal banking cost should go to zero as the stock of real reserves goes to infinity ($\lim_{m_t \to +\infty} \Gamma_\ell(\ell_t, m_t) = 0$, where ℓ_t denotes real loans and Γ the banking-cost function). The second condition is that banking-cost elasticities ($\Gamma_{\ell\ell}\ell/\Gamma_\ell$, $\Gamma_{mm}m/\Gamma_m$, $\Gamma_{\ell m}\ell/\Gamma_m$, and $\Gamma_{\ell m}m/\Gamma_\ell$) should be bounded from above. These two conditions are met, in particular, when the banking-cost function Γ is isoelastic, which happens when the loan-production and banker-labor-disutility functions are themselves isoelastic.

D.5. Defense of the Non-Satiation Assumption

In Diba and Loisel (2020), we present in detail our model with banks and show in particular that this model can account, in qualitative terms, for three key features of US inflation during the Great Recession: no significant deflation, little inflation volatility, and no significant inflation following quantitative-easing (QE) policies. These results, like our resolution of NK puzzles and paradoxes in the present paper, rest on the assumption that demand for bank reserves was not fully satiated in the US. For this reason, in Diba and Loisel (2020), we address in detail two types of arguments that go against our non-satiation view.

First, some observers may make a case for satiation noting that the federal-funds rate and Treasury-bill (T-bill) returns were below the IOR rate for several years in the aftermath of the crisis. We do not think this contradicts our claim that reserves still had a positive marginal convenience yield during this period. Most of the trading activity in the federal-funds market over this period involved banks borrowing funds from entities that do not have direct access to the IOR rate (particularly from Federal Home Loan Banks). Given the presence of such eager lenders, the federal-funds rate had to be below the IOR rate to incentivize the borrowers (banks with direct access to the IOR rate). As to T-bill returns, the low rates could reflect strong demand by non-bank entities – using T-bills as, e.g., collateral or international reserve asset. We formalize this counter-argument in Diba and Loisel (2020) by introducing government bonds providing liquidity services into our model with banks, and showing that the resulting model reconciles the observed negative spread between T-bill and IOR rates with our non-satiation assumption.

The second argument making a case for satiation of demand for reserves is the fact that large increases in reserve balances during the second and third rounds of quantitative easing (QE2 and QE3) had no apparent effect on expected inflation, as Reis (2016) points out. Our counter-argument is that this evidence may also be consistent with demand for reserves being close to satiation, rather than fully satiated. More specifically, we show in Diba and Loisel (2020) that in our model with banks, large increases in the money supply (say, doubling the stock of reserves) can have very small inflationary effects (around twenty basis points) if the demand for reserves is close to satiation and the monetary expansion is perceived as temporary (say, balance-sheet normalization is expected to occur in about five years). Distinguishing between the two possibilities (arbitrarily close to satiation versus fully satiated demand) may be difficult in practice. In fact, in contrast to Reis's (2016) evidence about expected inflation, Krishnamurthy and Lustig (2019) find statistically significant effects of monetary policy, during QE2 and QE3, on the convenience yield of US Treasury bills and the foreign-exchange value of the dollar.

Appendix E: CIA Model

In this appendix, we consider a sticky-price CIA model with leisure serving as the credit good. We show that the log-linearized reduced form of this model cannot smoothly converge to the log-linearized reduced form of the basic NK model, neither by gradually removing the monetary friction, nor by gradually satiating the demand for money. As a result, we cannot use this model to select an equilibrium of the basic NK model under an exogenous policy rate.

Whenever possible, we use the same notation as with the MIU model in Appendix B. For simplicity, we abstract from preference, supply, and government-purchases shocks (i.e. $\zeta_t = \varphi_{1,t} = \varphi_{2,t} = \varphi_{3,t} = \varphi_{4,t} = 1$ and $g_t = g \geq 0$), as we do not need them to make our point.

In our CIA model, households choose b_t, c_t, h_t, and m_t to maximize

$$U_t = \mathbb{E}_t \left\{ \sum_{k=0}^{\infty} \beta^k \left[u\left(c_{t+k}\right) - v\left(h_{t+k}\right) \right] \right\}$$

subject to their budget constraint

$$b_t + m_t \leq \frac{I_{t-1}}{\Pi_t} b_{t-1} + \frac{I_{t-1}^m}{\Pi_t} \left(m_{t-1} - c_{t-1}\right) + \frac{w_{t-1}}{\Pi_t} h_{t-1} + \tau_t$$

and their cash-in-advance constraint

$$c_t \leq m_t,$$

taking all prices as given. The monetary friction is that households need to use cash to buy goods in the goods exchange, and they can acquire cash only in the financial exchange that takes place before the goods exchange within the same period. Since all of consumption is subject to the CIA constraint, the model has no parameter that we can shrink to gradually remove the monetary friction. Therefore, we cannot make the model converge smoothly to the basic NK model by gradually removing the monetary friction.

In fact, the only parameters of the model that do not exist in the basic NK model are the steady-state interest rate on money I^m and the steady-state gross growth rate of money μ. So, the only possibility for the model to smoothly converge to the basic NK model would be to gradually shrink the spread between the steady-state interest rate on money I^m and the steady-state interest rate on bonds μ/β, and as a result gradually satiate the demand for money. In what follows, we show that the steady state of the model would then smoothly converge to the steady state of the basic NK model, but its reduced form would not smoothly converge to the reduced form of the basic NK model – essentially because the CIA constraint remains binding along the way for sufficiently small shocks (since I^m remains below μ/β along the way).

To do so, we start by deriving the first-order conditions of households' maximization problem (stated above):

$$u'(c_t) = \beta I_t^m \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\Pi_{t+1}} \right\} + \widetilde{\lambda}_t,$$

$$\lambda_t = \beta I_t^m \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\Pi_{t+1}} \right\} + \widetilde{\lambda}_t,$$

$$\lambda_t = \beta I_t \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\Pi_{t+1}} \right\},$$

$$v'\left(h_t\right) = \beta w_t \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\Pi_{t+1}} \right\},$$

where λ_t and $\widetilde{\lambda}_t$ denote the Lagrange multipliers associated with the budget and cash-in-advance constraints respectively. We then turn to the other equilibrium conditions of the model. Firms are subject to Calvo's (1983) constraints on the frequency at which they can change their prices. They receive cash from consumers and hoard it until the next period. Thus, in the specific case of perfectly flexible prices ($\theta = 0$), firm i chooses

$P_t(i)$ to maximize

$$\mathbb{E}_t\left\{\frac{\beta\lambda_{t+1}}{\lambda_t\Pi_{t+1}}\right\}\left[I_t^m\frac{P_t(i)y_t(i)}{P_t}-w_th_t(i)\right]$$

subject to the production function

$$y_t\left(i\right)=f\left[h_t\left(i\right)\right]$$

and the demand schedule

$$y_t\left(i\right)=\left[\frac{P_t(i)}{P_t}\right]^{-\varepsilon}y_t.$$

Using the Euler equation above, and the symmetry between firms, we can write the first-order condition of this flexible-price maximization problem as

$$\left(\frac{\varepsilon}{\varepsilon-1}\right)w_t=I_t^m f'(h_t).$$

Finally, the bond-market-clearing condition is

$$b_t=0,$$

the money-market-clearing condition is

$$m_t=\frac{M_t}{P_t},$$

and the goods-market-clearing condition is

$$c_t+g=y_t.$$

As with our MIU model, we set μ to one, so that the set of steady states is the same under sticky prices ($\theta>0$) as under flexible prices ($\theta=0$). So, we can use the first-order condition of firms' optimization problem under flexible prices (above) to characterize this set. When all variables are constant over time (in particular $h\equiv h_t=h_{t+1}$), the equilibrium conditions above imply

$$\frac{v'(h)}{f'(h)}=\beta I^m\left(\frac{\varepsilon-1}{\varepsilon}\right)u'[f(h)-g]. \tag{E.1}$$

Under standard assumptions on u, v and h, the left-hand side of (E.1) is increasing in h, from 0 (as $h=0$) to $+\infty$ (as $h\to+\infty$), while its right-hand side is decreasing in h, from $+\infty$ (as $h\to\underline{h}$, where $\underline{h}$ is defined by $f(\underline{h})=g$) to 0 (as $h\to+\infty$). Therefore, there exists a unique solution in h to (E.1), and hence a unique steady state of the model. As I^m gradually goes to $1/\beta$, (E.1) smoothly converges to

$$\frac{v'(h)}{f'(h)}=\left(\frac{\varepsilon-1}{\varepsilon}\right)u'[f(h)-g],$$

which is the equation that the steady-state value of labor h satisfies in the basic NK model. Therefore, the steady state of our CIA model smoothly converges to the steady state of the basic NK model as I^m gradually goes to $1/\beta$.

Since I^m goes to $1/\beta$ without ever reaching $1/\beta$, the CIA constraint remains binding along the way (for sufficiently small shocks). Therefore, log-linearizing the equilibrium conditions around the unique steady state leads to the following reduced form:

$$\begin{aligned}
\widehat{y}_t &= \mathbb{E}_t\left\{\widehat{y}_{t+1}\right\}-\frac{1}{\sigma}\left(i_t-\mathbb{E}_t\left\{\pi_{t+1}\right\}\right),\\
\pi_t &= \beta\mathbb{E}_t\left\{\pi_{t+1}\right\}+\kappa\left[\widehat{y}_t+\delta_i\left(i_t-i_t^m\right)\right],\\
\widehat{y}_t &= \nu\widehat{m}_t,
\end{aligned}$$

where $\widehat{m}_t = \widehat{M}_t - \widehat{P}_t$, $\pi_t = \widehat{P}_t - \widehat{P}_{t-1}$, and all parameters are positive:

$$\begin{aligned}
\sigma &\equiv \frac{-u''y}{u'} > 0, \\
\kappa &\equiv \frac{(1-\theta)(1-\beta\theta)}{\theta\left[1-\frac{\varepsilon f f''}{(f')^2}\right]}\left[-\frac{u''y}{u'} + \frac{v''h}{v'}\frac{f}{f'h} - \frac{ff''}{(f')^2}\right] > 0, \\
\delta_i &\equiv \left[-\frac{u''y}{u'} + \frac{v''h}{v'}\frac{f}{f'h} - \frac{ff''}{(f')^2}\right]^{-1} > 0, \\
\nu &\equiv \frac{c}{y} \in (0,1).
\end{aligned}$$

This reduced form does not converge to the reduced form of the basic NK model as I^m gradually goes to $1/\beta$. Thus, the reduced form of our CIA model does not smoothly converge to the reduced form of the basic NK model.

Appendix F: Discounting Models

"Discounting models" provide an alternative to our approach for solving the forward-guidance puzzle. In this appendix, we establish three points about a class of discounting models. The first two points generalize Cochrane's (2016) comments on Gabaix (2020). More specifically, the first point is that discounting models do not solve our paradox of flexibility (Definition 3). While we have no evidence against (nor in favor of) their implication that greater price flexibility magnifies the contraction and the deflation at the ZLB, we think that their implication about a discontinuity at the flexible-price limit – highlighted in our Definition 3 – seems implausible. As we will show, this discontinuity comes from the fact that discounting models do not deliver determinacy under an exogenous interest rate when prices are flexible.

The second point is that discounting models cannot solve the forward-guidance puzzle without generating a *negative* long-term relationship between the inflation rate and the interest rate on bonds. To our knowledge, existing empirical evidence does not support the precise *one-to-one* long-term relationship implied by our monetary models. Nonetheless, the existence of a *positive* long-term relationship between the inflation rate and nominal interest rates is a standard presumption of our textbooks, and is broadly reflected in cross-country data.

Our third point illustrates a limitation of our equilibrium-selection argument in Sections 3-4. Although discounting models converge to the basic NK model as we shrink the underlying friction, we cannot use them to uniquely select the equilibrium presented in Subsection 3.2 (nor any other equilibrium). This is because discounting models do not deliver determinacy under an exogenous interest rate beyond some point, as we approach the basic-NK-model limit. This illustrates more broadly that our equilibrium-selection argument must start with a model that has higher-order dynamics than the second-order dynamics of the basic NK model.

F.1. A Class of Discounting Models

We consider a class of models whose reduced form, in the absence of shocks other than interest-rate shocks, is made of an IS equation and a Phillips curve of type

$$y_t = \xi_1 \mathbb{E}_t\{y_{t+1}\} - \frac{\xi_2}{\sigma}\mathbb{E}_t\{i_t - \pi_{t+1}\}, \quad \text{(F.1)}$$

$$\pi_t = \beta\xi_3(\theta)\mathbb{E}_t\{\pi_{t+1}\} + \kappa(\theta)[y_t - \xi_4(\theta)\mathbb{E}_t\{y_{t+1}\}], \quad \text{(F.2)}$$

where $\beta \in (0,1)$, $\sigma > 0$, $\xi_1 > 0$, $\xi_2 > 0$, and, for all $\theta \in (0,1)$, $\xi_3(\theta) \geq 0$, $\xi_4(\theta) \in [0,1)$, and $\kappa(\theta) > 0$, with $\lim_{\theta\to 0} \xi_3(\theta) < +\infty$ and $\lim_{\theta \to 0} \kappa(\theta) = +\infty$.[4] This class of reduced forms nests the reduced form of the basic NK model as a special case in which $\xi_1 = \xi_2 = \xi_3(\theta) = 1$ and $\xi_4(\theta) = 0$. More generally, this class allows the coefficients of $\mathbb{E}_t\{y_{t+1}\}$ and $\mathbb{E}_t\{\pi_{t+1}\}$ to be smaller ("positive discounting") or larger ("negative discounting") than in the basic NK model, and also allows for a $\mathbb{E}_t\{y_{t+1}\}$ term in the Phillips curve. In particular, this class encompasses the reduced forms of three models that have been shown to be able to solve the forward-guidance puzzle: (i) Gabaix's (2020) benchmark model, in which $(\xi_1, \xi_3(\theta)) \in (0,1)^2$ and $\xi_4(\theta) = 0$; (ii) Angeletos and Lian's (2018) model, in which $(\xi_1, \xi_2, \xi_3(\theta), \xi_4(\theta)) \in (0,1)^4$; and (iii) Bilbiie's (2019) model with external price-adjustment costs, in which $(\xi_1, \xi_2) \in (0,1)^2$ and $\xi_3(\theta) = \xi_4(\theta) = 0$. In addition, it also encompasses the reduced forms of: (iv) Bilbiie's (2019) model with internal price-adjustment costs, in which $(\xi_1, \xi_2) \in (0,1)^2$, $\xi_3(\theta) = 1$, and $\xi_4(\theta) = 0$; (v) McKay et al.'s (2017) model, in which also $(\xi_1, \xi_2) \in (0,1)^2$, $\xi_3(\theta) = 1$, and $\xi_4(\theta) = 0$; (vi) Ravn and Sterk's (2018) model with risk-neutral equity investors, in which $\xi_3(\theta) = 1$ and $\xi_4(\theta) \in (0,1)$; and (vii) Woodford's (2019) model with exponentially distributed planning horizons and no learning, in which $\xi_1 = \xi_2 = \xi_3(\theta) \in (0,1)$ and $\xi_4(\theta) = 0$.[5]

F.2. Paradox of Flexibility

Like the basic NK model, and unlike our monetary models, discounting models exhibit the paradox of flexibility (as stated in Definition 3).[6] They make inflation and output explode in response to future shocks as the degree of price stickiness θ goes to zero. To establish this result, we assume that the interest rate is set exogenously from date 1 to some date $T \geq 2$, and that the economy is at its steady state at date $T+1$; and we show that the responses of $|\pi_1|$ and $|y_1|$ to an interest-rate change at any date $k \in \{2, ..., T\}$ go to infinity as $\theta \to 0$:

Proposition 10: *Discounting models exhibit the paradox of flexibility: if* $i_k = i^* \neq 0$, $i_t = 0$ *for all* $t \in \{1, ..., T\} \setminus \{k\}$, *and* $y_{T+1} = \pi_{T+1} = 0$, *then* $\lim_{\theta\to 0} |\pi_1| = \lim_{\theta \to 0} |y_1| = +\infty$.

Proof: We start with the case in which $\xi_3(\theta) > 0$ or $\xi_4(\theta) > 0$. In this case, the system made of the IS equation (F.1) and the Phillips curve (F.2) can be rewritten as

$$\mathbb{E}_t \left\{ \begin{bmatrix} y_{t+1} \\ \pi_{t+1} \end{bmatrix} \right\} = \mathbf{P} \begin{bmatrix} y_t \\ \pi_t \end{bmatrix} + \mathbf{Z} i_t \tag{F.3}$$

with

$$\mathbf{P} \equiv \frac{1}{\varphi(\theta)} \begin{bmatrix} \kappa(\theta)\xi_2 + \beta\sigma\xi_3(\theta) & -\xi_2 \\ \kappa(\theta)\sigma[\xi_4(\theta) - \xi_1] & \sigma\xi_1 \end{bmatrix} \quad \text{and} \quad \mathbf{Z} \equiv \frac{\xi_2}{\varphi(\theta)} \begin{bmatrix} \beta\xi_3(\theta) \\ \kappa(\theta)\xi_4(\theta) \end{bmatrix},$$

where $\varphi(\theta) \equiv \beta\sigma\xi_1\xi_3(\theta) + \kappa(\theta)\xi_2\xi_4(\theta) > 0$. The characteristic polynomial of $\mathbf{P}$ is

$$\mathcal{P}(X) \equiv X^2 - \frac{\sigma\xi_1 + \kappa(\theta)\xi_2 + \beta\sigma\xi_3(\theta)}{\varphi(\theta)} X + \frac{\sigma}{\varphi(\theta)}.$$

Since $\mathcal{P}(0) \neq 0$, $\mathbf{P}$ is invertible. Iterating the dynamic equation (F.3) forward to date T, and using the terminal condition $y_{T+1} = \pi_{T+1} = 0$ and the invertibility of $\mathbf{P}$, we get

$$\begin{bmatrix} y_1 \\ \pi_1 \end{bmatrix} = -\mathbf{P}^{-(k-1)} \mathbf{Z} i^*.$$

[4] We focus on discrete-time discounting models for the sake of comparability with our monetary models, but we have no reason to expect that continuous-time discounting models behave differently. Indeed, Michaillat and Saez (2019) show that their continuous-time discounting model has the same three properties as the ones listed above.

[5] However, it does not encompass the reduced forms of McKay et al.'s (2016) and Del Negro et al.'s (2015) models, which involve some discounting too but are more complex.

[6] They may *attenuate* this paradox, though, as Angeletos and Lian (2018) show in the context of their discounting model.

For any $X \in \mathbb{R}$, we have

$$\frac{1}{\xi_2} \lim_{\theta \to 0} \left[\frac{\varphi(\theta)\mathcal{P}(X)}{\kappa(\theta)} \right] = \left[\lim_{\theta \to 0} \xi_4(\theta) \right] X^2 - X.$$

One root of the polynomial on the right-hand side of this equation is zero. Therefore, one root of $\mathcal{P}(X)$ converges towards zero as $\theta \to 0$, which implies in turn that $\lim_{\theta \to 0} ||\mathbf{P}^{-1}|| = +\infty$. Using the fact that $||\mathbf{Z}||$ is bounded away from zero as $\theta \to 0$, we conclude that $\lim_{\theta \to 0} |y_1| = \lim_{\theta \to 0} |\pi_1| = +\infty$.

In the alternative case in which $\xi_3(\theta) = \xi_4(\theta) = 0$, the system made of the IS equation (F.1) and the Phillips curve (F.2) implies the following dynamic equation in inflation:

$$\left[\xi_1 + \frac{\kappa(\theta)\xi_2}{\sigma} \right] \mathbb{E}_t \{\pi_{t+1}\} = \pi_t + \frac{\kappa(\theta)\xi_2}{\sigma} i_t. \tag{F.4}$$

Iterating this dynamic equation forward to date T, and using the terminal condition $\pi_{T+1} = 0$, we get

$$\pi_1 = -\left[\xi_1 + \frac{\kappa(\theta)\xi_2}{\sigma} \right]^{k-1} \frac{\kappa(\theta)\xi_2 i^*}{\sigma},$$

so that $\lim_{\theta \to 0} |\pi_1| = +\infty$. Using the Phillips curve (F.2) with $\xi_3(\theta) = \xi_4(\theta) = 0$, we then get $\lim_{\theta \to 0} |y_1| = +\infty$. ■

The preceding proof is based on two properties of discounting models: (i) these models generate indeterminacy under a permanently exogenous policy rate when prices are sufficiently flexible, as their dynamic system then has one stable eigenvalue not matched by any predetermined variable, and (ii) this stable eigenvalue goes to zero as prices are made more and more flexible. As in the basic NK model in Section 2, this stable eigenvalue magnifies the effects of future conditions (at date k) on initial outcomes (at date 1), and these effects grow explosively as this eigenvalue goes to zero – thus giving rise to the paradox of flexibility.

Indeterminacy under sufficiently flexible prices, in turn, follows by continuity from indeterminacy under perfectly flexible prices. Under perfectly flexible prices, the Phillips curve (F.2) collapses to the dynamic equation $y_t = [\lim_{\theta \to 0} \xi_4(\theta)] \mathbb{E}_t\{y_{t+1}\}$, which pins down y_t uniquely if $\lim_{\theta \to 0} \xi_4(\theta) \neq 1$. Under an exogenous policy rate i_t, the IS equation (F.1) then pins down expected future inflation $\mathbb{E}_t\{\pi_{t+1}\}$, but not current inflation π_t. Thus, discounting models may deliver determinacy under a permanently exogenous policy rate for some degrees of price stickiness, but cannot do it for sufficiently small degrees of price stickiness.

In our monetary models, by contrast, the interest rate pegged at the ZLB is the IOR rate i_t^m, not the interest rate on bonds i_t. Setting exogenously the IOR rate and the nominal stock of reserves – two monetary-policy instruments under the direct control of central banks – makes the (market-determined) interest rate on bonds evolve according to a shadow Wicksellian rule, as we have explained. This shadow Wicksellian rule ensures determinacy for any degree of price stickiness, and in particular for perfectly flexible prices – thus solving the paradox of flexibility.

F.3. Fisher Effect

Discounting models cannot deliver determinacy under a permanently exogenous policy rate without making the inflation rate and the interest rate *negatively* related to each other in the long term. Therefore, they cannot both solve the forward-guidance puzzle and imply a long-term relationship consistent in sign (let alone in size) with the standard Fisher effect.[7] The following proposition formalizes this point:

Proposition 11: *In discounting models, if setting the policy rate exogenously delivers local-equilibrium determinacy, then a permanent increase in the policy rate leads to a permanent decrease in the inflation rate.*

[7] Gabaix (2020), however, adds price indexation and "inflation guidance" to his benchmark discounting model and shows that the resulting model (which does not belong to the class of discounting models we consider) can both solve the forward-guidance puzzle and make inflation respond positively to the nominal interest rate in the long term.

Proof: We start with the case in which $\xi_3(\theta) > 0$ or $\xi_4(\theta) > 0$. In this case, under a permanent peg $i_t = i^*$, the system made of the IS equation (F.1) and the Phillips curve (F.2) can be rewritten as (F.3) with $i_t = i^*$. If the peg ensures local-equilibrium determinacy, then $\mathcal{P}(X)$, the characteristic polynomial of $\mathbf{P}$ (derived in Appendix F.2), must have no root inside the unit circle, because the system has no predetermined variable. In particular, $\mathcal{P}(X)$ must have no root inside the real-number interval $[0,1]$, which requires that $\mathcal{P}(0)\mathcal{P}(1) > 0$, i.e. equivalently

$$\sigma(1-\xi_1)[1-\beta\xi_3(\theta)] - \kappa(\theta)\xi_2[1-\xi_4(\theta)] > 0. \tag{F.5}$$

In the unique local equilibrium, the (constant) inflation rate is easily obtained as

$$\pi_t = \pi^* \equiv \frac{-\kappa(\theta)\xi_2[1-\xi_4(\theta)]i^*}{\sigma(1-\xi_1)[1-\beta\xi_3(\theta)] - \kappa(\theta)\xi_2[1-\xi_4(\theta)]}.$$

Given (F.5), π^* is negatively related to i^*.

In the alternative case in which $\xi_3(\theta) = \xi_4(\theta) = 0$, under a permanent peg $i_t = i^*$, the system made of the IS equation (F.1) and the Phillips curve (F.2) implies the dynamic equation (F.4) with $i_t = i^*$. Therefore, for the peg to ensure determinacy, we need

$$\sigma(1-\xi_1) - \kappa(\theta)\xi_2 > 0. \tag{F.6}$$

In the unique local equilibrium, the (constant) inflation rate is easily obtained as

$$\pi_t = \pi^* \equiv \frac{-\kappa(\theta)\xi_2 i^*}{\sigma(1-\xi_1) - \kappa(\theta)\xi_2}.$$

Given (F.6), π^* is negatively related to i^*. ■

The preceding proof is simple, but mechanical. In what follows, we offer an interpretation of this result that involves a shadow interest-rate rule and the Taylor principle. The question (negatively) answered by Proposition 11 is whether the system made of the modified IS equation (F.1), the modified Phillips curve (F.2), and the permanent peg $i_t = i^*$ can have a unique stationary solution and make inflation, in this unique stationary solution, depend positively on i^*. This question will receive exactly the same answer if that system is replaced by the system made of the standard IS equation (1), the modified Phillips curve (F.2), and the shadow interest-rate rule

$$i_t = \xi_2 i^* + \sigma(1-\xi_1)\mathbb{E}_t\{y_{t+1}\} + (1-\xi_2)\mathbb{E}_t\{\pi_{t+1}\}. \tag{F.7}$$

Indeed, the two systems have exactly the same implications for local-equilibrium determinacy and the dynamics of inflation and output (they differ only in terms of the implied dynamics for i_t). So consider the latter system. The Taylor principle (as defined by Woodford, 2003, Chapter 4) states that a necessary condition for local-equilibrium determinacy is that the modified Phillips curve (F.2) and the shadow interest-rate rule (F.7) should make the interest rate react more than one-to-one to the inflation rate in the long term, that is to say

$$\zeta \equiv \frac{\sigma(1-\xi_1)[1-\beta\xi_3(\theta)]}{\kappa(\theta)[1-\xi_4(\theta)]} + (1-\xi_2) > 1. \tag{F.8}$$

In the unique local equilibrium, the (constant) interest rate i and the (constant) inflation rate π are therefore linked to each other by the relationship $i = \xi_2 i^* + \zeta\pi$, where $\zeta > 1$. Now, the standard IS equation (1) implies that they should be equal to each other: $i = \pi$. As a consequence, we get

$$\pi = \frac{-\xi_2 i^*}{\zeta - 1}.$$

Thus, the necessary condition for local-equilibrium determinacy (F.8) imposed by the Taylor principle requires that π be *negatively* related to i^*.

This conflict between the Taylor principle and the Fisher effect does not arise in our monetary models. First, the interest rate pegged at the ZLB in these models is the (directly controlled) IOR rate i_t^m, not the (market-determined) interest rate on bonds i_t. Under exogenous monetary-policy instruments, the interest rate on bonds evolves according to a shadow Wicksellian rule that always ensures determinacy. Second, these models generate the standard Fisher effect, i.e. a *one-to-one* long-term relationship between the inflation rate and the interest rate on bonds. Indeed, the money-demand equations (7), (C.3), and (D.2) imply that a permanent change in nominal-money growth $M_t - M_{t-1} = \delta^*$ leads to the same permanent change in inflation $\pi_t = \delta^*$. In turn, the IS equations (1) and (C.1) imply that $M_t - M_{t-1} = \pi_t = \delta^*$ leads to the same permanent change in the interest rate on bonds $i_t = \delta^*$.[8]

F.4. Convergence to the Basic NK Model

Although discounting models converge to the basic NK model as we shrink the underlying friction (e.g., the degree of bounded rationality in Gabaix, 2020, information frictions in Angeletos and Lian, 2018, market incompleteness in Bilbiie, 2019), we cannot use them to uniquely select the equilibrium presented in Subsection 3.2 (nor any other equilibrium). The reason is that discounting models no longer deliver determinacy under an exogenous interest rate as they approach the basic NK model:

Proposition 12: *Discounting models generate indeterminacy under a permanently exogenous policy rate when* $(\xi_1, \xi_2, \xi_3(\theta), \xi_4(\theta))$ *is sufficiently close to* $(1, 1, 1, 0)$.

Proof: We focus on the case in which $\xi_3(\theta)$ is sufficiently close to 1 for $\xi_3(\theta) > 0$. In this case, the system made of the IS equation (F.1) and the Phillips curve (F.2) can be rewritten as (F.3), and its characteristic polynomial under a permanently exogenous policy rate is $\mathcal{P}(X)$ (as defined in Appendix F.2). It is straightforward to check that as $(\xi_1, \xi_2, \xi_3(\theta), \xi_4(\theta))$ goes to $(1, 1, 1, 0)$, $\mathcal{P}(X)$ converges to $\mathcal{P}_b(X)$ for any $X \in \mathbb{R}$, and therefore the two roots of $\mathcal{P}(X)$ converge to the two roots $\rho_b \in (0, 1)$ and $\omega_b > 1$ of $\mathcal{P}_b(X)$. For $(\xi_1, \xi_2, \xi_3(\theta), \xi_4(\theta))$ sufficiently close to $(1, 1, 1, 0)$, thus, only one root of $\mathcal{P}(X)$ lies outside the unit circle. With only one eigenvalue outside the unit circle for two non-predetermined variables ($\mathbb{E}_t\{y_{t+1}\}$ and $\mathbb{E}_t\{\pi_{t+1}\}$), discounting models then generate indeterminacy. ■

The basic reason for this result is that the reduced form (F.1)-(F.2) of discounting models involves inflation and output, but not the price level, like the reduced form (1)-(2) of the basic NK model. As a result, under a permanently exogenous policy rate, the characteristic polynomial $\mathcal{P}(X)$ of their dynamic system is of degree two, like the characteristic polynomial $\mathcal{P}_b(X)$ of the dynamic equation of the basic NK model.

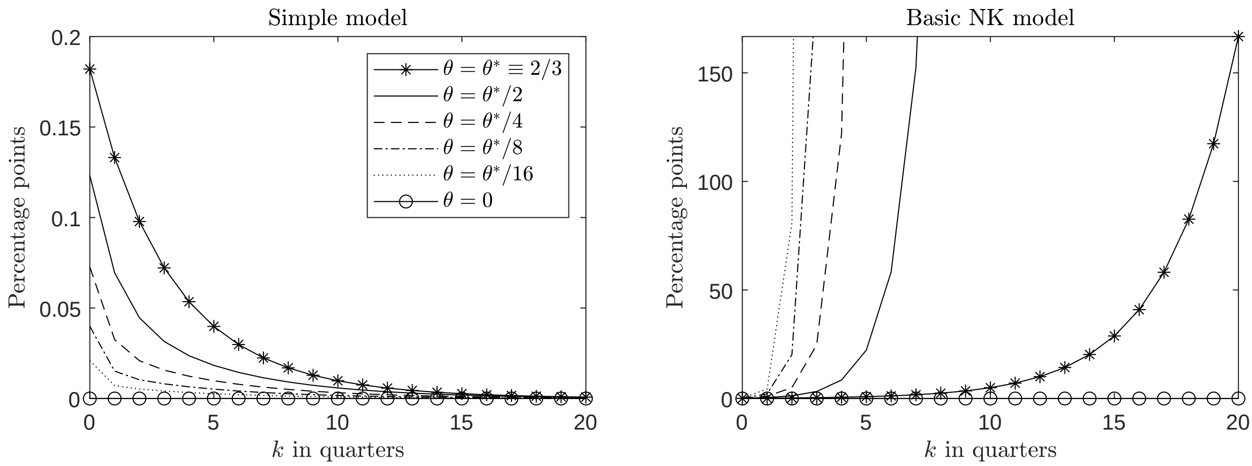
The reduced forms of our monetary models, by contrast, involve not only inflation and output, but also the price level p_t through the real stock of reserves $m_t = M_t - p_t$ in the money-demand equations (7), (C.3), and (D.2). As a result, under permanently exogenous monetary-policy instruments, their dynamic equation cannot be written in terms of inflation and involve only $\mathbb{E}_t\{\pi_{t+2}\}$, $\mathbb{E}_t\{\pi_{t+1}\}$, and π_t. Instead, it has to be written in terms of the price level, and to involve $\mathbb{E}_t\{p_{t+2}\}$, $\mathbb{E}_t\{p_{t+1}\}$, p_t, and p_{t-1}. As a consequence, the characteristic polynomial $\mathcal{P}(X)$ of our monetary models' dynamic equation is of degree three.

As our monetary models converge to the basic NK model, the roots ρ, ω_1, and ω_2 of their characteristic polynomial converge respectively to $\rho_b \in (0, 1)$, 1, and $\omega_b > 1$, where the limit value 1 of ω_1 simply reflects the identity $\pi_t = p_t - p_{t-1}$. As long as our monetary models do not exactly coincide with the basic NK model, ω_1 remains outside the unit circle. With two eigenvalues outside the unit circle (ω_1 and ω_2) for two non-predetermined variables ($\mathbb{E}_t\{p_{t+2}\}$ and $\mathbb{E}_t\{p_{t+1}\}$), therefore, our monetary models ensure determinacy even when they are arbitrarily close to the basic NK model.

Put differently, the price-level term in our shadow Wicksellian rules – which directly comes from the price-level term in the money-demand equations – acts as an *error-correction* term that makes the price level stationary and hence determinate in our monetary models. As these models converge to the basic NK

[8]Under the assumption that non-optimized prices are indexed to steady-state inflation, the Phillips curves (2), (C.2), and (D.1) remain valid and residually determine the permanent change in output.

Figure G.1: Effect of a policy-rate cut at date $t+k$ on output at date t



Note: The figure displays the effect on y_t of announcing at date t a one-percentage-point-per-annum cut in i^m_{t+k} (for the simple model) or i_{t+k} (for the basic NK model), as a function of $k \in \{0, ..., 20\}$. Parameter values are the same as for Figure 1 in the main text. More specifically, benchmark parameter values are set as in Galí (2008, Chapter 3): $\beta = 0.99$, $\sigma = 1$, $\chi_y = 1$, $\chi_i = 4$, and $\kappa = \lambda[(1-\theta)(1-\beta\theta)/\theta] = 0.13$, where $\lambda = 3/4$ and $\theta = \theta^* \equiv 2/3$. As θ takes the values $\theta^*/2$, $\theta^*/4$, $\theta^*/8$, and $\theta^*/16$, κ takes respectively the values 1.00, 3.13, 7.57, and 16.54.

model, the coefficient of the price level in the shadow Wicksellian rules goes to zero. But as long as our monetary models do not exactly coincide with the basic NK model, this coefficient remains positive, and both stationarity and determinacy of the price level ensue.

Appendix G: Simple Model (Numerical Illustrations)

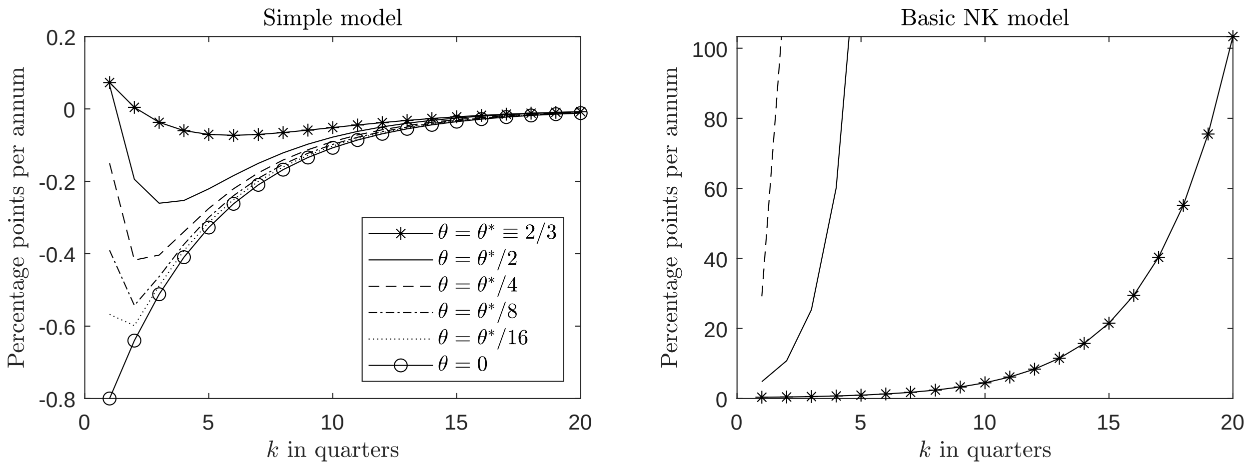
G.1. Additional Numerical Illustrations under our Benchmark Calibration

In Section 3 of the main text, Figure 1 illustrates numerically the effects of forward guidance (i.e. future policy-rate cuts) on inflation in our simple model, and compares these effects to the implications of the standard NK equilibrium. In this appendix, we illustrate and discuss the effects of forward guidance on output, and the effects of anticipated changes in fiscal policy on inflation and output $-$ both in the equilibrium of our simple model and in the standard equilibrium of the basic NK model. We continue to use our benchmark calibration taken from Galí (2008, Chapter 3), which sets $\theta = 2/3$ (corresponding to "3-quarter price rigidity"); we also report the effects of cutting θ in half, step by step, to make prices more flexible.

Figure G.1 shows the effects of forward guidance on output in the two models. As before, our policy experiment is to cut the policy rate by 25 basis points (one percentage point per annum) in Quarter $t+k$, and we display the effects on output in Quarter t (when the rate cut is announced). The right panel in Figure G.1 replicates the implausible implications of the basic NK model. The left panel shows that our model does not share these implications. The rate cut has small effects on output (less than 0.2 percent of steady-state output to begin with), and the effects die off quickly as we delay the rate cut. Moreover, these effects decline smoothly as we make prices more flexible; they converge to the flexible-price ($\theta = 0$) effects.

To analyze the effects of fiscal policy, we add government purchases to Galí's calibration. We set the share of government purchases in output to 0.3 in the steady state. We follow Galí's calibration for the structural parameters (like the intertemporal elasticity of substitution) and adjust the reduced-form parameters (like the coefficient $1/\sigma$ on the real interest rate in the IS equation) to reflect the introduction of government purchases. Our policy experiment is an increase in government purchases, amounting to one percent of steady-state output, occurring (only) in Quarter $t+k$ and announced in Quarter t. Figures G.2 and G.3

Figure G.2: Effect of government purchases at date $t+k$ on inflation at date t



Note: The figure displays the effect on π_t of announcing at date t a one-percent-of-steady-state-output increase in g_{t+k}, as a function of $k \in \{1, ..., 20\}$. The steady-state share of government purchases in output is set to 0.3, and benchmark structural-parameter values are set as in Galí (2008, Chapter 3), implying $\beta = 0.99$, $\sigma = 1.43$, $\delta_g = 0.42$, $\chi_y = 1$, $\chi_i = 4$, and $\kappa = \lambda[(1-\theta)(1-\beta\theta)/\theta] = 0.15$, where $\lambda = 0.86$ and $\theta = \theta^* \equiv 2/3$. As θ takes the values $\theta^*/2$, $\theta^*/4$, $\theta^*/8$, and $\theta^*/16$, κ takes respectively the values 1.15, 3.58, 8.65, and 18.90.

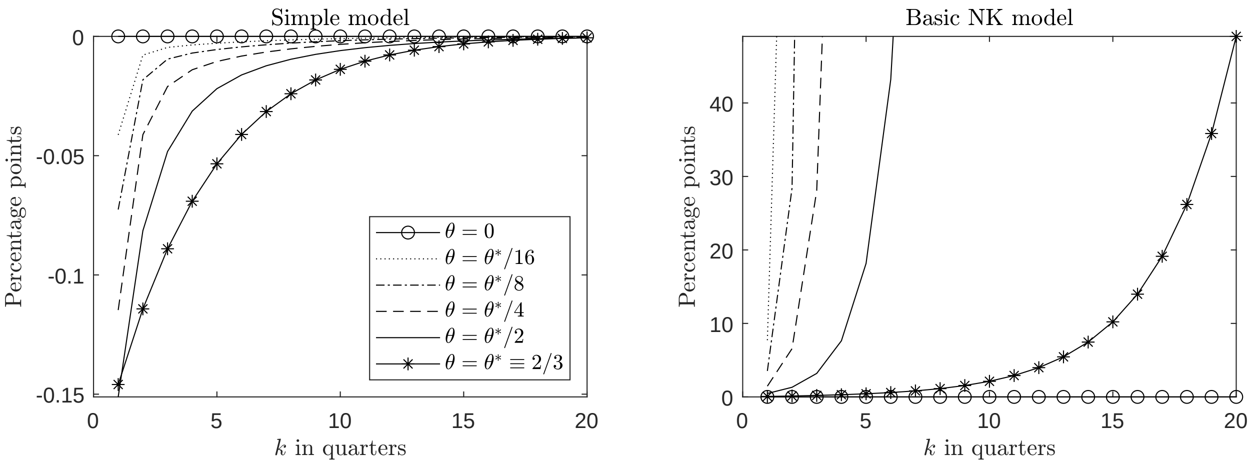
display the effects on inflation and output in Quarter t. Once again, the comparison between the left and the right panels shows that our model's equilibrium does not share the puzzling implications of the basic NK model's standard equilibrium: the effects of anticipated fiscal policy die out as we delay the policy intervention, and they converge to the flexible-price values as we make prices more and more flexible.

Another notable difference between our model's equilibrium and the basic NK model's standard equilibrium under our benchmark calibration is that anticipated fiscal expansions have a contractionary effect on output in our model.[9] Several contributions (e.g., Christiano et al., 2011) suggest that anticipated fiscal expansions can have large positive output multipliers at the ZLB according to the basic NK model. The right-hand panel of Figure G.3 confirms this implication of the basic NK model. This implication arises from a feedback loop first described in Farhi and Werning (2016). As we explain in Subsection 3.2 of the main text, this feedback loop works back in time via the IS equation and the Phillips curve: given that $\pi_{T+1} = y_{T+1} = 0$, a fiscal expansion at date T raises inflation at date T, which lowers the real interest rate at date $T-1$, which raises output and inflation at date $T-1$, and so on. This feedback loop is also present in our model, but π_{T+1} and y_{T+1} are endogenously determined when the fiscal expansion is announced. As a result, expected future fiscal expansions can reduce current output in our model (as is the case under the calibration we use for Figure G.3). Intuitively, these contractionary effects of anticipated fiscal expansions may come from wealth effects that also arise in standard Real-Business-Cycle models: consumers realize that the future fiscal expansion reduces their permanent income, and they respond by lowering current consumption.

The effects of anticipated fiscal expansions on inflation may be dominated either by the wealth effect we mention above (which is deflationary) or by an inflationary effect that we can trace back to staggered price setting. The latter effect arises because the fiscal expansion is expected to raise prices in the future, and this motivates current price setters to set higher prices too. Under our benchmark calibration, the effects of anticipated fiscal expansions on inflation (displayed in the left panel of Figure G.2, for various horizons k and price-stickiness degrees θ) are small, and mostly negative.

[9] Our analytical derivations in the main text show that this is always the case when we use our model to go to the basic-NK-model limit. Figure G.3 makes the point numerically under our benchmark calibration, without taking the model to the basic-NK-model limit.

Figure G.3: Effect of government purchases at date $t+k$ on output at date t



Note: The figure displays the effect on y_t of announcing at date t a one-percent-of-steady-state-output increase in g_{t+k}, as a function of $k \in \{1, ..., 20\}$. Parameter values are the same as for Figure G.2 above.

G.2. Numerical Sensitivity Analysis

The quantitative impressions conveyed by Figure 1 in the main text and Figures G.1-G.3 in the previous appendix are not particularly sensitive to Galí's (2008, Chapter 3) choices about the parameters of the basic NK model, nor to his (standard) assumption of a unitary income elasticity of money demand. The value taken by the interest semi-elasticity of money demand χ_i, however, does matter for the quantitative impression conveyed by our results about the effects of forward guidance. The value of χ_i affects both the magnitude and the persistence of the effects of future changes in the IOR rate on current inflation and output. Our choice of $\chi_i = 4$, following Galí, represents a middle-of-the-range value compared to estimates that we could take from the empirical literature on money demand.

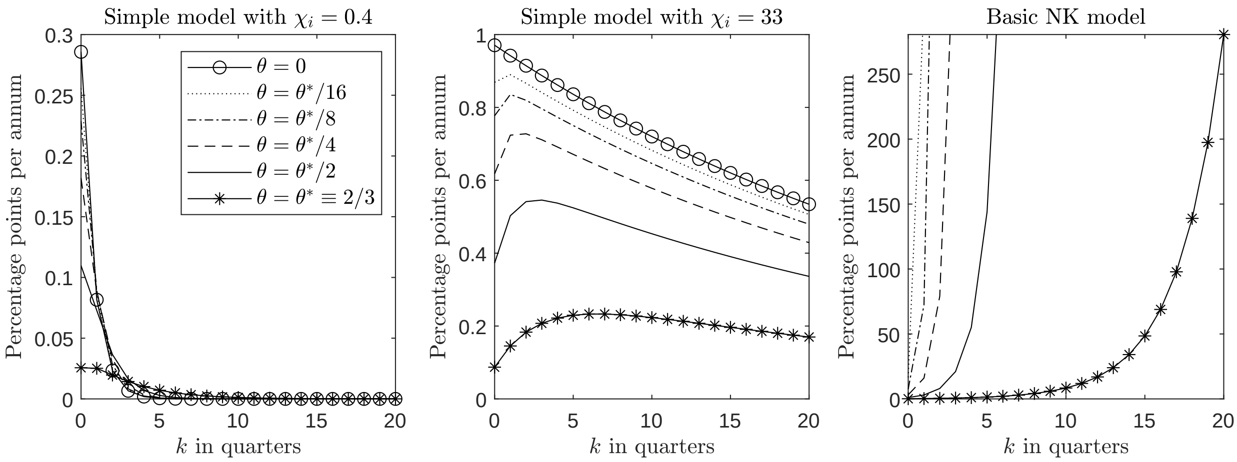
Semi-log specifications of money demand typically yield small estimates of χ_i based on US data. The estimates in Stock and Watson (1993) and Cochrane (2018), for example, suggest semi-elasticities close to -0.1 on an annual basis.[10] Given the quarterly frequency of our model, these estimates correspond to $\chi_i = 0.4$ (one order of magnitude smaller than the value we use for Figure 1). By contrast, log-log specifications of money demand, estimated on US or cross-country data, suggest interest elasticities around $-1/4$ (e.g., Teles and Zhou, 2005) or $-1/3$ (e.g., Teles et al., 2016). If we set the opportunity cost of holding money to one percent per quarter, an elasticity of $-1/3$ implies $\chi_i = 33$ (one order of magnitude larger than the value we use for Figure 1). Figure G.4 shows how the quantitative effects of forward guidance on inflation vary when we set χ_i to 0.4 or 33.

The policy experiment and the parameter values (other than the value of χ_i) used for Figure G.4 are the same as earlier for Figures 1 and G.1. The right panel in Figure G.4 replicates the implausible implications of the basic NK model. The left panel shows the results for our simple model with $\chi_i = 0.4$, and the middle panel shows the results with $\chi_i = 33$. The left panel suggests that the inflationary effects of anticipated IOR-rate cuts are tiny (below 3 basis points to begin with, and dying off quickly). The middle panel suggests that forward guidance has a sizable and more persistent effect on inflation (announcing that the IOR rate will be cut by one percentage point in 20 quarters raises current inflation by 17 basis points).

Beyond this quantitative difference, however, both the left and middle panels of Figure G.4 also illustrate the analytical results that are the main focus of our paper: the inflationary effects go to zero as we cut the IOR rate in the more distant future ($k \to +\infty$), and they converge to the flexible-price effects as we make

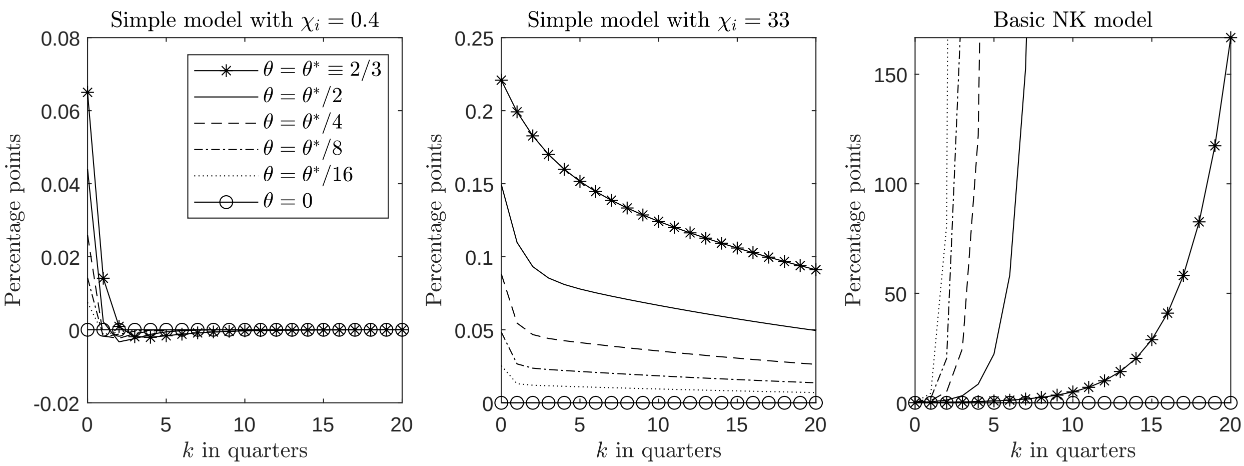
[10] Ball's (2001) estimate of -0.05 is even closer to zero.

Figure G.4: Effect of a policy-rate cut at date $t+k$ on inflation at date t for alternative values of χ_i



Note: The figure displays the effect on π_t of announcing at date t a one-percentage-point-per-annum cut in i^m_{t+k} (for the simple model) or i_{t+k} (for the basic NK model), as a function of $k \in \{0, ..., 20\}$. Parameter values (except the value of χ_i) are the same as for Figure 1 in the main text and Figure G.1 above.

Figure G.5: Effect of a policy-rate cut at date $t+k$ on output at date t for alternative values of χ_i



Note: The figure displays the effect on y_t of announcing at date t a one-percentage-point-per-annum cut in i^m_{t+k} (for the simple model) or i_{t+k} (for the basic NK model), as a function of $k \in \{0, ..., 20\}$. Parameter values (except the value of χ_i) are the same as for Figure 1 in the main text and Figures G.1 and G.4 above.

prices more flexible ($\theta \to 0$). Whatever the calibration, our simple model exhibits neither the forward-guidance puzzle nor the paradox of flexibility.

Figure G.5 shows the effects of forward guidance on output (under the same policy experiment and parameter values we describe above). Again, the quantitative impressions we get are sensitive to the value of the semi-elasticity χ_i. The effects are tiny if we set $\chi_i = 0.4$, but more noteworthy and persistent if we set $\chi_i = 33$.

Of course, our simple model is not really suitable for a quantitative assessment of the effects that one may associate with forward-guidance policies. Nonetheless, we suspect that the sensitivity of quantitative results to the specification of money demand may also be present in richer (larger-scale) models. So, we suspect that the unsettled state of empirical research on money demand may hinder sharp answers to interesting policy questions in this context.

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