New Principles For Stabilization Policy

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Abstract: In a broad class of discrete-time rational-expectations models, I consider stabilization-policy rules making the policy instrument react with coefficient $\phi \in \mathbb{R}$ to a (past, current, or expected future) variable at horizon $h \in \mathbb{Z}$, possibly among other variables, possibly with inertia. Using two complex-analysis theorems, I establish analytically some simple, easily interpretable, necessary or sufficient conditions on ϕ and h for these rules to ensure local-equilibrium determinacy. These conditions lead to new, general principles for stabilization policy in terms of whether, and how strongly or weakly, to react to any variable, at any horizon, in any model, with any policy instrument. Building on these conditions, I characterize the scope of validity of (a generalized version of) the long-run Taylor principle as a condition for determinacy. I apply all these results to standard interest-rate rules in 134 quantitative monetary-policy models, and find the new principles to be (either typically or occasionally) quantitatively relevant.

Keywords: stabilization policy, local-equilibrium determinacy, Taylor principle, backward- or forward-looking rule, inertial rule, robust rule.

JEL codes: E32, E52.

1 Introduction

Dynamic rational-expectations models are widely used in macroeconomics. It is well known that these models can have "sunspot equilibria" in which the economy fluctuates around a steady state because of self-fulfilling expectations. Such fluctuations may notably explain the relatively large macroeconomic volatility in the 1960s and 1970s in the US, as argued by Clarida et al. (2000) and Lubik and Schorfheide (2004). Since these fluctuations are typically detrimental to welfare, a natural goal for stabilization policy is

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to eliminate these equilibria by ensuring "local-equilibrium determinacy" (i.e. existence and uniqueness of a stationary solution to the locally log-linearized model).

A large number of papers have thus studied, in various specific contexts, the conditions under which a policy-instrument rule ensures determinacy; that is, in discrete time, the inequality conditions on the coefficients of the rule for the resulting dynamic system to satisfy Blanchard and Kahn's (1980) determinacy conditions. Probably the best known result along these lines is about the so-called "Taylor principle" for monetary policy. Since Taylor (1993), monetary policy is commonly modeled by a simple interest-rate rule; in its simplest version, the Taylor principle states that the rule should make the interest rate react more than one-for-one to the inflation rate (when it reacts only to the inflation rate). This principle has been found to be necessary and/or sufficient for determinacy in some simple prominent models and for several alternative inflation horizons in the rule (see, e.g., Woodford, 2003, Chapter 4).

These determinacy conditions, however, have so far been studied – analytically or numerically – only on a model-by-model, rule-by-rule basis; no general determinacy condition has yet been established. Some tentative patterns emerge from this literature, but no clear-cut result stands out; the Taylor principle, in particular, is a good guide for determinacy in many monetary-policy models, but a poor one in others (see, e.g., Benhabib et al., 2001, Bilbiie, 2008, and a dozen other references in Holden, 2024). For monetary policy as for other stabilization policies, we still lack a general picture and a good understanding of determinacy outcomes depending on the coefficients and time horizons of the variables in the rule. We lack them essentially because the literature has been able to derive analytical determinacy conditions only in simple models and for simple rules with short horizons. The main difficulty in getting more general analytical results is that Blanchard and Kahn's (1980) conditions are about the roots of the characteristic polynomial of the dynamic system; and these roots depend on (the coefficients and horizons of) the policy-instrument rule in a complicated way.

In this paper, I use two complex-analysis theorems to overcome this difficulty and establish analytically some general, necessary or sufficient conditions for determinacy in dynamic rational-expectations models. These conditions depend on the coefficients and horizons of the policy-instrument rule in a simple and easily interpretable way. They lead to new principles for stabilization policy in terms of whether, and how strongly or weakly, to react to any variable, at any horizon, in any model, with any policy instrument.

More specifically, I consider a broad class of (locally log-linearized) discrete-time infinite-horizon rational-expectations models. For simplicity, I first focus on (locally log-linearized) rules that make the policy instrument react to a single variable (or linear combination of variables) with coefficient $\phi \in \mathbb{R}$. The time horizon of this variable is $h \in \mathbb{Z}$: the policy instrument reacts to the $|h|$-period-lagged variable (when $h \leq -1$), the current variable (when $h = 0$), or the current expectation of the h-period-ahead variable (when $h \geq 1$). A negative horizon, making the rule backward-looking, may be due to "inside lags" (as are called recognition, decision, and implementation lags, which delay the reaction of policy to the state of the economy). A positive horizon, making the rule forward-looking, captures a reaction to forecasts or expectations (e.g., for monetary policy, the central bank's inflation forecasts, or market- or survey-based measures of inflation expectations).

The determinacy status of the dynamic system composed of the model and the rule can be either "determinacy" (unique stationary solution), or "multiplicity" (infinity of stationary solutions), or "explosiveness" (no stationary solution).[1] I characterize this determinacy status as a function of the coefficient ϕ and the horizon h in the rule.

I show that there exists a positive threshold $\underline{\phi}$ such that for any $|\phi| < \underline{\phi}$, the determinacy status is independent of h and is the same as under a policy-instrument peg ($\phi = 0$). Intuitively, for $|\phi|$ sufficiently small, the structural equations "dominate" the rule in the system's dynamics: the rule does not change the system's dynamics enough, relatively to a peg, to affect the determinacy status. There also exist a higher threshold $\bar{\phi}$ and a horizon $h^* \in \mathbb{Z}$ such that for any $|\phi| > \bar{\phi}$, there is explosiveness if $h \leq h^* - 1$, determinacy if $h = h^*$, and multiplicity if $h \geq h^* + 1$. Intuitively, for $|\phi|$ sufficiently large, it is conversely the rule that dominates the structural equations in the system's dynamics: a sufficiently large weight $|\phi|$ on outcomes before (resp. after) horizon h^* favors exploding (resp. imploding) paths and leads to explosiveness (resp. multiplicity).

For any $|\phi| \in (\underline{\phi}, \bar{\phi})$, there is explosiveness (resp. multiplicity) if $-h$ (resp. h) is sufficiently large, reflecting again the fact that a sufficiently large weight $|\phi|$ on outcomes sufficiently distant in the past (resp. the future) favors exploding (resp. imploding) paths. The set of horizons $h \in \mathbb{Z}$ such that determinacy obtains for at least one value of $\phi \in \mathbb{R}$ may be bounded or not, below or above; I establish necessary or sufficient conditions for these outcomes to obtain. I also identify ranges of ϕ values, within $(-\bar{\phi}, -\underline{\phi}) \cup (\underline{\phi}, \bar{\phi})$, for which there is no determinacy for any $h \in \mathbb{Z}$.

[1] If the determinacy status is explosiveness, then any equilibrium path that starts close to the steady state eventually leaves the neighborhood of the steady state within which the log-linear approximation is valid – after which non-linearities kick in, and the path may for instance converge to a limit cycle, as in Beaudry et al. (2017, 2020).

Building on these results, I study the validity of the Taylor principle as a condition for determinacy. I consider Woodford's (2001, 2003) version of the Taylor principle, also called the long-run Taylor principle, which has a broader scope than the simpler version described above. I provide a formal, general definition of this principle, which applies to any stabilization-policy model and any variable in the rule, and which comes down to an inequality of type $\phi > \phi_1$, where $\phi_1 \in [\underline{\phi}, \bar{\phi}]$. I characterize circumstances under which this principle is (not) necessary, (not) sufficient, or (not) locally sufficient for determinacy (where "locally," in this context, means "for ϕ in a neighborhood of ϕ_1").

In particular, I show that the validity of the Taylor principle as a condition for determinacy across different horizons depends crucially on whether $\phi_1 = \bar{\phi}$, or $\phi_1 \in (\underline{\phi}, \bar{\phi})$, or $\phi_1 = \underline{\phi}$. The Taylor principle can be necessary and locally sufficient for determinacy in any of these three cases, but only for a single horizon in the first case (the horizon $h = h^*$), for finitely many horizons in the second case, and for infinitely many horizons in the third case (the horizons $h < h_1$ or $h > h_1$, where $h_1 \in \mathbb{R}$). The distinction between the three cases sheds light on some intriguing results in the monetary-policy literature about the "*inverted Taylor principle*" $\phi < \phi_1$ being necessary and sufficient for determinacy (Benhabib et al., 2001, and Bilbiie, 2008).

All these results still hold for rules involving several variables, one of which is a variable with coefficient ϕ and horizon h, provided that the coefficients and horizons of the other variables in the rule are taken as given.[2] They also still hold for inertial rules, i.e. rules that involve some past values of the policy instrument. I determine how rule inertia affects the thresholds $\underline{\phi}$, $\bar{\phi}$, ϕ_1, h^* and h_1. In particular, I show that for an important subclass of models and rules, increasing the inertia coefficient $\rho \in (0, 1)$ widens (unboundedly as $\rho \to 1$) the range of horizons for which the Taylor principle is necessary and locally sufficient for determinacy.

I illustrate all these analytical results graphically with 5 small-scale monetary-policy models borrowed from the literature. To show that the results can be *quantitatively* relevant, however, I also apply them to 134 medium- or large-scale monetary-policy models of the Macroeconomic Model Data Base (MMB) described in Wieland et al. (2012, 2016). For interest-rate rules reacting to inflation only or to both inflation and output, with or without inertia, I find that the threshold values $\underline{\phi}$, ϕ_1, h^* and h_1 are typically of the same order of magnitude as standard values of ϕ and h in the literature, while $\bar{\phi}$ can be of

[2]The results then hold in exactly the same terms as above, except that "under a peg" should be replaced by "for $\phi = 0$."

the same order of magnitude but is typically one or several orders of magnitude larger. I discuss the implications of these quantitative results in the main text.

A few remarks may serve to put my contribution in the context of the literature. The paper is, to my knowledge, the first to establish *general* determinacy conditions about the coefficients and horizons of policy-instrument rules, which hold regardless of the dimension of the dynamic system. In particular, the concepts of $\underline{\phi}$, $\bar{\phi}$, h^* and h_1 (which underpin these determinacy conditions) are new. The literature has derived determinacy conditions analytically only in simple models and for simple rules with short horizons (so that the dimension of the dynamic system, i.e. the degree of the characteristic polynomial, is typically not higher than three).[3] Early examples of such contributions include Benhabib et al. (2001), Bullard and Mitra (2002), Carlstrom and Fuerst (2002) and Woodford (2003, Chapter 4).[4] Recent examples include Acharya and Dogra (2020), Bilbiie (2024) and Gabaix (2020).

The two complex-analysis theorems that I use to establish my general results are those of Rouché (1862) and Erdős and Turán (1950). One of these theorems is not new to economics: Bhattarai et al. (2014) use (another version of) Rouché's theorem to derive a sufficient condition for determinacy in a monetary-policy model with partial price indexation and habit formation in consumption. There are, in substance, three key differences between their sufficient condition for determinacy and my sufficient conditions for determinacy, or multiplicity, or explosiveness.

The first difference is that Bhattarai et al. (2014) are after the weakest possible sufficient condition for determinacy in the context of their model and their rule; to that aim, they use a stronger version of Rouché's theorem, established by Glicksberg (1976); the analytical condition that they get depends on each coefficient of the rule in a complicated and opaque way (even though this condition is numerically found to be only slightly stronger than the simple long-run Taylor principle). By contrast, I am after some simple and easily interpretable sufficient conditions; I get them using the standard version of Rouché's theorem, and applying it differently; these analytical conditions depend on the coefficient ϕ of the policy-instrument rule in a simple and transparent way, through the thresholds $\underline{\phi}$, $\bar{\phi}$ and ϕ_1. The other side of the coin, however, is that my sufficient conditions are not the weakest possible, and they are about one coefficient at a time

[3] Exceptionally, the dimension of the dynamic system can go up to four (Ascari et al., 2017) or even five (Bhattarai et al., 2014).

[4] Benhabib et al. (2001) conduct most of their analysis in continuous time; so, their concepts of backward- and forward-looking rules differ from mine (in particular, their backward-looking rules amount to inertial rules, as Benhabib et al., 2003, note).

(taking the other coefficients as given, if the rule has several coefficients).

The second difference is that my conditions characterize the determinacy status for all horizons $h \in \mathbb{Z}$, while Bhattarai et al. (2014) focus on a few specific short horizons (namely 0 and 1), like the rest of the literature. The third difference, finally, is that I establish my sufficient conditions for a generic rule in a generic model, in order to derive general principles for stabilization policy, while Bhattarai et al. (2014) establish theirs for a specific interest-rate rule in a specific monetary-policy model.

Some of the results I establish are conditional on whether the model delivers multiplicity, determinacy, or explosiveness under a policy-instrument peg. One comes across the three types of models in the monetary-policy literature. Standard New Keynesian models typically deliver *multiplicity* under an interest-rate peg; this property is emphasized by Cochrane (2011); Giannoni and Woodford (2002) and Woodford (2003, Chapter 8) call it the "Sargent-Wallace property," after Sargent and Wallace (1975). Older models often deliver *explosiveness* under an interest-rate peg; this property is emphasized by Cochrane (2011), who calls these models "Old Keynesian." More recently, models have been developed that can deliver *determinacy* under an interest-rate peg (and, as a result, can solve some New Keynesian puzzles and paradoxes at the zero lower bound). Examples include the heterogenous-agents models of Acharya and Dogra (2020) and Bilbiie (2008, 2024), and the bounded-rationality model of Gabaix (2020).

My results about *positive* horizons offer an explanation for the propensity of forward-looking interest-rate rules to generate multiplicity in New Keynesian models, as found in, e.g., Levin et al. (2003). Existing results on this front are mostly numerical and sparsely distributed across calibrated or estimated models and rules; my analytical results generalize them to a broad class of models and a broad class of rules (making the policy instrument react to any expected future variable).

My results about *negative* horizons matter in the presence of inside lags. Benhabib (2004) analyzes the implications of inside lags for determinacy in a simple monetary-policy model (analytically in continuous time, numerically in discrete time). In Loisel (2024), I investigate the ability of stabilization policy to ensure determinacy and to control the anticipation and convergence rates in the presence of inside or outside lags; the approach I take there (starting from a targeted characteristic polynomial and deriving a corresponding, arbitrarily complex policy-instrument rule) is radically different from the one I am taking here, and does not lead to any simple principle for stabilization policy.

As I discuss in the main text, the application of my general results to MMB models provides guidelines for finding a robust interest-rate rule, in the sense of an interest-rate rule delivering determinacy across a wide range of alternative monetary-policy models. Levin et al. (1999, 2003), Levin and Williams (2003), and Taylor and Williams (2011), among others, look for interest-rate rules that are robust in a broader sense (which goes beyond determinacy issues). Unlike them, I use the numerical values of analytical coefficient and horizon thresholds (rather than numerical simulations over a grid of rule-coefficient values): this approach provides more structured insights and facilitates the comparison across many models.[5] Relatedly, my general results shed light on the poor performance of superinertial rules in Old Keynesian models (Rudebusch and Svensson, 1999; Levin and Williams, 2003), and on the degree of superinertia of "robustly optimal rules" (Giannoni and Woodford, 2002, 2003, 2005; Woodford, 2003, Chapter 8).

Most of the literature on policy-instrument rules is about monetary policy. My results apply more generally to any stabilization policy. In particular, fiscal policy can also raise various indeterminacy issues, as is well known since Leeper (1991) and Schmitt-Grohé and Uribe (1997). Finally, I establish not only determinacy conditions, but also multiplicity conditions and explosiveness conditions. Clarida et al. (2000) and Lubik and Schorfheide (2004) have famously argued that US macroeconomic volatility before 1979 may be due to multiplicity. Beaudry et al. (2017, 2020) argue that recent US macroeconomic data are consistent with explosiveness (and convergence to a limit cycle).

The rest of the paper is organized as follows. Section 2 illustrates some of the main results of the paper in the basic New Keynesian model, with a rule making the interest rate react to inflation. Section 3 generalizes the analysis to a broad class of models and a broad class of rules. Section 4 applies these general results to standard interest-rate rules in 134 quantitative monetary-policy models. I then conclude and provide a technical appendix.

2 A basic New Keynesian illustration

In this section, I illustrate some of the main results of the paper in a simple and well known monetary-policy context: the basic New Keynesian (NK) model, with a rule making the interest rate react to inflation. The analysis is a specific case, in terms of model and rule, of the more general analysis conducted in the next section.

[5] Using a different strategy (less directly related to my paper), Holden (2024) proposes a robust interest-rate rule that pins down inflation uniquely and controls its value in any model with a Fisher equation (arising from the mere existence of a real-bond market).

2.1 Model and rule

I refer the reader to Woodford (2003) and Galí (2015) for a detailed presentation of the basic NK model. In this model, at each date $t \in \mathbb{Z}$, the private sector sets inflation π_t and output y_t according to the following (locally log-linearized) IS equation and Phillips curve:

$$\begin{aligned} y_t &= \mathbb{E}_t\{y_{t+1}\} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t\{\pi_{t+1}\}\right), &(1)\\ \pi_t &= \beta\mathbb{E}_t\{\pi_{t+1}\} + \kappa y_t, &(2)\end{aligned}$$

where $\mathbb{E}_t\{.\}$ denotes the date-t rational-expectations operator, and $\sigma > 0$, $\beta \in (0,1)$ and $\kappa > 0$ are three parameters. I abstract from exogenous shocks in the structural equations (1) and (2), as they are irrelevant for determinacy issues. The policymaker is a central bank setting the short-term nominal interest rate i_t. I assume for now that the central bank reacts only to the past, current, or expected future inflation rate; i.e., I consider the following (locally log-linearized) interest-rate rule:

$$i_t = \phi\mathbb{E}_t\left\{\pi_{t+h}\right\}, \qquad \text{(Rule 1)}$$

where $(\phi, h) \in \mathbb{R} \times \mathbb{Z}$ (with $\mathbb{E}_t\left\{\pi_{t+h}\right\} = \pi_{t+h}$ when $h \leq 0$). I call ϕ and h the coefficient and horizon of inflation in the rule − or, with slight abuse of language, the coefficient and horizon of the rule.

Using the Phillips curve (2) and Rule 1 to replace y_t, y_{t+1}, and i_t in the IS equation (1), I get the dynamic equation

$$\mathbb{E}_t\left\{Q(L)\pi_{t+2}\right\} + \phi\mathbb{E}_t\left\{\pi_{t+h}\right\} = 0, \qquad (3)$$

where $Q(z) := (\sigma/\kappa)[\beta - (1+\beta+\kappa/\sigma)z + z^2] \in \mathbb{R}[z]$ and L is the lag operator.[6] Let ν denote the number of non-predetermined variables of this dynamic equation.[7] Let $P(z)$ denote the *reciprocal* polynomial of this dynamic equation's characteristic polynomial.[8] Under an interest-rate peg ($\phi = 0$), we have $\nu = 2$ and $P(z) = Q(z)$. When the interest rate is not pegged ($\phi \neq 0$), we generically have $\nu = \max(2, h)$,[9] and

$$P(z) = Q(z)z^{\max(0,h-2)} + \phi z^{\max(0,2-h)}. \qquad (4)$$

[6] Throughout the paper, $\mathbb{R}[z]$ denotes the set of polynomials in z with real-number coefficients.

[7] Throughout the paper, the non-predetermined variables of a dynamic system are defined, following Blanchard and Kahn (1980), as the non-predetermined elements of the vector $\mathbf{Z}_t$ when the dynamic system is written in a first-order form of type $\mathbb{E}_t\left\{\mathbf{Z}_{t+1}\right\} = \mathbf{M}\mathbf{Z}_t$, where $\mathbf{M}$ is a square matrix.

[8] For any $\tilde{P}(z) \in \mathbb{R}[z]$ of degree d, the reciprocal polynomial of $\tilde{P}(z)$ is $z^d\tilde{P}(z^{-1})$. I work with the reciprocal polynomial of the characteristic polynomial, rather than with the characteristic polynomial itself, as the former is more convenient to use than the latter in conjunction with the lag operator.

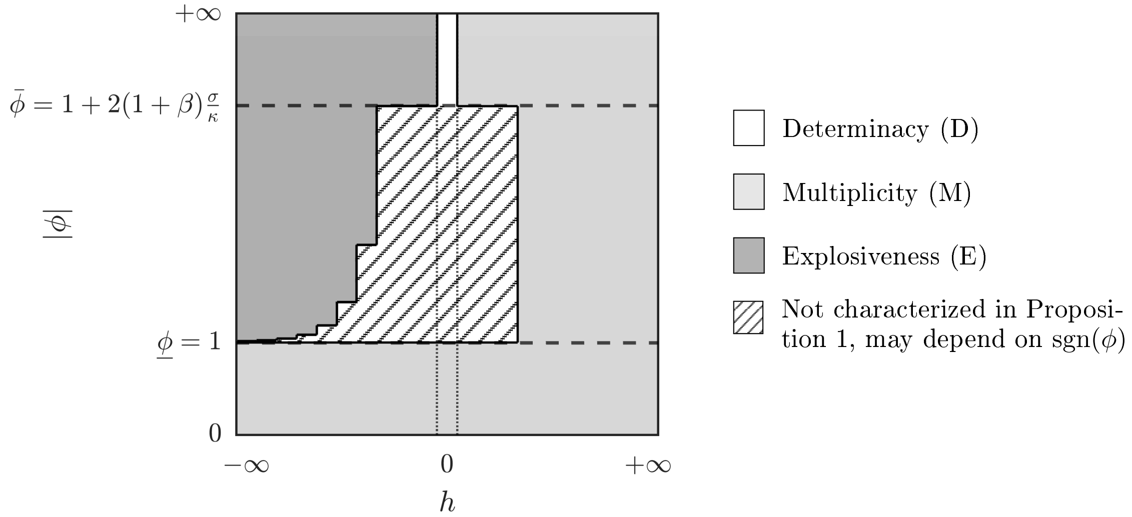
[9] This result is "generic" in the sense of holding for all $(\phi, h) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{Z}$ except $(\phi, h) = (-\beta\sigma/\kappa, 2)$. If $(\phi, h) = (-\beta\sigma/\kappa, 2)$, then the coefficient of $\mathbb{E}_t\left\{\pi_{t+2}\right\}$ in the dynamic equation is 0, and we get $\nu = 1$ instead of $\nu = 2$. I study such zero-measure cases in Loisel (2009); I ignore them in the present paper.

Finally, let $\mathcal{C}$ denote the circle of radius 1 centered at the origin of the complex plane, and p the number of roots of $P(z)$ inside $\mathcal{C}$ (counting multiplicity). As follows from Blanchard and Kahn (1980), the dynamic equation has an infinity of stationary solutions if $p \leq \nu - 1$, a unique stationary solution if $p = \nu$, and no stationary solution if $p \geq \nu + 1$. I say that the "determinacy status" $S(\phi, h)$ of the system composed of the structural equations (1)-(2) and Rule 1 is equal to M (for "multiplicity") in the first case, D (for "determinacy") in the second case, and E (for "explosiveness") in the third case.

2.2 Determinacy status, independently of sgn(ϕ)

I now determine the determinacy status $S(\phi, h)$ for $|\phi|$ sufficiently small or large and any h, as well as for $|h|$ sufficiently large and any ϕ. I find that for these values of ϕ and h, the determinacy status does not depend on the sign of ϕ, i.e. $S(\phi, h) = S(-\phi, h)$. I will state the results below, in Proposition 1. Before stating the results, however, I represent them diagrammatically in Figure 1, in order to guide the reading and facilitate the understanding of Proposition 1. Figure 1 shows the determinacy status $S(\phi, h)$ in the pseudo half-plane $(h, |\phi|) \in \mathbb{Z} \times \mathbb{R}_+$, according to Proposition 1.

Figure 1: Determinacy status for the basic NK model and Rule 1, independently of sgn(ϕ)



Proposition 1, which may look a bit hermetic at first sight, is actually nothing else than a mathematical formulation of Figure 1:[10]

[10]In this proposition and in the rest of the paper, I use the shortcut "$\forall\, |\phi|$ ..." for "$\forall \phi \in \mathbb{R}$ such that $|\phi|$ ...".

Proposition 1 (Determinacy status, independently of sgn(ϕ), in the basic NK illustration): *Consider the basic NK model (1)-(2) and Rule 1. Let* $\underline{\phi} := \min_{z \in \mathcal{C}} |Q(z)| = 1$ *and* $\bar{\phi} := \max_{z \in \mathcal{C}} |Q(z)| = 1 + 2(1+\beta)\sigma/\kappa$*. Then, as represented diagrammatically in Figure 1:*

(a) $\forall\, |\phi| < \underline{\phi}$, $\forall h \in \mathbb{Z}$, $S(\phi, h) = M$;

(b) $\forall\, |\phi| > \bar{\phi}$, *(i)* $\forall h \leq -1$, $S(\phi, h) = E$, *(ii)* $S(\phi, 0) = D$, *(iii)* $\forall h \geq 1$, $S(\phi, h) = M$;

(c) $\exists \bar{h} \in \mathbb{Z}$, $\forall\, |\phi| \in (\underline{\phi}, \bar{\phi})$, $\forall h \geq \bar{h}$, $S(\phi, h) = M$;

(d) $\exists \underline{h} : (\underline{\phi}, \bar{\phi}) \to \mathbb{Z}$, *(i)* $\forall\, |\phi| \in (\underline{\phi}, \bar{\phi})$, $\forall h \leq \underline{h}\,(|\phi|)$, $S(\phi, h) = E$, *(ii)* $\forall \varepsilon \in (0, \bar{\phi} - \underline{\phi})$, $\underline{h}$ *is bounded on* $(\underline{\phi} + \varepsilon, \bar{\phi})$.

Proof: See the Appendix. The proof uses two complex-analysis theorems, those of Rouché (1862) and Erdős and Turán (1950). ■

Points (a)-(b) of Proposition 1 are about the determinacy status $S(\phi, h)$ for a sufficiently small or large $|\phi|$ (bottom and top horizontal bands of Figure 1). To get the intuition for these two points, note that the polynomial $P(z)$ in (4), which characterizes the system's dynamics, is the sum of two terms: the term $Q(z)z^{\max(0,h-2)}$, coming from the structural equations and independent of ϕ, and the term $\phi z^{\max(0,2-h)}$, coming from the rule and proportional to ϕ.

For $|\phi| < \underline{\phi} := \min_{z \in \mathcal{C}} |Q(z)|$ (Point (a) of Proposition 1 and bottom horizontal band of Figure 1), the former term is larger in modulus than the latter term on the entire circle $\mathcal{C}$: $\forall z \in \mathcal{C}$, $\left|Q(z)z^{\max(0,h-2)}\right| > \left|\phi z^{\max(0,2-h)}\right|$. In this sense, the structural equations *dominate* the rule in the system's dynamics. As a result, the rule does not change the system's dynamics enough, relatively to an interest-rate peg, to affect the determinacy status; and this status remains the same as under an interest-rate peg – i.e., multiplicity.

Conversely, for $|\phi| > \bar{\phi} := \max_{z \in \mathcal{C}} |Q(z)|$ (Point (b) of Proposition 1 and top horizontal band of Figure 1), it is the rule that dominates the structural equations in the system's dynamics: $\forall z \in \mathcal{C}$, $\left|\phi z^{\max(0,2-h)}\right| > \left|Q(z)z^{\max(0,h-2)}\right|$. As a result, the determinacy status depends only on the horizon h in the rule. A large weight $|\phi|$ on past inflation ($h \leq -1$) favors exploding paths and leads to explosiveness; a large weight $|\phi|$ on expected future inflation ($h \geq 1$) favors imploding paths and leads to multiplicity; and a large weight $|\phi|$ on current inflation ($h = 0$) strikes the right balance between exploding and imploding paths, and leads to determinacy.

Point (c) of Proposition 1 is about the determinacy status for $|\phi| \in (\underline{\phi}, \bar{\phi})$ and h sufficiently large (right-hand side of the central horizontal band in Figure 1). To get the intuition for this point, let z_o denote the root of $Q(z)$ in $(1, +\infty)$, with the subscript "o" standing for "outside $\mathcal{C}$."[11] Under an interest-rate peg ($\phi = 0$), we have a multiplicity of equilibrium paths that converge over time to zero at rate z_o^{-1}. When the interest rate is not pegged ($\phi \neq 0$), these paths are no longer equilibrium paths: they do not satisfy the dynamic equation (3) because of the (now non-zero) term $\phi\mathbb{E}_t\{\pi_{t+h}\}$ in this equation. When h is large, however, they are "*close to satisfying*" the dynamic equation, as the term $\phi\mathbb{E}_t\{\pi_{t+h}\}$ is, on these paths, proportional to z_o^{-h} and hence close to zero. As a result, by continuity, there are neighboring paths that do satisfy the dynamic equation; i.e., there are equilibrium paths that converge over time to zero at a rate close to z_o^{-1}.[12] So, the determinacy status is multiplicity for h sufficiently large.

Point (d) of Proposition 1 is about the determinacy status for $|\phi| \in (\underline{\phi}, \bar{\phi})$ and $-h$ sufficiently large (left-hand side of the central horizontal band in Figure 1). For $|\phi| \in (\underline{\phi}, \bar{\phi})$, the structural equations do not dominate the rule on the entire circle $\mathcal{C}$ (since $|\phi| > \underline{\phi}$), nor does the rule dominate the structural equations on the entire circle $\mathcal{C}$ (since $|\phi| < \bar{\phi}$). As $h \to -\infty$, the roots of $P(z)$ distribute themselves between outside and inside $\mathcal{C}$ *in proportion of* the share of $\mathcal{C}$ on which the structural equations dominate the rule and the share of $\mathcal{C}$ on which the rule dominates the structural equations; so, the number of roots of $P(z)$ inside $\mathcal{C}$ grows unboundedly ($p \to +\infty$) and eventually exceeds the constant number of non-predetermined variables ($\nu = 2$), leading to explosiveness. Determinacy may still obtain for arbitrarily large $-h$'s, but *only if* $|\phi|$ is arbitrarily close to $\underline{\phi}$, i.e. only if the portion of $\mathcal{C}$ on which the rule dominates the structural equations is arbitrarily small.

In both Points (b) and (d), thus, a *sufficiently large weight* $|\phi|$ on *sufficiently ancient inflation rates* favors exploding paths and leads to explosiveness (left-hand side of the top and central horizontal bands in Figure 1). The two points suggest some *substitutability* between "sufficiently large weight" and "sufficiently ancient inflation rates." In Point (b), the weight $|\phi|$ is higher than $\bar{\phi}$ (i.e. the rule dominates the structural equations on the entire circle $\mathcal{C}$), and even the most recent inflation rates are enough to generate explosiveness. In Point (d), conversely, for very ancient inflation rates, even a weight $|\phi|$ hardly higher than $\underline{\phi}$ (i.e. even a very small portion of $\mathcal{C}$ on which the rule dominates the structural equations) is enough to generate explosiveness.

[11] Since $Q(0) = \beta\sigma/\kappa > 0$, $Q(1) = -1 < 0$, and $\lim_{z\in\mathbb{R}, z\to+\infty} Q(z) = +\infty$, $Q(z)$ has one root in $(0, 1)$ and another in $(1, +\infty)$.

[12] As $h \to +\infty$, these equilibrium paths uniformly converge to those under an interest-rate peg, as the rate at which they converge over time to zero converges to z_o^{-1}. In this sense, arbitrarily large horizons in the rule preserve all the local equilibria existing under an interest-rate peg.

2.3 Determinacy status, depending on sgn(ϕ)

Proposition 1 and the associated Figure 1 focus on regions of the pseudo plane $(h, \phi) \in \mathbb{Z} \times \mathbb{R}$ in which the determinacy status does not depend on the sign of ϕ. In the remaining, hatched, central region of Figure 1, however, the determinacy status may $-$ and does $-$ depend on the sign of ϕ. It is well known, since Bullard and Mitra (2002) and Woodford (2003, Chapter 4), that $S(\phi, h) = D$ for $1 < \phi < 1 + 2(1+\beta)\sigma/\kappa$ (i.e. $\underline{\phi} < \phi < \bar{\phi}$) and $h \in \{-1, 0, 1\}$. So, for $\phi > 0$, the hatched region of Figure 1 includes a determinacy zone. By contrast, for $\phi < 0$, the hatched region does not include any determinacy zone, as the following proposition shows:

Proposition 2 (Determinacy status for $\phi \in (-\bar{\phi}, -\underline{\phi})$ in the basic NK illustration): *Consider the basic NK model (1)-(2) and Rule 1. Then* $\forall \phi \in (-\bar{\phi}, -\underline{\phi})$, $\forall h \in \mathbb{Z}$, $S(\phi, h) \neq D$.

Proof: See Online Appendix A.1.[13] ■

Recall that determinacy obtains if and only if $p = \nu$ (where, as a reminder, p denotes the number of roots of $P(z)$ inside $\mathcal{C}$, and ν the number of non-predetermined variables). The proof of Proposition 2 simply shows that $p - \nu$ is odd, and hence non-zero, for all $(\phi, h) \in (-\bar{\phi}, -\underline{\phi}) \times \mathbb{Z}$. So, as (ϕ, h) moves within this set, the only possible changes in the determinacy status $S(\phi, h)$ are direct jumps from multiplicity to explosiveness or vice-versa.

Proposition 2 might be considered of limited practical relevance, as empirical estimates of ϕ are, of course, typically positive in the literature. The proposition is nonetheless useful to emphasize that the determinacy status depends on the sign of ϕ in the hatched region of Figure 1. It also prepares the ground for more general results in Section 3, which are of a similar nature but may apply to positive values of ϕ.

2.4 Numerical example

In order to illustrate Propositions 1-2 and the next propositions numerically and graphically, I consider Woodford's (2003, Chapter 4) calibration of the basic NK model:

[13] Unlike the proof of Proposition 1, which is in the Appendix of the paper, the proofs of Propositions 2-8 are relegated to an Online Appendix. The reason is that these proofs do not bring any substantial new insight, compared to their brief discussion in the main text and/or to the proof of Proposition 1.

$(\beta, \kappa, \sigma) = (0.99, 0.022, 0.16)$, the period being one quarter.[14] I call "Model 1" the resulting calibrated model, as it is the first of several calibrated models that I will consider in the paper.

The results obtained for Model 1 and Rule 1 are presented in Figure 2. This figure shows the determinacy status $S(\phi, h)$ in the pseudo plane $(h, \phi) \in \mathbb{Z} \times \mathbb{R}$ with a log scale for ϕ.[15] The horizon value $\lfloor h_1 \rfloor$ featuring in Figure 2a will be introduced and commented upon in the next subsection.

Figure 2: Determinacy status for Model 1 and Rule 1

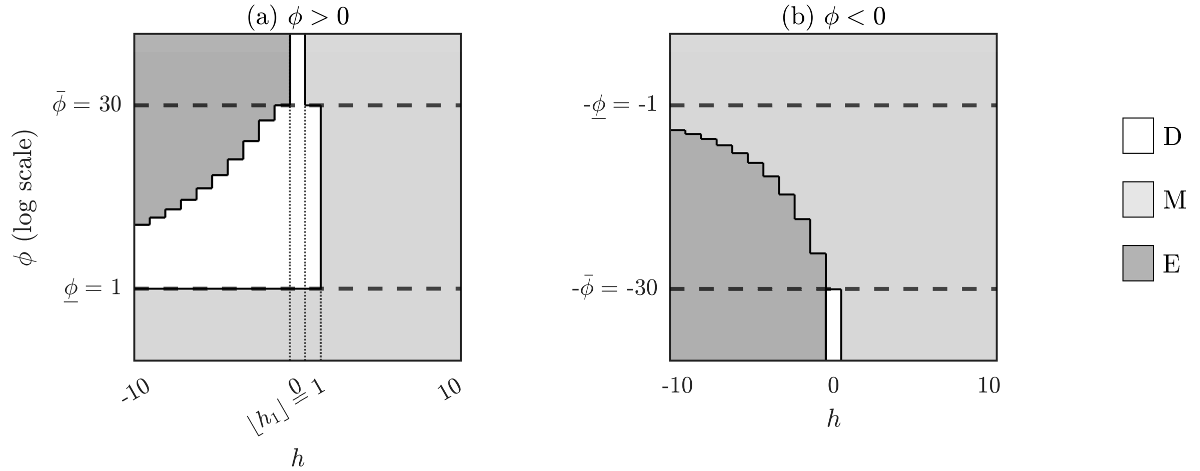


Figure 2 provides a numerical illustration of the diagrammatic Figure 1 (which summarizes Proposition 1). The difference between Figures 2a and 2b makes clear that the determinacy status in the hatched, central region of Figure 1 depends on the sign of ϕ; in particular, Figure 2b illustrates Proposition 2.

Even though the basic NK model is clearly not quantitative, two numerical features of Figure 2 are worth emphasizing. First, the upper coefficient threshold $\bar{\phi}$ is one order of magnitude larger than standard values of ϕ in the literature (which are often between 1 and 2). Second, the horizon threshold at and above which the rule can no longer deliver determinacy (i.e., the lowest integer $\bar{h}$ in Point (c) of Proposition 1) is only two quarters; so, in this numerical example, a forward-looking monetary policy can ensure determinacy only if it is hardly forward-looking.

[14] Galí's (2015, Chapter 3) calibration, $(\beta, \kappa, \sigma) = (0.99, 0.125, 1)$, leads to qualitatively and quantitatively similar results.

[15] Throughout the paper, I normalize the dimensions of numerical-figure panels as follows: (i) each panel is square; (ii) in each panel, the horizontal band corresponding to $\phi \in (\underline{\phi}, \bar{\phi})$ or $\phi \in (-\bar{\phi}, -\underline{\phi})$ is vertically centered and has a 16:9 widescreen aspect ratio. I do not report the implied lowest and largest ϕ values in the panel, as they are uninformative.

2.5 Taylor principle

I now examine the validity of the Taylor principle as a condition for determinacy. By "Taylor principle", throughout the paper, I mean the long-run Taylor principle, first proposed by Woodford (2001, 2003) and widely used thereafter (e.g. in Galí, 2015, Chapter 4). I will provide a formal, general definition of the long-run Taylor principle in Section 3. In the current section, I only need to state this principle in the specific context of the basic NK model. In this context, loosely speaking, the long-run Taylor principle states that if the inflation rate were permanently higher by one percentage point, then the system composed of the Phillips curve (2) and the rule considered should make the interest rate permanently higher by more than one percentage point. Under Rule 1, this principle straightforwardly translates into the familiar inequality $\phi > 1$.

Proposition 1 and the associated Figure 1 have two straightforward implications for the Taylor principle in the basic NK model under Rule 1. First, for $\phi \geq 0$, the Taylor principle $\phi > 1$ is *necessary* for determinacy for all $h \in \mathbb{Z}$. Second, for all $h \in \mathbb{Z} \setminus \{0\}$, the Taylor principle is *not sufficient* for determinacy. The following proposition states a third result about the Taylor principle, which does not directly follow from Proposition 1. This result is, in part, about the Taylor principle being "locally sufficient" for determinacy, in the sense of ϕ just above 1 delivering determinacy (i.e. $\exists \varepsilon > 0$, $\forall \phi \in (1, 1+\varepsilon)$, $S(\phi, h) = D$).

Proposition 3 (Taylor principle in the basic NK illustration): *Consider the basic NK model (1)-(2) and Rule 1 with $\phi \geq 0$. Let $h_1 := 2 - Q'(1)/Q(1) = 1 + (1-\beta)\sigma/\kappa$. Then the Taylor principle $\phi > 1$ is necessary and locally sufficient for determinacy if and only if $h < h_1$.*

Proof: See Online Appendix A.2. ■

Proposition 3 can be understood and interpreted as follows. For any $\phi \in (0, 1)$ and any $h \in \mathbb{Z}$, the degree of indeterminacy is one: $P(z)$ lacks one root inside the unit circle $\mathcal{C}$ to deliver determinacy (just like under a peg). As ϕ goes from below 1 to above 1, one root of $P(z)$ crosses the unit circle $\mathcal{C}$ (at point 1). When $h < h_1$, the root goes from outside to inside $\mathcal{C}$, so the determinacy status changes from multiplicity to determinacy (reflecting the fact that increasing the weight on inflation rates sufficiently distant in the past favors exploding paths). Alternatively, when $h > h_1$, the root goes from inside to outside $\mathcal{C}$, so the determinacy status remains multiplicity (reflecting the fact that increasing the weight on inflation rates sufficiently distant in the future favors imploding paths).[16]

Proposition 3 is illustrated numerically in Figure 2a, where $\lfloor h_1 \rfloor = 1$. More generally, for standard calibrations of the basic NK model, the horizon threshold $h_1 = 1 + (1-\beta)\sigma/\kappa$ is typically between 1 and 2, as β is typically set to 0.99 (on a quarterly basis). For Woodford's (2003, Chapter 4) and Galí's (2015, Chapter 3) calibrations, for instance, h_1 takes the values 1.07 and 1.08, which are much closer to 1 than to 2. So, in the basic NK model under Rule 1, the Taylor principle is typically not a valid determinacy condition for horizons of two or more quarters.

Conversely, the Taylor principle is always a valid determinacy condition for negative horizons (in the basic NK model under Rule 1), as illustrated again in Figure 2a. As a consequence, a central bank reacting to inflation can be arbitrarily backward-looking and still ensure determinacy − provided that ϕ is above and arbitrarily close to 1.

2.6 Rule inertia

The interest-rate rules considered in the literature are often inertial: they set the interest rate conditionally on its past value(s). I now introduce some inertia into Rule 1 in the following way:

$$i_t = \rho i_{t-1} + (1-\rho)\,\phi \mathbb{E}_t \{\pi_{t+h}\}, \tag{Rule 2}$$

where $\rho \in (0,1)$ and $(\phi, h) \in \mathbb{R} \times \mathbb{Z}$. Following standard practice in the literature, I have multiplied the coefficient ϕ by the scale factor $1-\rho$ in this rule, in order to "normalize" the long-run Taylor principle to $\phi > 1$. The next proposition describes how this inertia affects the determinacy status and the validity of the Taylor principle as a condition for determinacy:

Proposition 4 (Rule inertia in the basic NK illustration): *Propositions 1-3 still hold for Rule 2 instead of Rule 1, if $Q(z)$ is replaced by $Q(z)(1-\rho z)/(1-\rho)$ in these propositions. This replacement leaves $\underline{\phi}$ unchanged, multiplies $\bar{\phi}$ by $(1+\rho)/(1-\rho)$, and adds $\rho/(1-\rho)$ to h_1.*

Proof: see Online Appendix A.3. ■

Replacing the non-inertial Rule 1 with the inertial Rule 2 thus amounts to replacing $Q(z)$ with $Q(z)(1-\rho z)/(1-\rho)$ in the expression (4) of $P(z)$; I will explain why below. In turn,

[16] In a (longer) working-paper version of this paper (Loisel, 2022), I also show that $\forall h > h_1$, $\forall \phi \in \mathbb{R}$, $S(\phi, h) = M$: so, $\lceil h_1 \rceil$ is the smallest integer $\bar{h}$ in Point (c) of Proposition 1.

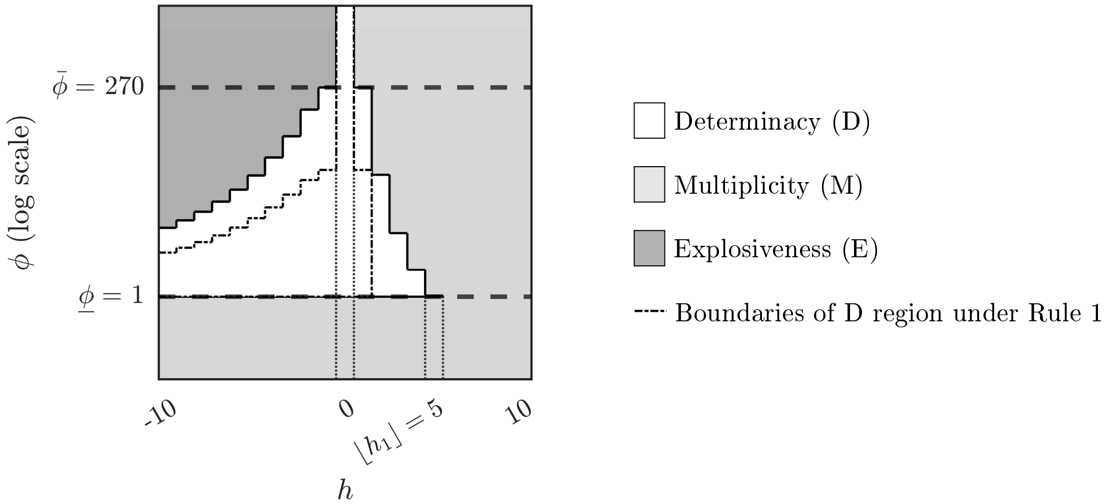
the replacement of $Q(z)$ by $Q(z)(1-\rho z)/(1-\rho)$ affects the previous results *quantitatively* (notably by increasing $\bar{\phi}$ and h_1, and increasing them unboundedly as $\rho \to 1$), but not *qualitatively*: the general form of Figure 1 remains the same (Proposition 1), we still have indeterminacy for all $(\phi, h) \in (-\bar{\phi}, -\underline{\phi}) \times \mathbb{Z}$ (Proposition 2), and the Taylor principle is still necessary and locally sufficient for determinacy if and only if the horizon h is below a certain threshold h_1 (Proposition 3).[17]

Proposition 4 is illustrated numerically in Figure 3. This figure shows the determinacy status for Model 1 and Rule 2 with $\rho = 0.8$ and $\phi > 0$. For this value of ρ (which is a standard value in the literature, when the period is one quarter), as we move from Rule 1 to Rule 2, $\bar{\phi}$ increases ninefold (as $(1+\rho)/(1-\rho) = 9$), from 30 to 270, and h_1 increases by 4 quarters (as $\rho/(1-\rho) = 4$), from 1 to 5 quarters. More generally, the whole determinacy region extends both upwards and rightwards as we move from Rule 1 to Rule 2, as apparent in Figure 3 (where, for comparison purposes, I have also shown the boundaries of the determinacy region under Rule 1). It extends upwards because inertia makes it harder for the rule to dominate the structural equations on any portion of $\mathcal{C}$ (since $|(1-\rho z)/(1-\rho)| \geq 1$ for any $z \in \mathcal{C}$). And it extends rightwards because inertia, by increasing the weight on past outcomes, tends to favor exploding paths and, as a result, changes the determinacy status from multiplicity to determinacy on the immediate right of the determinacy region obtained under Rule 1 (thus increasing h_1).

I have so far commented upon and interpreted the second part of Proposition 4, but not its first part yet. Why does replacing Rule 1 with Rule 2 amount, in the first place, to replacing $Q(z)$ with $Q(z)(1-\rho z)/(1-\rho)$ in the expression (4) of $P(z)$? Recall that $Q(z)$ is the reciprocal polynomial of the characteristic polynomial under a peg. Why does introducing inertia *into the rule* amount to changing the dynamics *under a peg*? The reason is that introducing some inertia into the reaction of the interest rate to the state of the economy (i.e. replacing i_t with $(1-\rho L)i_t/(1-\rho)$ in the rule), leaving unchanged the reaction of the economy to the interest rate (i.e. leaving the structural equations unchanged), is equivalent, as far as the determinacy status is concerned, to introducing

[17] Of course, if we replace Rule 1 not with Rule 2, but instead with the rule $i_t = \rho i_{t-1} + \phi \mathbb{E}_t\{\pi_{t+h}\}$ (which is Rule 2 without the scale factor $1-\rho$), then Propositions 1-3 still hold with this time $\underline{\phi}$ multiplied by $1-\rho$, $\bar{\phi}$ multiplied by $1+\rho$, the Taylor principle changed to $\phi > 1-\rho$, and h_1 still increased by $\rho/(1-\rho)$. As $\rho \to 1$, this rule converges to the "first-difference rule" $i_t = i_{t-1} + \phi \mathbb{E}_t\{\pi_{t+h}\}$, which has the same implications for the determinacy status as the "Wicksellian rule" $i_t = \phi \mathbb{E}_t\{p_{t+h}\}$ (where p_t denotes the price level). So, by continuity, the determinacy status under this rule converges, as $\rho \to 1$, to the determinacy status under the Wicksellian rule, which I characterize in a (longer) working-paper version of this paper (Loisel, 2022). In particular, at the limit, the Taylor principle is $\phi > 0$ and h_1 is infinite; so, under the Wicksellian rule, for any horizon $h \in \mathbb{Z}$, a sufficiently small positive coefficient ϕ ensures determinacy.

Figure 3: Determinacy status for Model 1 and Rule 2 with $\rho = 0.8$ and $\phi > 0$



the same inertia into the reaction of the economy to the interest rate (i.e. replacing x_t with $(1-\rho L)x_t/(1-\rho)$ for all $x_t \in \{y_t, \pi_t, y_{t+1}, \pi_{t+1}\}$ in the structural equations, and hence replacing $Q(z)$ with $Q(z)(1-\rho z)/(1-\rho)$), leaving unchanged the reaction of the interest rate to the state of the economy (i.e. leaving the rule unchanged).

2.7 Discussion

In this section, I have studied the determinacy status in the basic NK model under Rules 1 and 2, as a function of the coefficient ϕ and the horizon h of inflation in the rule. In the next section, I will study the determinacy status for a broad class of models and a broad class of rules, as a function of the coefficient and horizon of *one variable* in the rule, taking as given the coefficients and horizons of the *other variables* in the rule (if any). For instance, for interest-rate rules of type

$$i_t = \rho i_{t-1} + (1-\rho)\left(\phi_\pi \mathbb{E}_t\left\{\pi_{t+h_\pi}\right\} + \phi_y \mathbb{E}_t\left\{y_{t+h_y}\right\}\right),$$

where $\rho \in [0,1)$, $(\phi_\pi, \phi_y) \in \mathbb{R}^2$ and $(h_\pi, h_y) \in \mathbb{Z}^2$, the results will characterize the determinacy status as a function of (ϕ_π, h_π) for a given (ϕ_y, h_y), or as a function of (ϕ_y, h_y) for a given (ϕ_π, h_π), but not as a function of $(\phi_\pi, \phi_y, h_\pi, h_y)$.[18] Thus, the new principles for stabilization policy that I establish in this paper do not require the policymaker to react to only one variable (as the analysis above, based on Rules 1 and 2, might suggest); rather, they are about how much and at what horizon the policymaker should react to one variable, taking as given how much and at what horizon the policymaker reacts to other variables.

[18] Studying the determinacy status as a function of $(\phi_\pi, \phi_y, h_\pi, h_y)$ would lead to intricate and model-specific results, whereas I am after simple and general results.

What qualitative principles for monetary policy have we learned from this basic NK illustration? To achieve determinacy, the interest rate should not react too weakly to inflation, nor too strongly (except at the horizon 0). It should not react to inflation too far away in the future, nor too far away in the past (except with a coefficient just above 1). It should essentially not react negatively to inflation. And the Taylor principle is a locally good guide for determinacy, but only up to a certain positive horizon threshold, which increases unboundedly with the inertia coefficient. How general are all these principles? Do they extend to other models, other rules, other policy instruments? I address these questions in the next section.

3 General analysis

In this section, I generalize the results of the previous section to a broad class of dynamic rational-expectations models, and to rules making the policy instrument react to any variable (or linear combination of variables) at horizon h with coefficient ϕ, possibly among other variables, possibly with inertia.

3.1 Models and rules

At each date $t \in \mathbb{Z}$, the private sector sets an n-dimension vector of endogenous variables $\mathbf{X}_t$ according to the following (locally log-linearized) structural equations:

$$\mathbb{E}_t \left\{ \mathbf{\Delta} \left(L^{-1} \right) \left[\mathbf{A} \left(L \right) \mathbf{X}_t + L^{-\gamma} \mathbf{B} \left(L \right) i_t \right] \right\} = \mathbf{0}, \tag{5}$$

where again i_t denotes the policy instrument at date t, L the lag operator, and $\mathbb{E}_t\{.\}$ the date-t rational-expectations operator. I abstract again from exogenous shocks, as they are irrelevant for determinacy issues. These structural equations are parameterized by $n \in \mathbb{N} \setminus \{0\}$, $\gamma \in \mathbb{N}$, $\mathbf{A}(z) \in \mathbb{R}^{n \times n}[z]$, $\mathbf{B}(z) \in \mathbb{R}^{n \times 1}[z] \setminus \{\mathbf{0}\}$, $\mathbf{\Delta}(z) = \text{diag}\left(z^{\delta_1}, ..., z^{\delta_n}\right) \in \mathbb{R}^{n \times n}[z]$, and $(\delta_1, ..., \delta_n) \in \mathbb{N}^n$.[19] I assume that $\det[\mathbf{A}(0)] \neq 0$; this assumption is made without any loss in generality because any system of independent structural equations of type (5) that does not satisfy this assumption can be equivalently rewritten as a system of type (5) that does.

The policymaker sets the policy instrument i_t according to a rule. I assume for now that

[19] Throughout the paper, letters in bold denote vectors and matrices that have potentially more than one element. $\mathbf{0}$ denotes a vector or a matrix whose elements are all equal to zero and whose dimensions depend on the specific context in which it is used. For any $(n_1, n_2) \in (\mathbb{N} \setminus \{0\})^2$, $\mathbb{R}^{n_1 \times n_2}[z]$ denotes the set of polynomials in z whose coefficients are $n_1 \times n_2$ matrices with real-number elements.

the rule is not inertial and involves a single variable (I will relax these assumptions later in the section). So, for now, the policymaker follows the (log-linearized) rule

$$i_t = \phi \mathbb{E}_t \{v_{t+h}\}, \tag{6}$$

where $\phi \in \mathbb{R}$ and $h \in \mathbb{Z}$ (with again $\mathbb{E}_t \{v_{t+h}\} = v_{t+h}$ when $h \leq 0$), and where v_t can be any linear combination of current and past endogenous variables:

$$v_t := \mathbf{V}(L)\mathbf{X}_t \tag{7}$$

with $\mathbf{V}(z) \in \mathbb{R}^{1\times n}[z]$. I make the following non-restrictive assumption on $\mathbf{V}(z)$:

$$W(z) := \det \begin{bmatrix} \mathbf{A}(z) & \mathbf{B}(z) \\ \mathbf{V}(z) & 0 \end{bmatrix} \neq 0.$$

If this assumption were not satisfied, then v_t could be expressed as a linear combination of (a backward-looking version of) the structural equations, and would therefore be exogenous.[20]

3.2 Preliminaries

As in Section 2, let ν denote the number of non-predetermined variables of the system (5)-(6), and $P(z)$ the reciprocal polynomial of the characteristic polynomial of this system. In addition, let $\omega \in \mathbb{N}$ denote the multiplicity of 0 as a root of $W(z)$ (with $\omega = 0$ if $W(0) \neq 0$), and let $\delta := \sum_{j=1}^{n} \delta_j$, $m := \omega - \gamma$, $Q(z) := \det[\mathbf{A}(z)]$ and $R(z) := -z^{-\omega} W(z)$. I start by establishing a useful preliminary result:

Lemma 1: *Consider a model of type (5) and a rule of type (6). Then: (a) if $\phi = 0$, then $\nu = \delta$ and $P(z) = Q(z)$; (b) if $\phi \neq 0$, then generically $\nu = \delta + \max(0, h - m)$ and*

$$P(z) = Q(z) z^{\max(0,h-m)} + \phi R(z) z^{\max(0,m-h)}. \tag{8}$$

Proof: See Online Appendix A.4. ■

This lemma generalizes similar preliminary results obtained in Section 2: in the specific context of the basic NK model under Rule 1, we had $\delta = 2$, $m = 2$, $Q(z) = (\sigma/\kappa)[\beta - (1 + \beta + \kappa/\sigma)z + z^2]$ and $R(z) = 1$.[21]

[20] In the basic NK model, for instance, imposing $W(z) \neq 0$ rules out the variable $v_t = \pi_{t-1} - \beta\pi_t - \kappa y_{t-1}$, which corresponds to a backward-looking version of the Phillips curve (2). This variable is exogenous because it can be rewritten, using the Phillips curve (2), as the expectation error $v_t = -\beta(\pi_t - \mathbb{E}_{t-1}\{\pi_t\})$, which implies that $\mathbb{E}_t\{v_{t+h}\} = 0$ for $h \geq 1$.

[21] The qualifier "generically" in the lemma refers again to the fact that $\nu = \delta + \max(0, h - m)$ for all $(\phi, h) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{Z}$ except $(\phi, h) = (-Q(0)/R(0), m)$, as discussed in Footnote 9.

The polynomials $Q(z)$ and $R(z)$ will play a key role in my analysis. None of them depends on the coefficient ϕ or the horizon h of the rule. $Q(z)$ is the reciprocal polynomial of the characteristic polynomial *under a policy-instrument peg* (i.e. under the policy-instrument rule $i_t = 0$), while $R(z)$ is the reciprocal polynomial of the characteristic polynomial *under the "targeting rule"* $v_t = 0$. In (8), if we abstract from the factors $z^{\max(0,h-m)}$ and $z^{\max(0,m-h)}$ (which capture lags and leads), then $P(z)$ is a weighted sum of $Q(z)$ and $R(z)$, and the relative weight of $R(z)$ in this sum is the coefficient ϕ. Loosely speaking, as $|\phi|$ moves from zero to infinity, $P(z)$ moves from $Q(z)$ to $R(z)$, and the dynamics of the system move from the dynamics under a peg ($i_t = 0$) to the dynamics under the targeting rule $v_t = 0$. In this paper, I focus on the "regular case" in which $Q(z)$ and $R(z)$ have no roots *exactly* on $\mathcal{C}$ (i.e. no roots of modulus *exactly* equal to 1). I study non-regular cases in a (longer) working-paper version (Loisel, 2022).

As in Section 2, let p denote the number of roots of $P(z)$ inside $\mathcal{C}$ (counting multiplicity). Blanchard and Kahn's (1980) *root-counting* condition for determinacy is $p = \nu$. I assume throughout the paper that Blanchard and Kahn's (1980) *no-decoupling* condition is met (and I check that it is met in all the illustrations and applications I consider in the paper).[22] So, the determinacy status is multiplicity if $p \leq \nu - 1$, determinacy if $p = \nu$, and explosiveness if $p \geq \nu + 1$.

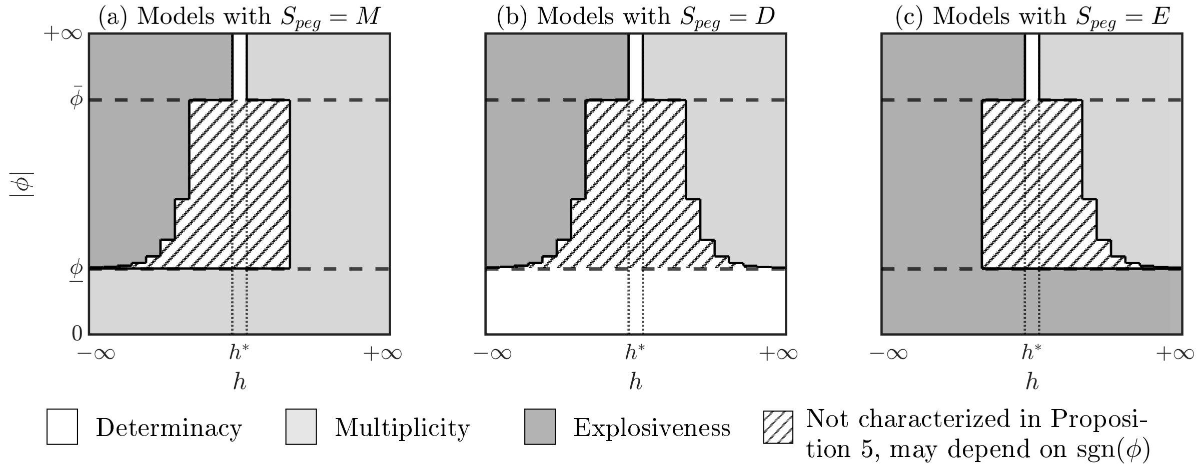
3.3 Determinacy status, independently of sgn(ϕ)

I start by generalizing Proposition 1 to the class of models (5) and the class of rules (6). I will state the results below, in Proposition 5. Before stating the results, however, I represent them diagrammatically in Figure 4, in order to guide the reading and facilitate the understanding of Proposition 5 (like in Section 2, with Figure 1 and Proposition 1). Figure 4 shows the determinacy status $S(\phi, h)$ in the pseudo half-plane $(h, |\phi|) \in \mathbb{Z} \times \mathbb{R}_+$, according to Proposition 5. In this figure, S_{peg} denotes the *determinacy status under a peg* ($S_{peg} := S(0, h)$ for any $h \in \mathbb{Z}$), while $\underline{\phi}$, $\bar{\phi}$, and h^* will be defined in Proposition 5.

Proposition 5, which may look a bit hermetic at first sight (and in which r denotes the number of roots of $R(z)$ inside $\mathcal{C}$, counting multiplicity), is actually nothing else than a mathematical formulation of Figure 4:

[22] The "no-decoupling condition" requires that the system should not be "decoupled" in the sense of Sims (2007). It is formulated as a matrix-rank condition in Blanchard and Kahn (1980, Page 1308), and is often called the "rank condition" in the literature. Sims' (2007) bare-bones example of a system meeting the root-counting condition but not the no-decoupling condition is $x_t = 1.1x_{t-1} + \varepsilon_t$ and $\mathbb{E}_t\{y_{t+1}\} = 0.9y_t + \nu_t$.

Figure 4: Determinacy status for models of type (5) and rules of type (6), independently of sgn(ϕ)



Proposition 5 (Determinacy status, independently of sgn(ϕ), in the general framework): *Consider a model of type (5) and a rule of type (6). Let* $\underline{\phi} := \min_{z \in \mathcal{C}} |Q(z)/R(z)|$, $\bar{\phi} := \max_{z \in \mathcal{C}} |Q(z)/R(z)|$, *and* $h^* := m + r - \delta$. *Then, as represented diagrammatically in Figure 4:*

(a) $\forall\, |\phi| < \underline{\phi}$, $\forall h \in \mathbb{Z}$, $S(\phi, h) = S_{peg}$*;*

(b) $\forall\, |\phi| > \bar{\phi}$, *(i)* $\forall h \leq h^* - 1$, $S(\phi, h) = E$, *(ii)* $S(\phi, h^*) = D$, *(iii)* $\forall h \geq h^* + 1$, $S(\phi, h) = M$*;*

(c) $\exists \bar{h} : (\underline{\phi}, \bar{\phi}) \to \mathbb{Z}$, *(i)* $\forall\, |\phi| \in (\underline{\phi}, \bar{\phi})$, $\forall h \geq \bar{h}\,(|\phi|)$, $S(\phi, h) = M$, *(ii)* $\forall \varepsilon \in (0, \bar{\phi} - \underline{\phi})$, $\bar{h}$ *is bounded on* $(\underline{\phi} + \varepsilon, \bar{\phi})$, *(iii) if* $S_{peg} = M$, *then* $\bar{h}$ *is bounded on* $(\underline{\phi}, \bar{\phi})$*;*

(d) $\exists \underline{h} : (\underline{\phi}, \bar{\phi}) \to \mathbb{Z}$, *(i)* $\forall\, |\phi| \in (\underline{\phi}, \bar{\phi})$, $\forall h \leq \underline{h}\,(|\phi|)$, $S(\phi, h) = E$, *(ii)* $\forall \varepsilon \in (0, \bar{\phi} - \underline{\phi})$, $\underline{h}$ *is bounded on* $(\underline{\phi} + \varepsilon, \bar{\phi})$, *(iii) if* $S_{peg} = E$, *then* $\underline{h}$ *is bounded on* $(\underline{\phi}, \bar{\phi})$.

Proof: See Online Appendix A.5. ■

The intuitions behind Proposition 5 are identical or similar to those behind Proposition 1. In Point (a), as $|\phi| < \underline{\phi}$, the structural equations *dominate* the rule in the system's dynamics, and the determinacy status remains the same as under a peg. Compared to Point (a) of Proposition 1, the novelty is that the determinacy status under a peg, S_{peg}, is no longer necessarily M: it can now also be D or E. In Point (b), as $|\phi| > \bar{\phi}$, the rule *dominates* the structural equations in the system's dynamics and makes the determinacy status depend only on h. Compared to Point (b) of Proposition 1, the novelty is that the pivotal horizon h^* can now be different from zero.

Points (c)-(i), (c)-(ii), (d)-(i), and (d)-(ii) of Proposition 5 generalize Point (d) of Proposition 1. For any given $|\phi| \in (\underline{\phi}, \bar{\phi})$, as $h \to -\infty$ (resp. as $h \to +\infty$), the roots of $P(z)$ distribute themselves between inside and outside $\mathcal{C}$ (resp. between outside and inside $\mathcal{C}$) *in proportion of* the share of $\mathcal{C}$ on which the rule dominates the structural equations and the share of $\mathcal{C}$ on which the structural equations dominate the rule; so, the number of inside roots increases less than one-for-one with $|h|$; since the number of non-predetermined variables remains constant (resp. increases one-for-one with h), we eventually get more (resp. fewer) inside roots than non-predetermined variables, and hence explosiveness (resp. multiplicity). Determinacy may still obtain for arbitrarily large $|h|$'s, but *only if* $|\phi|$ is arbitrarily close to $\underline{\phi}$, i.e. only if the portion of $\mathcal{C}$ on which the rule dominates the structural equations is arbitrarily small.

Finally, Points (c)-(iii) and (d)-(iii) of Proposition 5 generalize Point (c) of Proposition 1. For $|\phi| \in (\underline{\phi}, \bar{\phi})$, large positive (resp. negative) horizons h do not much "*perturb*" the imploding (resp. exploding) equilibrium paths obtained under a peg, as the term $\phi \mathbb{E}_t\{v_{t+h}\}$ (i.e. the right-hand side of the rule) is small on these paths; so, these horizons *preserve* the determinacy status obtained under a peg if this status is multiplicity (resp. explosiveness). I do not develop the intuition further, as it is the same intuition as for Point (c) of Proposition 1 (discussed in detail in Subsection 2.2).

3.4 Simple numerical illustrations

A first simple numerical illustration of Proposition 5 (more specifically of the diagrammatic Figure 4a) is provided by Figure 2 in Section 2. This figure, which I have already commented upon, shows the determinacy status $S(\phi, h)$ for Model 1 and Rule 1.

In order to provide additional illustrations of Proposition 5 (not only of Figure 4a, but also of Figures 4b and 4c), and also in order to illustrate the next propositions numerically and graphically, I consider, in addition to Model 1, four other simple calibrated monetary-policy models. Table 1 presents the overall five models: two imply $S_{peg} = M$, two $S_{peg} = D$, and one $S_{peg} = E$. The table also indicates, for each model, the *degree of indeterminacy under a peg*, $d_{peg} := \delta - q \in \mathbb{Z}$, where q denotes the number of roots of $Q(z)$ inside $\mathcal{C}$ counting multiplicity ($S_{peg} = M$ if $d_{peg} \geq 1$, $S_{peg} = D$ if $d_{peg} = 0$, and $S_{peg} = E$ if $d_{peg} \leq -1$).

All these models share the following "canonical" features: they have two simple structural equations; these equations are an IS equation and a Phillips curve; and the two endogenous variables set by the private sector are output and inflation. The IS equation and

Table 1: Five simple calibrated monetary-policy models

No.	Model	Calibration	S_{peg}	d_{peg}
1	Basic NK Model	Woodford (2003)	M	1
2	McKay et al. (2017)	McKay et al. (2017)	M	1
3	Gabaix (2020)	Gabaix (2020)	D	0
4	Bilbiie (2008)	Bilbiie (2008)	D	0
5	Svensson (1997) and Ball (1999)	Ball (1999)	E	-1

the Phillips curve of Models 2-4 are

$$\begin{aligned} y_t &= \alpha \mathbb{E}_t\{y_{t+1}\} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t\{\pi_{t+1}\}\right), \\ \pi_t &= \beta \mathbb{E}_t\{\pi_{t+1}\} + \kappa y_t, \end{aligned}$$

with $(\alpha, \beta, \sigma, \kappa)$ equal to $(0.97, 0.99, 2.67, 0.02)$ in Model 2, $(0.85, 0.792, 5, 0.11)$ in Model 3, and $(1, 0.99, -0.11, 0.228)$ in Model 4. These models introduce, into the basic NK model, income risk and borrowing constraints (Model 2), bounded rationality (Model 3), or limited asset-markets participation (Model 4). Compared to the basic NK model (in which α is implicitly equal to 1), Model 2 "discounts" the IS equation (i.e. reduces α), Model 3 discounts both the IS equation and the Phillips curve (i.e. reduces both α and β), and Model 4 inverts the slope of the IS equation (i.e. makes σ negative). Finally, unlike Models 1-4, Model 5 is non-micro-founded and purely backward-looking; its IS equation and Phillips curve are

$$\begin{aligned} y_t &= \lambda y_{t-1} - \mu\left(i_{t-1} - \pi_{t-1}\right), \\ \pi_t &= \pi_{t-1} + \chi y_{t-1}, \end{aligned}$$

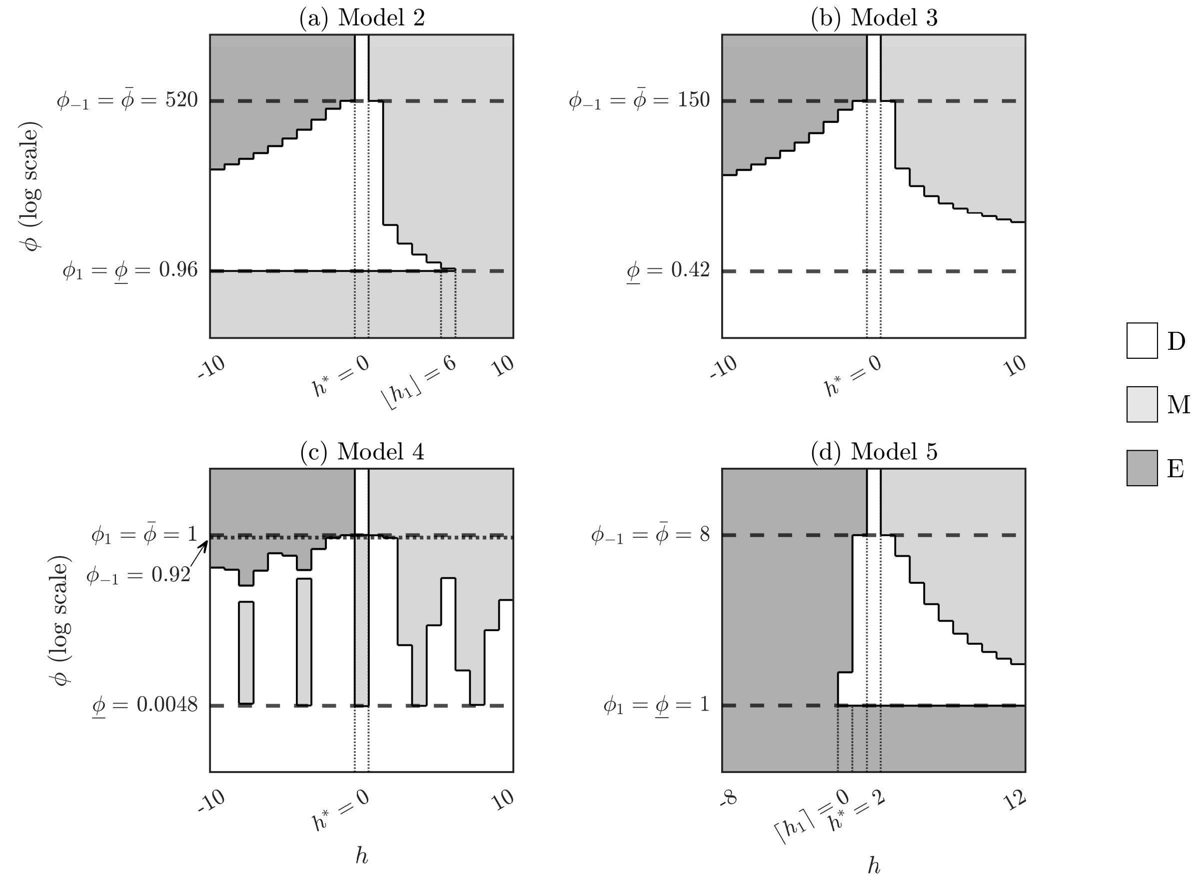
with $(\lambda, \mu, \chi) = (0.8, 1, 0.4)$.

Figure 5 shows the determinacy status $S(\phi, h)$ for Models 2-5 and Rule 1 with $\phi > 0$.[23] It provides simple numerical illustrations of the diagrammatic Figures 4a, 4b and 4c. The coefficient and horizon values ϕ_1, ϕ_{-1}, $\lfloor h_1 \rfloor$ and $\lceil h_1 \rceil$ featuring in Figure 5 will be introduced and commented upon in the next subsections.

Note that the "topology" of the E, D and M regions is simple in Figures 5a, 5b and 5d: each region is connected, and the borders between regions are monotonic functions linking h to ϕ. However, the topology is more complex in Figure 5c (even though Model 4 looks as simple as Models 2, 3 and 5): the M region is disconnected, and the borders between regions are non-monotonic, with "lace patterns."

[23] I relegate the results for $\phi < 0$ to Online Appendix A.9.

Figure 5: Determinacy status for Models 2-5 and Rule 1 with $\phi > 0$



Even though Models 2-5 are clearly not quantitative, two numerical features of Figure 5 are worth emphasizing. First, compared to standard values of ϕ in the literature (often between 1 and 2), the lower threshold $\underline{\phi}$ is of the same order of magnitude or several orders of magnitude smaller, while the upper threshold $\bar{\phi}$ is of the same order of magnitude or one or several orders of magnitude larger. Second, in Figure 5d, the horizon threshold at and below which the rule can no longer deliver determinacy is minus one period; so, in this example, a central bank that would react to inflation with a delay of at least one period (say because of data-publication lags) would necessarily fail to ensure determinacy, no matter how strongly or weakly it reacts to inflation.

3.5 Determinacy status for some ϕ intervals

Proposition 5 is silent about the determinacy status in the hatched regions of Figure 4, which are the only regions in which the determinacy status may depend on the sign of ϕ (i.e. in which we may have $S(\phi, h) \neq S(-\phi, h)$). These hatched regions lie in the two horizontal bands $\phi \in (\underline{\phi}, \bar{\phi})$ and $\phi \in (-\bar{\phi}, -\underline{\phi})$ of the pseudo plane $(h, \phi) \in \mathbb{Z} \times \mathbb{R}$.

I now identify, within these two horizontal bands (or coinciding with them), some horizontal bands that contain no determinacy zones. In other words, I identify some intervals of ϕ values, inside $(\underline{\phi}, \bar{\phi})$ and $(-\bar{\phi}, -\underline{\phi})$, for which $S(\phi, h) \neq D$ for all $h \in \mathbb{Z}$. I obtain the following proposition, which generalizes Proposition 2 along several dimensions:

Proposition 6 (Determinacy status for some ϕ intervals in the general framework): *Consider a model of type (5) and a rule of type (6). Let* $\phi_1 := -Q(1)/R(1)$, $\phi_{-1} := |Q(-1)/R(-1)|$, $\Phi_a := \{\phi_1, \text{sgn}(\phi_1)\phi_{-1}\}$ *and* $\Phi_b := \{\phi_1, -\text{sgn}(\phi_1)\phi_{-1}\}$. *Then:*
(a) if d_{peg} *is even and* $|\phi_1| < \phi_{-1}$, *then* $\forall \phi \in (\min \Phi_a, \max \Phi_a)$, $\forall h \in \mathbb{Z}$, $S(\phi, h) \neq D$;
(b) if d_{peg} *is odd and* $|\phi_1| < \phi_{-1}$, *then* $\forall \phi \in (\min \Phi_b, \max \Phi_b)$, $\forall h \in \mathbb{Z}$, $S(\phi, h) \neq D$;
(c) if d_{peg} *is odd and* $|\phi_1| > \phi_{-1}$, *then* $\forall \phi \in (-\phi_{-1}, \phi_{-1})$, $\forall h \in \mathbb{Z}$, $S(\phi, h) \neq D$.

Proof: See Online Appendix A.6. ■

Recall that determinacy obtains if and only if $p = \nu$ (where, as a reminder, p denotes the number of roots of $P(z)$ inside $\mathcal{C}$, and ν the number of non-predetermined variables). The proof of Proposition 6 simply shows that $p - \nu$ is odd, and hence non-zero, inside the ϕ intervals mentioned in the proposition (for all $h \in \mathbb{Z}$). The endpoints of these ϕ intervals are ϕ_1, ϕ_{-1}, or $-\phi_{-1}$ (which all belong to $[-\bar{\phi}, -\underline{\phi}] \cup [\underline{\phi}, \bar{\phi}]$). The reason is the following: for a given h (and hence a given ν), as ϕ varies, the *parity* of $p - \nu$ changes if and only if a *real* root of $P(z)$ crosses the unit circle, that is to say if and only if ϕ goes through a value such that $P(1) = 0$ or $P(-1) = 0$. Now, we have $P(1) = 0$ if and only if $\phi = \phi_1$, and $P(-1) = 0$ only if $\phi \in \{\phi_{-1}, -\phi_{-1}\}$, as can be readily checked.

In the basic NK model under Rule 1 (studied in Section 2), we have $d_{peg} = 1$, $\phi_1 = \underline{\phi}$ and $\phi_{-1} = \bar{\phi}$. So, Point (b) of Proposition 6 applies, and we get no determinacy for all $(\phi, h) \in (-\bar{\phi}, \underline{\phi}) \times \mathbb{Z}$, and hence in particular for all $(\phi, h) \in (-\bar{\phi}, -\underline{\phi}) \times \mathbb{Z}$, which simply amounts to Proposition 2. Similarly, Points (a) and (b) of Proposition 6 also imply no determinacy for all $(\phi, h) \in (-\bar{\phi}, -\underline{\phi}) \times \mathbb{Z}$ in Models 2, 3 and 5 under Rule 1, as the first two lines of Table 2 explain.[24]

Proposition 6 has some implications for Models 1-5 under Rule 1 with a *negative* coefficient $\phi \in (-\bar{\phi}, -\underline{\phi})$, but not with a *positive* coefficient $\phi \in (\underline{\phi}, \bar{\phi})$. In general, however, it can have implications for monetary-policy models under rules that make the interest rate react to inflation and/or output with a *positive* coefficient $\phi \in (\underline{\phi}, \bar{\phi})$. To illustrate this point in a simple way, I consider a rule that makes the interest rate react only to

[24]Proposition 6 does not apply to Model 4 under Rule 1, because it does not cover the case in which d_{peg} is even and $|\phi_1| > \phi_{-1}$. In this case, there may be no value of $\phi \in (-\bar{\phi}, -\underline{\phi}) \cup (\underline{\phi}, \bar{\phi})$ such that $S(\phi, h) \neq D$ for all $h \in \mathbb{Z}$, as Figures 5c (in the main text) and A.2c (in Online Appendix A.9) illustrate.

Table 2: Some illustrations of Proposition 6

Model(s)	d_{peg}	Rule	ϕ_1	ϕ_{-1}	Point of Prop. 6	ϕ interval with no D	Figure(s)
1, 2, 5	odd	1	$= \underline{\phi}$	$= \bar{\phi}$	b	$(-\bar{\phi}, \underline{\phi})$	2, A.2
3	even	1	$= -\underline{\phi}$	$= \bar{\phi}$	a	$(-\bar{\phi}, -\underline{\phi})$	A.2
1	odd	(9)	$= \bar{\phi}$	$\in (\underline{\phi}, \bar{\phi})$	c	$(-\phi_{-1}, \phi_{-1})$	6a, A.3
2	odd	(9)	$\in (\underline{\phi}, \bar{\phi})$	$= \bar{\phi}$	b	$(-\bar{\phi}, \phi_1)$	6b, A.3

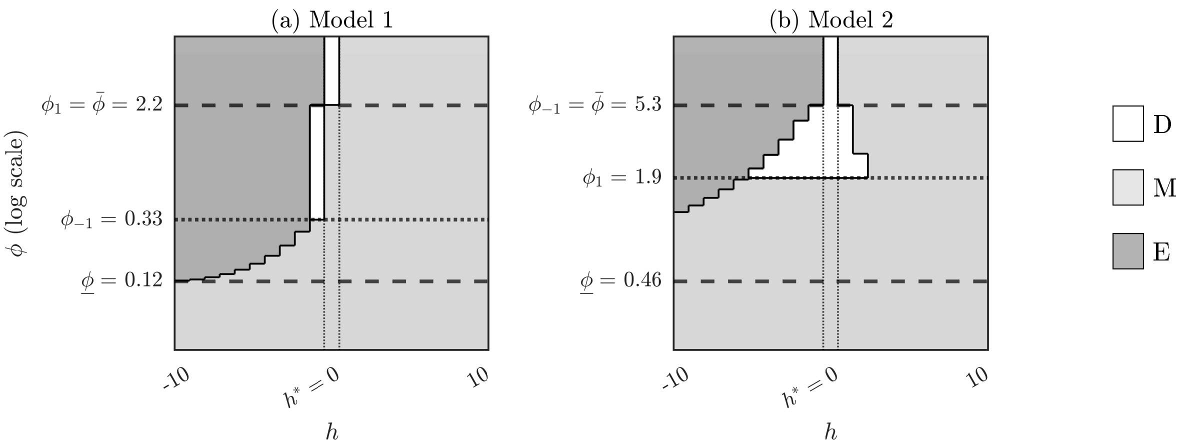
Note: Figures A.2 and A.3 can be found in Online Appendix A.9.

output:

$$i_t = \phi \mathbb{E}_t \{y_{t+h}\}, \tag{9}$$

where $(\phi, h) \in \mathbb{R} \times \mathbb{Z}$. Figure 6 shows the determinacy status $S(\phi, h)$ for Models 1-2 and this rule with $\phi > 0$.[25] In both Figures 6a and 6b, there is no determinacy for any $h \in \mathbb{Z}$ for a finite range of ϕ values just above $\underline{\phi}$. This result is a consequence of Proposition 6, as the last two lines of Table 2 explain.

Figure 6: Determinacy status for Models 1-2 and Rule (9) with $\phi > 0$



In such a case (of no determinacy for any $h \in \mathbb{Z}$ for ϕ just above $\underline{\phi}$), Proposition 5 and the associated Figure 4 straightforwardly imply that the determinacy region is both left-bounded and right-bounded in the pseudo half-plane $(h, \phi) \in \mathbb{Z} \times \mathbb{R}_+$. Therefore, both sufficiently backward-looking stabilization policies and sufficiently forward-looking ones necessarily fail to deliver determinacy, no matter how strongly or weakly the policy instrument reacts to the state of the economy. This result is illustrated in a particularly stark way in Figure 6a, where determinacy obtains only for the horizons -1 and 0.[26]

[25] I relegate the results for $\phi < 0$ to Online Appendix A.9.

[26] For Models 1-2 under Rule (9), Proposition 6 implies no determinacy for any $h \in \mathbb{Z}$ not only for ϕ just above $\underline{\phi}$, but also for ϕ just below $-\underline{\phi}$, as clear from Table 2. So, Proposition 5 and the associated Figure 4 imply that the determinacy region is left- and right-bounded in the *whole* pseudo plane $(h, \phi) \in \mathbb{Z} \times \mathbb{R}$ (not only in the pseudo half-plane $(h, \phi) \in \mathbb{Z} \times \mathbb{R}_+$).

3.6 Taylor principle

I now study the validity of the Taylor principle as a condition for determinacy. I start by providing a formal, general definition of Woodford's (2001, 2003) long-run Taylor principle. Woodford (2001, 2003) mostly discusses this principle in the specific context of the basic NK model under several alternative parametric families of interest-rate rules. His discussion suggests that this principle should be thought of, in a broader context, as an inequality condition obtained by replacing, in the equality condition for the dynamic system to have an eigenvalue equal to 1, the equality sign with an inequality sign.[27]

The system (5)-(6) has an eigenvalue equal to 1 if and only if $P(1) = 0$, that is to say if and only if $\phi = \phi_1$. In all the examples considered by Woodford (2001, 2003), ϕ_1 is positive and the Taylor principle is $\phi > \phi_1$, not $\phi < \phi_1$ (i.e., the policy instrument should react to the variable sufficiently strongly, not sufficiently weakly). So, I propose the following definition of the Taylor principle:

Definition 1 (Taylor principle): *If $\phi_1 > 0$, then the Taylor principle is $\phi > \phi_1$.*

This definition is a generalization of the definition considered in Section 2. The latter definition was tailored to the specific context of the basic NK model; it defined the Taylor principle as a more-than-one-for-one permanent reaction of the interest rate to a permanent change in inflation. Definition 1, more generally, defines the Taylor principle as higher-than-ϕ_1 coefficient ϕ in the policy-instrument rule, where ϕ_1 is the only value of ϕ that makes the dynamic system have an eigenvalue equal to 1.[28]

Proposition 5 and the associated Figure 4 have a straightforward implication for the Taylor principle, independently of the value of ϕ_1: for $h \in \mathbb{Z} \setminus \{h^*\}$, the Taylor principle is *not sufficient* for determinacy. The following proposition focuses instead on results that are conditional on the value of ϕ_1; in this proposition, I use the initialism "TP"

[27] In the words of Woodford (2003, Chapter 4, Page 256, Footnote 27): "*One observes quite generally – in the case of* any *family of policy rules that involve feedback only from inflation and output, regardless of how many lags of these might be involved – that the boundary between sets of coefficients that satisfy the Taylor principle and those that do not will consist of coefficients for which there is an eigenvalue exactly equal to* 1*. (...) It follows that a real eigenvalue crosses the unit circle as the sign of the inequality corresponding to the Taylor principle changes. This boundary is therefore one at which the number of unstable eigenvalues increases by one. Often this results in moving from a situation of indeterminacy to determinacy, though I do not seek to establish general conditions for this.*"

[28] Recent examples of Definition 1's Taylor principle in the literature, outside the context of the basic NK model, include the "income-risk augmented Taylor principle" of Acharya and Dogra (2020), the "HANK Taylor principle" of Bilbiie (2024), and the "modified Taylor principle" of Gabaix (2020) (as long as $\phi_1 > 0$).

for "Taylor principle", and I use the expression "the Taylor principle is locally sufficient for determinacy" in the same sense as in Subsection 2.5 (i.e. in the sense $\exists\varepsilon > 0$, $\forall\phi \in (\phi_1, \phi_1 + \varepsilon)$, $S(\phi, h) = D$):[29]

Proposition 7 (Taylor principle in the general framework): *Consider a model of type (5) and a rule of type (6) with $\phi \geq 0$. Then:*

(a) if $\phi_1 = \bar{\phi}$, then: (i) $\forall h \leq h^* - 1$, the TP is sufficient for E, (ii) for $h = h^*$, the TP is sufficient for D, (iii) $\forall h \geq h^* + 1$, the TP is sufficient for M;

(b) if $\phi_1 \in (\underline{\phi}, \bar{\phi})$, then: (i) the TP is locally sufficient for D for finitely many or no h's, (ii) if $\phi_1 < \phi_{-1}$ and d_{peg} is even, then $\forall h \in \mathbb{Z}$, the TP is not locally sufficient for D, (iii) if $\phi_1 < \phi_{-1}$ and d_{peg} is odd, then $\forall h \in \mathbb{Z}$, the TP is necessary for D;

(c) if $\phi_1 = \underline{\phi}$, then: the TP is necessary and locally sufficient for D if and only if ($d_{peg} = 1$ and $h < h_1$) or ($d_{peg} = -1$ and $h > h_1$), where $h_1 := m + R'(1)/R(1) - Q'(1)/Q(1)$.

Proof: See Online Appendix A.7. ■

Points (a) and (b)(i) of this proposition straightforwardly follow from Proposition 5 and Figure 4, while Points (b)(ii) and (b)(iii) straightforwardly follow from Proposition 6. Point (c) is a generalization of Proposition 3 and can be understood as follows. When $\phi_1 = \underline{\phi}$, the degree of indeterminacy $\nu - p$ is equal to d_{peg} for any $\phi \in (0, \phi_1)$ and any $h \in \mathbb{Z}$. As ϕ goes from below ϕ_1 to above ϕ_1, one root of $P(z)$ crosses $\mathcal{C}$ (at point 1), so the degree of indeterminacy increases or decreases by one (and thus the determinacy status cannot become D if $|d_{peg}| \neq 1$). When $h < h_1$, the root goes from outside to inside $\mathcal{C}$, so the determinacy status changes from M to D (resp. remains E) if $d_{peg} = 1$ (resp. $d_{peg} = -1$), reflecting the fact that increasing the weight on outcomes sufficiently distant in the past favors exploding paths. Alternatively, when $h > h_1$, the root goes from inside to outside $\mathcal{C}$, so the determinacy status remains M (resp. changes from E to D) if $d_{peg} = 1$ (resp. $d_{peg} = -1$), reflecting the fact that increasing the weight on outcomes sufficiently distant in the future favors imploding paths.

Proposition 7 emphasizes that the validity of the Taylor principle as a condition for determinacy across different horizons depends crucially on whether (a) $\phi_1 = \bar{\phi}$, or (b) $\phi_1 \in (\underline{\phi}, \bar{\phi})$, or (c) $\phi_1 = \underline{\phi}$. The Taylor principle can be necessary and locally sufficient for determinacy in any of the three Cases (a), (b) and (c), but only for a *single* horizon in Case (a) (the horizon $h = h^*$, as illustrated in Figure 6a), only for *finitely many* horizons

[29] Point (c) of the proposition rests on the following additional regularity assumption: if $1 \in \operatorname{argmin}_{z\in\mathcal{C}} |Q(z)/R(z)|$, then $\operatorname{argmin}_{z\in\mathcal{C}} |Q(z)/R(z)| = \{1\}$.

in Case (b) (as illustrated in Figure 6b), and only for an *infinity* of horizons in Case (c) (the horizons $h < h_1$ or $h > h_1$, as illustrated in Figures 2a, 5a and 5d).

In Case (a), the Taylor principle $\phi > \phi_1 = \bar{\phi}$ *makes the rule dominate* the structural equations in the system's dynamics; as a result, the degree of indeterminacy $\nu - p$ increases one-for-one with the horizon h in the rule, and determinacy obtains only for a single horizon (h^*). In Case (c), by contrast, the Taylor principle $\phi > \phi_1 = \underline{\phi}$ *prevents the structural equations from dominating* the rule in the system's dynamics; locally, for ϕ just above ϕ_1, the structural equations are close to dominating the rule, so the degree of indeterminacy $\nu - p$ and the determinacy status do not change with the horizon h in the rule, except only once, when h crosses the threshold h_1.

The distinction $\phi_1 = \underline{\phi}$ vs. $\phi_1 = \bar{\phi}$ also sheds light on some contrasting results in the monetary-policy literature about the Taylor principle as a guide for determinacy. In the basic NK model under Rule 1, we have $\phi_1 = 1$, and for $h = 1$ the Taylor principle $\phi > \phi_1$ is necessary and locally sufficient for determinacy (as illustrated in Figure 2a). In Bilbiie's (2008) model under Rule 1, we also have $\phi_1 = 1$, but for $h = 1$ the necessary and sufficient condition for determinacy is the "*inverted Taylor principle*" $\phi < \phi_1$ (as highlighted by Bilbiie, 2008, and as illustrated in Figure 5c). Key to understand these contrasting results is the fact that $\phi_1 = \underline{\phi}$ in the former setup, while $\phi_1 = \bar{\phi}$ in the latter setup (and $h = 1 > h^*$ in both setups). Thus, in the former setup, the Taylor principle avoids multiplicity by preventing the structural equations from dominating the rule; in the latter setup, the inverted Taylor principle avoids multiplicity by preventing the rule from dominating the structural equations.

Similarly, Benhabib et al. (2001) show that in the standard flexible-price money-in-the-utility-function model, under Rule 1 with $h = 1$, the Taylor principle (resp. the inverted Taylor principle) is locally necessary and sufficient for determinacy if consumption and real money balances are complements (resp. substitutes). What distinguishes these two cases (complements and substitutes), and can explain their contrasting implications for the Taylor principle, is not the value of ϕ_1 (equal to 1 in both cases), but rather the fact that $\phi_1 = \bar{\phi}$ in one case, while $\phi_1 = \underline{\phi}$ in the other.[30]

[30] Other illustrations of the inverted Taylor principle $\phi < \phi_1$ being necessary and locally sufficient for determinacy (with $\phi_1 = \bar{\phi}$) can be found in Figures 5c and 6a for $h = -1$.

3.7 Rule inertia

I now turn to inertial rules, i.e. rules making the policy instrument react not only to a variable v_t of type (7) at horizon h with coefficient ϕ, but also to its own past values:

$$\rho(L)i_t = \rho(1)\phi\mathbb{E}_t\{v_{t+h}\}, \tag{10}$$

where $\phi \in \mathbb{R} \setminus \{0\}$, $h \in \mathbb{Z}$, and $\rho(z) \in \mathbb{R}[z]$ with $\rho(0) \neq 0$. In this paper, I focus on the "regular case" in which $\rho(z)$ has no roots *exactly* on $\mathcal{C}$ (i.e. no roots of modulus *exactly* equal to 1). I study non-regular cases in the (longer) working-paper version (Loisel, 2022).

Three remarks are in order about Rule (10). First, I allow $\rho(z)$ to have some roots *inside* $\mathcal{C}$, in which case the rule is "*superinertial*" in the sense of Giannoni and Woodford (2002) and Woodford (2003, Chapter 8); I denote by $d_{sup} \in \mathbb{N}$ the *degree of superinertia* of the rule, i.e. the number of roots of $\rho(z)$ inside $\mathcal{C}$ (counting multiplicity). Second, unlike previously, I now rule out the specific case $\phi = 0$, as Blanchard and Kahn's (1980) no-decoupling condition (discussed in Footnote 22) is violated in this case when $d_{sup} \geq 1$. Third, following standard practice, I have multiplied the coefficient ϕ by the scale factor $\rho(1)$ in the rule, in order to leave unchanged the threshold value ϕ_1 of the long-run Taylor principle.

Let $\tilde{S}$ denote the function $\mathbb{Z} \to \{M, D, E\}$ such that $\tilde{S}(d) = M$ if $d \geq 1$, $\tilde{S}(d) = D$ if $d = 0$, and $\tilde{S}(d) = E$ if $d \leq -1$. I obtain the following proposition, which generalizes Proposition 4 along several dimensions:

Proposition 8 (Rule inertia in the general framework): *Propositions 5-7 still hold for Rule (10) instead of Rule (6), if in these propositions $Q(z)$ is replaced by $Q(z)\rho(z)/\rho(1)$, d_{peg} by $d_{peg} - d_{sup}$, and S_{peg} by $\tilde{S}(d_{peg} - d_{sup})$. These replacements leave ϕ_1 and h^* unchanged, multiply ϕ_{-1} by $|\rho(-1)/\rho(1)|$, and add $-\rho'(1)/\rho(1)$ to h_1.*

Proof: See Online Appendix A.8. ■

Proposition 8 states that if Rule (10) is *not superinertial* ($d_{sup} = 0$), then replacing the non-inertial Rule (6) with the inertial Rule (10) simply amounts to replacing $Q(z)$ with $Q(z)\rho(z)/\rho(1)$ in the expression (8) of $P(z)$ (just like in Proposition 4, in which we had $\rho(z) = 1 - \rho z$ and hence $\rho(z)/\rho(1) = (1 - \rho z)/(1 - \rho)$). The intuition is the same as for Proposition 4: in essence, introducing some inertia into the reaction of the policy instrument to the state of the economy (i.e. replacing i_t by $\rho(L)i_t/\rho(1)$ in the rule), leaving unchanged the reaction of the economy to the policy instrument (i.e. leaving the structural equations unchanged), is equivalent, as far as the determinacy status is concerned, to introducing the same inertia into the reaction of the economy to the policy instrument (i.e. replacing $\mathbf{A}(L)\mathbf{X}_t$ by $\mathbf{A}(L)\rho(L)\mathbf{X}_t/\rho(1)$ in the structural equations, and hence replacing $Q(z)$ by $Q(z)\rho(z)/\rho(1)$), leaving unchanged the reaction of the policy instrument to the state of the economy (i.e. leaving the rule unchanged).

In turn, the replacement of $Q(z)$ by $Q(z)\rho(z)/\rho(1)$ affects the previous results *quantitatively*, notably by adding $-\rho'(1)/\rho(1)$ to h_1 (just like in Proposition 4, in which we had $\rho(z) = 1 - \rho z$ and hence $-\rho'(1)/\rho(1) = \rho/(1-\rho)$). However, because S_{peg} is not modified in Proposition 5 (since $d_{sup} = 0$), the results do not change *qualitatively*, in the sense that if the determinacy status was characterized by Figure 4a (resp. 4b, 4c) under Rule (6), then it is still characterized by Figure 4a (resp. 4b, 4c) under Rule (10).

Alternatively, if Rule (10) is *superinertial* ($d_{sup} \geq 1$), then replacing Rule (6) with Rule (10) amounts to replacing not only $Q(z)$ with $Q(z)\rho(z)/\rho(1)$, but also d_{peg} with $d_{peg} - d_{sup}$ and S_{peg} with $\tilde{S}(d_{peg} - d_{sup})$. In this case, the results change *qualitatively* in the following sense: if $S_{peg} = M$, then S_{peg} may be replaced by D or E in Proposition 5, and we may accordingly move from Figure 4a to Figure 4b or 4c. If $S_{peg} = D$, then S_{peg} is replaced by E in Proposition 5, and we move accordingly from Figure 4b to Figure 4c. These changes simply reflect the fact that superinertia generates exploding paths for sufficiently small $|\phi|$'s.

Because superinertia generates exploding paths for sufficiently small $|\phi|$'s, it can be used to offset multiplicity and deliver determinacy. More specifically, for models with $S_{peg} = M$, replacing a non-inertial rule (6) with a superinertial rule (10) whose degree of superinertia is equal to the degree of indeterminacy under a peg (i.e. $d_{sup} = d_{peg}$) moves us from Figure 4a to Figure 4b. So, for sufficiently small (but non-zero) $|\phi|$'s, the superinertial rule ensures determinacy for any horizon $h \in \mathbb{Z}$. This result echoes, and sheds light on, a result obtained by Giannoni and Woodford (2002, 2003, 2005) and Woodford (2003, Chapter 8) about the degree of superinertia of their "robustly optimal rules," which they find is equal to the degree of indeterminacy under a peg (in models with $S_{peg} = M$).

Alternatively, for models with $S_{peg} \in \{D, E\}$, replacing a non-inertial rule (6) with a superinertial rule (10) moves us from Figure 4b to Figure 4c or keeps us in Figure 4c. So, for sufficiently small (but non-zero) $|\phi|$'s, the superinertial rule leads to explosiveness for any horizon $h \in \mathbb{Z}$. This result offers an explanation for the propensity of superinertial rules to generate explosiveness in backward-looking models, i.e. models with $\delta = 0$ and hence with $S_{peg} \in \{D, E\}$ (Rudebusch and Svensson, 1999, and Levin and Williams, 2003).

3.8 Rules with several variables

The classes of rules of type (6) or (10), which I have considered so far, involve a single variable v_t (even though v_t can itself be defined as a linear combination of several variables). In the monetary-policy literature, however, interest-rate rules often involve several variables, like inflation and output. It is easy to extend the previous results to rules involving several variables. To see how, consider the extended class of rules of type

$$\rho(L)i_t = \rho(1)\left(\phi\mathbb{E}_t\left\{v_{t+h}\right\} + \sum\nolimits_{j=1}^{J}\phi_j\mathbb{E}_t\left\{v_{j,t+h_j}\right\}\right), \tag{11}$$

where $\rho(z) \in \mathbb{R}[z]$, $\rho(0) \neq 0$, $J \in \mathbb{N} \setminus \{0\}$, $(\phi, \phi_1, ..., \phi_J) \in \mathbb{R}^{J+1}$, $(h, h_1, ..., h_J) \in \mathbb{Z}^{J+1}$, and v_t,$v_{1,t}$,...,$v_{J,t}$ are variables of type (7). To analyze how the determinacy status for Model (5) and Rule (11) depends on (ϕ, h), for some given $\rho(z)$ and $(\phi_j, h_j)_{1\leq j\leq J}$, I can simply rewrite Rule (11) as

$$\begin{aligned}\rho(L)i_t &= \rho(1)\left(\tilde{i}_t + \sum\nolimits_{j=1}^{J}\phi_j\mathbb{E}_t\left\{v_{j,t+h_j}\right\}\right), &(12)\\ \tilde{i}_t &= \phi\mathbb{E}_t\left\{v_{t+h}\right\}, &(13)\end{aligned}$$

and then proceed as if (12) were an additional structural equation, i_t an additional variable set by the private sector, $\tilde{i}_t$ the policy instrument, and (13) the rule. So, I can reformulate the problem in a way that involves a modified model, composed of (5) and (12), and a modified rule, (13), which is of type (6); and I can then apply Propositions 5-7 to this modified model and that modified rule.

3.9 Discussion

What broad, qualitative principles for stabilization policy have we learned from this section? To achieve determinacy, the policy instrument should not react too weakly to the state of the economy (except if the model delivers determinacy under a peg), nor too strongly (except at the horizon h^*). It should not react to the state of the economy too far away in the future nor too far away in the past (except possibly with a coefficient below or just above $\underline{\phi}$). And depending on the value of ϕ_1, the Taylor principle $\phi > \phi_1$ may be a locally good guide for determinacy only for a single horizon (h^*), or for a finite number of horizons, or for an infinite number of horizons (all horizons up to h_1 or all horizons from h_1, where h_1 crucially depends on the inertia coefficients).

How *quantitatively* relevant are these principles? What are the values of the coefficient and horizon thresholds $\underline{\phi}$, $\bar{\phi}$, ϕ_1, ϕ_{-1}, h^* and h_1 in *quantitative* models under standard policy-instrument rules? The next section makes a first step in addressing these questions.

4 Quantitative application to monetary policy

In this brief section, I apply my general results to 134 quantitative monetary-policy models, under rules making the interest rate react to inflation alone, or to both inflation and output, with or without inertia. The 134 models are the 140 rational-expectations models of the Macroeconomic Model Data Base (MMB) described in Wieland et al. (2012, 2016), minus 6 models that do not satisfy at least one regularity condition under each of the rules considered.[31]

4.1 MMB models

The current MMB version (3.1) contains 140 rational-expectations models, which are all discrete-time, infinite-horizon models of type (5). These models differ in various dimensions: micro-founded or not, medium- or large-scale, closed- or open-economy, with nominal and real rigidities of this or that nature, with or without financial frictions, with a representative agent or heterogenous agents, calibrated or estimated, using US data or euro-area data or data from other countries, used or not used for policymaking (e.g. at the Federal Reserve Board, the European Central Bank or the International Monetary Fund). In all these models, the policy instrument is the short-term nominal interest rate, and the period is one quarter. Table 3 reports the distribution of d_{peg} across these models: most of them are such that $d_{peg} = 1$ (and hence $S_{peg} = M$).

Table 3: Distribution of d_{peg} across MMB models *(140 models)*

Value of d_{peg}	-1	0	1
Number of models	6	4	130

[31] Online Appendix A.10 lists the model-and-rule combinations that do not satisfy at least one regularity condition, and specifies which regularity condition they do not satisfy.

4.2 Rules 1 and 2

Under Rule 1, there are 10 MMB models that do not satisfy at least one regularity condition; so, I focus on the remaining 130 models. Figure 7 reports the distribution of the coefficient and horizon thresholds across these models under Rule 1, as well as the distribution of H_{TP} types (where H_{TP} denotes the set of horizons $h \in \mathbb{Z}$ such that the Taylor principle is necessary and locally sufficient for determinacy).[32]

As Figures 7a, 7d and 7f show, the values of $\underline{\phi}$, ϕ_1, h^* and $\lfloor h_1 \rfloor$ are typically of the same order of magnitude as standard values of ϕ and h in the literature. More specifically, these values are, for a majority of MMB models, equal or quantitatively close to their values in the calibrated basic NK model (Model 1), i.e. the values $\underline{\phi} = \phi_1 = 1$, $h^* = 0$ and $\lfloor h_1 \rfloor = 1$.[33] Similarly, H_{TP} is of type $\{h | h < h_1\}$ for a majority of MMB models (Figure 7e), just like in the basic NK model. In 59 models, the values of $\underline{\phi}$, ϕ_1, h^* and $\lfloor h_1 \rfloor$ are jointly equal to their values in the calibrated basic NK model, and H_{TP} is of type $\{h | h < h_1\}$. So, in these 59 models, the Taylor principle is $\phi > 1$, it is not sufficient for determinacy if $h \neq 0$, and it is necessary and locally sufficient for determinacy if and only if $h \leq 1$, just like in the calibrated basic NK model.

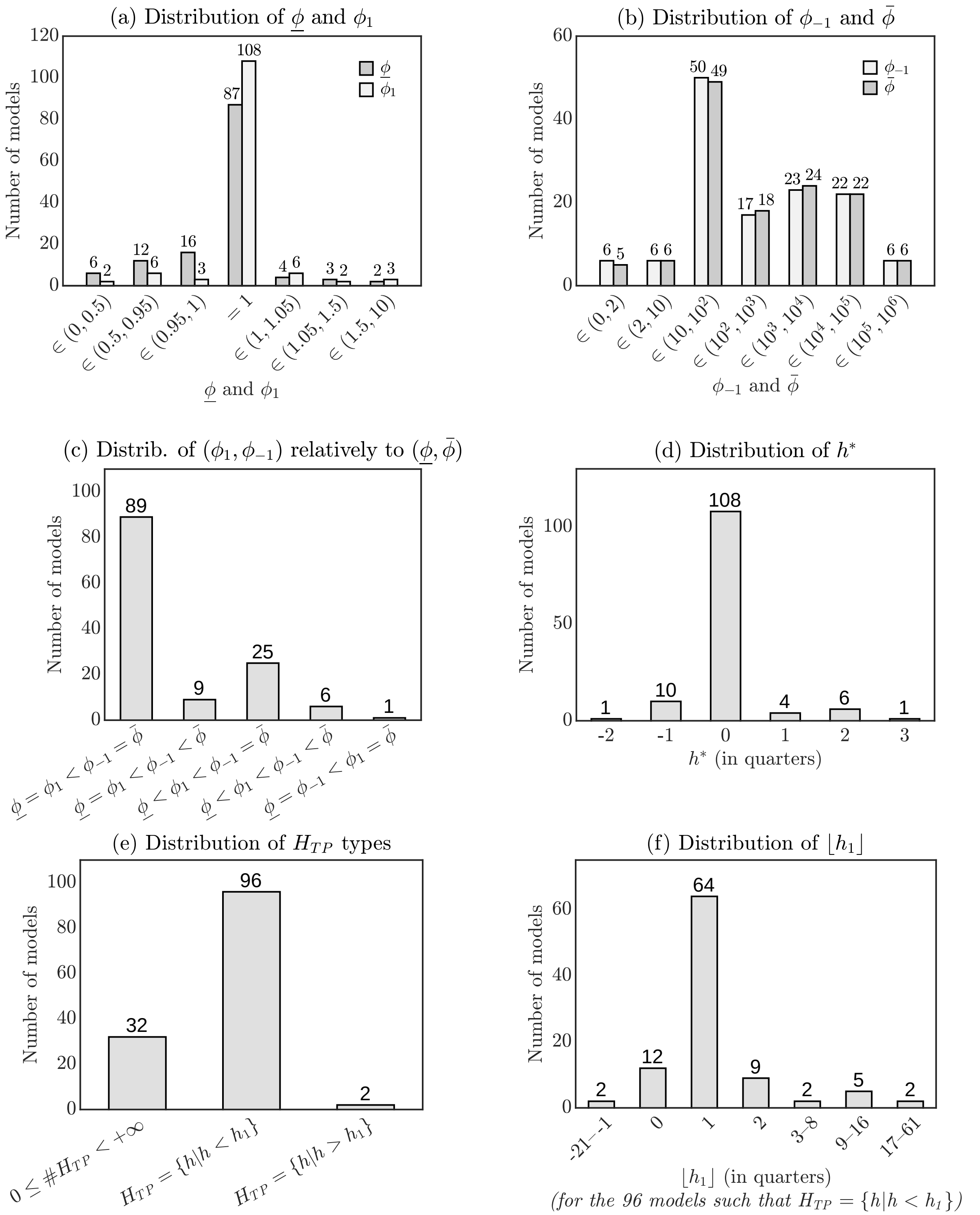
The values of ϕ_{-1} and $\bar{\phi}$, meanwhile, are typically one or several orders of magnitude larger than standard values of ϕ in the literature, as apparent in Figure 7b – although they are of the same order of magnitude in a few models: $\phi_{-1} \in (0, 2)$ in 6 models, and $\bar{\phi} \in (0, 2)$ in 5 models. Figure 7c shows that $(\phi_1, \phi_{-1}) = (\underline{\phi}, \bar{\phi})$ in most models, which implies that Case (b) in Proposition 6 and Case (c) in Proposition 7 are the most common of the three alternative cases considered in each proposition. In 114 models, we have $|d_{peg}| = 1$ and $\underline{\phi} \leq \phi_1 < \phi_{-1} = \bar{\phi}$; so, as Proposition 6 implies, there is no determinacy for any $(\phi, h) \in (-\bar{\phi}, \phi_1) \times \mathbb{Z}$ in these 114 models, just like in the basic NK model.

How do all these distributions change when the non-inertial Rule 1 is replaced by the inertial Rule 2? The answer can be largely deduced from Proposition 8 (and, thus, the new distributions do not need to be shown in figures). More specifically, Proposition 8 straightforwardly implies that for all models, ϕ_1 and h^* are unchanged, ϕ_{-1} is multiplied by $(1 + \rho)/(1 - \rho)$, and h_1 increases by $\rho/(1 - \rho)$. Proposition 8 also implies that $\underline{\phi}$

[32] Online Appendix A.11 describes in detail the method that I use to compute the threshold values.

[33] In practice, the values that I obtain for $\underline{\phi}$ and ϕ_1 may be extremely close to 1 but not exactly equal to 1, even when it can be shown that their true value is 1. In Figure 7a, I use the shortcut "$\underline{\phi} = 1$" (resp. "$\phi_1 = 1$") to denote the case in which the distance from $\underline{\phi}$ (resp. ϕ_1) to 1 is lower than 10^{-7}.

Figure 7: Some distributions across MMB models under Rule 1 *(130 models)*



is unchanged for the 98 models with $\phi_1 = \underline{\phi}$ under Rule 1; that $\bar{\phi}$ is multiplied by $(1+\rho)/(1-\rho)$ for the 114 models with $\phi_{-1} = \bar{\phi}$ under Rule 1; and that H_{TP} remains of type $\{h|h < h_1\}$ for the 96 models with H_{TP} of this type under Rule 1.[34] So, to sum

[34]The reason is that $\text{argmin}_{z\in\mathcal{C}} |(1-\rho z)/(1-\rho)| = \{1\}$ and $\text{argmax}_{z\in\mathcal{C}} |(1-\rho z)/(1-\rho)| = \{-1\}$. So, if $\text{argmin}_{z\in\mathcal{C}} |Q(z)/R(z)| = \{1\}$ (resp. $\text{argmax}_{z\in\mathcal{C}} |Q(z)/R(z)| = \{-1\}$), then $\text{argmin}_{z\in\mathcal{C}} |(1-\rho z)$

up, under Rule 2, $\underline{\phi}$ and ϕ_1 are still predominantly equal or close to 1; $\bar{\phi}$ and ϕ_{-1} are still typically very large; h^* is still 0 and H_{TP} still of type $\{h|h < h_1\}$ for a majority of models; and the main change is, thus, that h_1 is now $\rho/(1-\rho)$ quarters higher than under Rule 1. For $\rho = 0.8$, for instance, h_1 is 4 quarters higher than under Rule 1, and $\lfloor h_1 \rfloor$ is typically equal to 5 quarters, as opposed to 1 quarter under Rule 1.[35]

4.3 Other rules

I now introduce output into Rule 1 in two different ways:

$$i_t = \phi \mathbb{E}_t \left\{ \pi_{t+h} + (1/3) y_{t+h} \right\}, \qquad \text{(Rule 3)}$$

$$i_t = \phi \mathbb{E}_t \left\{ \pi_{t+h} \right\} + (1/2) y_t, \qquad \text{(Rule 4)}$$

where $(\phi, h) \in \mathbb{R} \times \mathbb{Z}$. Unlike Rule 4, Rule 3 makes the interest rate react to output at the same horizon as inflation, and with a proportional coefficient. In the specific case $(\phi, h) = (1.5, 0)$, Rules 3 and 4 coincide with each other and take the familiar form $i_t = 1.5\pi_t + 0.5y_t$.

Overall, the results under Rules 3 and 4 are similar to those under Rule 1 (shown in Figure 7). So, I relegate most of the corresponding figures to Online Appendix A.12. In a nutshell, $\underline{\phi}$ and ϕ_1 are still predominantly close to 1; $\bar{\phi}$ and ϕ_{-1} are still typically very large (although they are low again, between 0 and 2, in a few models); and for a majority of models, h^* is still 0 and H_{TP} still of type $\{h|h < h_1\}$ (Figure 8a).[36] The result that ϕ_1 is approximately unchanged in most models (and exactly unchanged in a few models), as we move from Rule 1 to Rules 3 and 4, simply reflects the fact that the long-run Phillips curve is approximately vertical in most models (and exactly vertical in a few models).

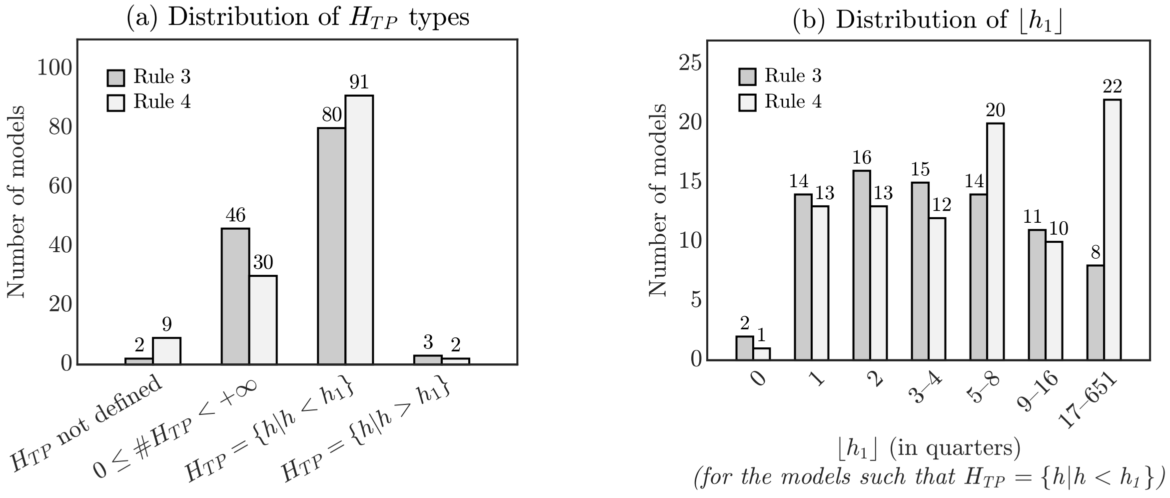
The most notable change is that the distribution of $\lfloor h_1 \rfloor$ has shifted rightwards (for models with $H_{TP} = \{h|h < h_1\}$): the median of $\lfloor h_1 \rfloor$ is now 3.5 quarters under Rule 3 and 6 quarters under Rule 4 (Figure 8b), as opposed to 1 quarter under Rule 1 (Figure 7f). So, the most notable effect of introducing output into Rule 1 is to push forward the maximum horizon for which the Taylor principle is necessary and locally sufficient for determinacy.

$Q(z)/[(1-\rho)R(z)]| = \{1\}$ (resp. $\text{argmax}_{z \in \mathcal{C}} |(1-\rho z)Q(z)/[(1-\rho)R(z)]| = \{-1\}$). Therefore, if $\phi_1 = \underline{\phi}$ (resp. $\phi_{-1} = \bar{\phi}$) under Rule 1, then $\phi_1 = \underline{\phi}$ (resp. $\phi_{-1} = \bar{\phi}$) also under Rule 2.

[35] In Online Appendix A.12, I show the distributions of H_{TP} types and $\lfloor h_1 \rfloor$ under Rule 2, and I discuss these distributions in more detail.

[36] For a few models in Figure 8a, H_{TP} is not defined because the Taylor principle itself is not defined, as $\phi_1 < 0$.

Figure 8: Some distributions across MMB models under Rules 3-4 *(131-132 models)*



Introducing inertia into Rules 3 and 4, i.e. more specifically replacing them with

$$i_t = \rho i_{t-1} + (1-\rho)\,\phi\mathbb{E}_t\left\{\pi_{t+h} + (1/3)y_{t+h}\right\}, \tag{Rule 5}$$

$$i_t = \rho i_{t-1} + (1-\rho)\left[\phi\mathbb{E}_t\left\{\pi_{t+h}\right\} + (1/2)y_t\right], \tag{Rule 6}$$

where $\rho \in (0,1)$ and $(\phi, h) \in \mathbb{R} \times \mathbb{Z}$, pushes $\lfloor h_1 \rfloor$ further forward (notably, but not only, for models with $H_{TP} = \{h|h < h_1\}$). This effect on $\lfloor h_1 \rfloor$ of replacing Rule 3 (resp. Rule 4) with Rule 5 (resp. Rule 6) is identical (resp. similar) to the effect on $\lfloor h_1 \rfloor$ of replacing Rule 1 with Rule 2; so, I relegate the corresponding figures to Online Appendix A.12.

4.4 Discussion

This application to interest-rate rules in MMB models shows that the new principles for stabilization policy can be quantitatively relevant, at least for conventional monetary policy, as the values that I obtain for the coefficient and horizon thresholds are typically (for $\underline{\phi}$, ϕ_1, h^* and h_1) or occasionally (for $\bar{\phi}$ and ϕ_{-1}) of the same order of magnitude as standard values of ϕ and h in the literature. Of course, much remains to be done to assess the quantitative relevance of these new principles more widely: beyond Rules 1-6, there are plenty of other interest-rate rules of type (11) to which the results could be applied, which differ in the variable v_t, the additional variables $v_{j,t}$, and the coefficients ϕ_j and horizons h_j of these additional variables. Moreover, there are other policy instruments than the interest rate, and other models than the MMB models.

The application also provides guidelines for finding a *robust* interest-rate rule, in the sense of an interest-rate rule delivering determinacy across a wide range of alternative monetary-policy models. Using five models (which now belong to the MMB) and a grid of rule-coefficient values, Levin et al. (2003) identified four characteristics of interest-rate rules that deliver determinacy: (i) "a relatively short inflation forecast horizon," (ii) "a moderate degree of responsiveness to the inflation forecast," (iii) "a substantial degree of policy inertia," and (iv) "an explicit response to the current output gap." My application shows that these four characteristics actually favor determinacy in *most* MMB models. In most MMB models, $d_{peg} = 1$ and both $\underline{\phi}$ and ϕ_1 are low to moderate, implying that Characteristics (i) and (ii) favor determinacy (given Propositions 5 and 6, and the associated Figure 4a). In most MMB models, H_{TP} is of type $\{h|h < h_1\}$, where $\lfloor h_1 \rfloor$ is typically equal to only 1 quarter for the non-inertial and output-free Rule 1, but increases (unboundedly) with the inertia coefficient and (significantly) with the presence of output in the rule (especially *current* output, like in Rules 4 and 6), implying that Characteristics (i), (iii) and (iv) favor determinacy.

Importantly, Propositions 5-8, which provide new insights into the forces at work behind determinacy, multiplicity and explosiveness, offer an explanation for *why* Characteristics (i)-(iii) favor determinacy. They also show that these three characteristics, qualitatively speaking, favor determinacy in a broad class of stabilization-policy models (not just MMB monetary-policy models) and for a broad class of variables in the policy-instrument rule (not just inflation in the interest-rate rule) − in essence, in all models of type (5) with $d_{peg} = 1$, and for all variables v_t of type (7).

5 Conclusion

This paper has established some simple, easily interpretable, necessary or sufficient conditions for determinacy for a broad class of policy-instrument rules in a broad class of dynamic rational-expectations models. These determinacy conditions lead to new, general principles for stabilization policy in terms of whether, and how strongly or weakly, to react to any variable, at any horizon, in any model, with any policy instrument. Building on these conditions, the paper has characterized the scope of validity of (a generalized version of) the long-run Taylor principle as a condition for determinacy. All these results can be applied to conventional monetary policy, unconventional monetary policy, fiscal policy, macroprudential policy, or any other stabilization policy. The paper has made a first step in this direction by applying the results to standard interest-rate rules in 134 quantitative monetary-policy models. This application shows that the new principles for stabilization policy are operational and can be quantitatively relevant. Overall, the paper thus opens new horizons for the study of stabilization policies, and paves the way for new qualitative and quantitative research.

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Appendix: Proof of Proposition 1

Point (a). To prove Point (a), I use the theorem of Rouché (1862). I refer the reader to Henrici (1988, Theorem 4.10b, Page 280) or Marden (1966, Page 2) for a general and modern statement of this theorem. Because I will apply it only to polynomials, I only need the following, more restrictive version of the theorem, where the term "Jordan curve" refers to a non-self-intersecting closed curve in the complex plane, and where the subscripts "b" and "s" stand respectively for "big" and "small":[37]

Theorem 1 (Rouché, 1862): *Let $\mathcal{J}$ be a Jordan curve, $P_b(z) \in \mathbb{C}[z]$, and $P_s(z) \in \mathbb{C}[z]$. If $\forall z \in \mathcal{J}$, $|P_b(z)| > |P_s(z)|$, then $P_b(z) + P_s(z)$ and $P_b(z)$ have the same number of roots inside $\mathcal{J}$ (counting multiplicity).*

[37] Throughout the Appendix, $\mathbb{C}[z]$ denotes the set of polynomials in z with complex coefficients.

Proof: See Henrici (1988, Page 280) or Marden (1966, Page 3). ■

I apply Rouché's theorem to $\mathcal{J} = \mathcal{C}$, $P_b(z) = Q(z)z^{\max(0,h-2)}$, and $P_s(z) = \phi z^{\max(0,2-h)}$ (with, thus, $P_b(z) + P_s(z) = P(z)$). For any $|\phi| < \underline{\phi} := \min_{\tilde{z}\in\mathcal{C}} |Q(\tilde{z})|$ and any $z \in \mathcal{C}$, we have

$$\left|Q(z)z^{\max(0,h-2)}\right| = |Q(z)| \geq \min_{\tilde{z}\in\mathcal{C}} |Q(\tilde{z})| = \underline{\phi} > |\phi| = \left|\phi z^{\max(0,2-h)}\right|.$$

So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{C}$ as $Q(z)z^{\max(0,h-2)}$. The latter polynomial has exactly $\max(1, h-1)$ roots inside $\mathcal{C}$, since $Q(z)$ has exactly one root inside $\mathcal{C}$.[38] Therefore, $p = \max(1, h-1) < \max(2, h) = \nu$, and we get $S(\phi, h) = M$ for any $h \in \mathbb{Z}$.

Point (b). To prove Point (b), I switch $P_b(z)$ and $P_s(z)$: i.e., I apply Rouché's theorem to $\mathcal{J} = \mathcal{C}$, $P_b(z) = \phi z^{\max(0,2-h)}$, and $P_s(z) = Q(z)z^{\max(0,h-2)}$. For any $|\phi| > \bar{\phi} := \max_{\tilde{z}\in\mathcal{C}} |Q(\tilde{z})|$ and any $z \in \mathcal{C}$, we have

$$\left|\phi z^{\max(0,2-h)}\right| = |\phi| > \bar{\phi} = \max_{\tilde{z}\in\mathcal{C}} |Q(\tilde{z})| \geq |Q(z)| = \left|Q(z)z^{\max(0,h-2)}\right|.$$

So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{C}$ as $\phi z^{\max(0,2-h)}$. The latter polynomial has exactly $\max(0, 2-h)$ roots inside $\mathcal{C}$; so, $p = \max(0, 2-h)$. Since $\nu = \max(2, h)$, we get: (i) if $h \leq -1$, then $p > \nu$ and $S(\phi, h) = E$; (ii) if $h = 0$, then $p = \nu$ and $S(\phi, h) = D$; and (iii) if $h \geq 1$, then $p < \nu$ and $S(\phi, h) = M$.

Point (c). For $h \geq 2$, we have $\nu = h$ and

$$P(z) = Q(z)z^{h-2} + \phi.$$

Let z_o denote the root of $Q(z)$ in $(1, +\infty)$, with the subscript "o" standing for "outside $\mathcal{C}$." Consider a Jordan curve $\mathcal{J}_o$ surrounding z_o and not intersecting nor surrounding $\mathcal{C}$. I apply Rouché's theorem to $\mathcal{J} = \mathcal{J}_o$, $P_b(z) = Q(z)z^{h-2}$, and $P_s(z) = \phi$. For any $|\phi| \in (\underline{\phi}, \bar{\phi})$, any

$$h \geq \bar{h} := 2 + \max\left\{0, \left\lceil \frac{\log(\bar{\phi}) - \log(\min_{\tilde{z}\in\mathcal{J}_o} |Q(\tilde{z})|)}{\log(\min_{\tilde{z}\in\mathcal{J}_o} |\tilde{z}|)} \right\rceil \right\},$$

and any $z \in \mathcal{J}_o$, we have

$$\left|Q(z)z^{h-2}\right| \geq \min_{\tilde{z}\in\mathcal{J}_o} \left|Q(\tilde{z})\,\tilde{z}^{h-2}\right| \geq \left(\min_{\tilde{z}\in\mathcal{J}_o} |Q(\tilde{z})|\right) \left(\min_{\tilde{z}\in\mathcal{J}_o} |\tilde{z}|\right)^{h-2} \geq \bar{\phi} > |\phi|,$$

[38] Since $Q(0) = \beta\sigma/\kappa > 0$, $Q(1) = -1 < 0$, and $\lim_{z\in\mathbb{R},z\to+\infty} Q(z) = +\infty$, $Q(z)$ has one root in $(0,1)$ and another in $(1, +\infty)$.

where the last but one inequality follows from the definition of $\bar{h}$. So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{J}_o$ as $Q(z)z^{h-2}$. The latter polynomial has exactly one root inside $\mathcal{J}_o$, which is z_o. Therefore, $P(z)$ has also exactly one root inside $\mathcal{J}_o$, and hence at least one root outside $\mathcal{C}$. Since the degree of $P(z)$ is h, we thus get $p \leq h-1 < h = \nu$, and consequently $S(\phi,h) = M$ for any $|\phi| \in (\underline{\phi}, \bar{\phi})$ and any $h \geq \bar{h}$.

Point (d). For $h \leq 2$, we have $\nu = 2$ and

$$P(z) = Q(z) + \phi z^{2-h}.$$

I proceed in four steps. In the first step, I show that for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$, all but one root of $P(z)$ converge uniformly to $\mathcal{C}$ as $h \to -\infty$. I get this result by applying Rouché's theorem twice. Consider an arbitrary $\epsilon \in (0, 1-z_i)$, where z_i denotes the root of $Q(z)$ in $(0,1)$, with the subscript "i" standing for "inside $\mathcal{C}$." For any $r \in \mathbb{R}_+$, let $\mathcal{C}_r$ denote the circle of radius r centered at the origin of the complex plane (so that in particular $\mathcal{C}_1 = \mathcal{C}$). I first apply Rouché's theorem to $\mathcal{J} = \mathcal{C}_{1-\epsilon}$, $P_b(z) = Q(z)$, and $P_s(z) = \phi z^{2-h}$. For any $|\phi| \in (\underline{\phi}, \bar{\phi})$, any

$$h \leq \underline{h}_{1-\epsilon} := 2 + \min\left\{0, \left\lfloor \frac{\log\left(\min_{\tilde{z} \in \mathcal{C}_{1-\epsilon}} |Q(\tilde{z})|\right) - \log\left(\bar{\phi}\right)}{-\log(1-\epsilon)} \right\rfloor\right\},$$

and any $z \in \mathcal{C}_{1-\epsilon}$, we have

$$|Q(z)| \geq \min_{\tilde{z} \in \mathcal{C}_{1-\epsilon}} |Q(\tilde{z})| \geq \bar{\phi}(1-\epsilon)^{2-h} > |\phi|(1-\epsilon)^{2-h} = \left|\phi z^{2-h}\right|,$$

where the second inequality follows from the definition of $\underline{h}_{1-\epsilon}$. So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{C}_{1-\epsilon}$ as $Q(z)$. The latter polynomial has exactly one root inside $\mathcal{C}_{1-\epsilon}$, which is z_i. Therefore, $P(z)$ has also exactly one root inside $\mathcal{C}_{1-\epsilon}$ for any $|\phi| \in (\underline{\phi}, \bar{\phi})$ and any $h \leq \underline{h}_{1-\epsilon}$.

I then apply Rouché's theorem to $\mathcal{J} = \mathcal{C}_{1+\epsilon}$, $P_b(z) = \phi z^{2-h}$, and $P_s(z) = Q(z)$. For any $|\phi| \in (\underline{\phi}, \bar{\phi})$, any

$$h \leq \underline{h}_{1+\epsilon} := 2 + \min\left\{0, \left\lfloor \frac{\log\left(\underline{\phi}\right) - \log\left(\max_{\tilde{z} \in \mathcal{C}_{1+\epsilon}} |Q(\tilde{z})|\right)}{\log(1+\epsilon)} \right\rfloor\right\},$$

and any $z \in \mathcal{C}_{1+\epsilon}$, we have

$$\left|\phi z^{2-h}\right| = |\phi|(1+\epsilon)^{2-h} > \underline{\phi}(1+\epsilon)^{2-h} \geq \max_{\tilde{z} \in \mathcal{C}_{1+\epsilon}} |Q(\tilde{z})| \geq |Q(z)|,$$

where the last but one inequality follows from the definition of $\underline{h}_{1+\epsilon}$. So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{C}_{1+\epsilon}$ as ϕz^{2-h}. Therefore, $P(z)$ has exactly $2-h$ roots inside $\mathcal{C}_{1+\epsilon}$ for any $h \leq \underline{h}_{1+\epsilon}$. Since the degree of $P(z)$ is $2-h$ when $h \leq 0$, we eventually get that for any $|\phi| \in (\underline{\phi}, \bar{\phi})$ and any $h \leq \min\left(0, \underline{h}_{1-\epsilon}, \underline{h}_{1+\epsilon}\right)$, all but one root of $P(z)$ lie between $\mathcal{C}_{1-\epsilon}$ and $\mathcal{C}_{1+\epsilon}$. We conclude that for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$, all but one root of $P(z)$ converge uniformly to $\mathcal{C}$ as $h \to -\infty$.

In the second step, I show that for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$, the roots of $P(z)$ uniformly converging to $\mathcal{C}$ as $h \to -\infty$ converge in distribution to the uniform distribution on $\mathcal{C}$. This result is a direct consequence of the following theorem (stated but not proved in Marden, 1966, Page 193):

Theorem 2 (Erdős and Turán, 1950): *Let* $\tilde{P}(z) = \sum_{k=0}^{d} \tilde{p}_k z^k \in \mathbb{C}[z]$ *with* $\tilde{p}_0 \tilde{p}_d \neq 0$. *Let* $\varphi_k \in [0, 2\pi)$ *for* $1 \leq k \leq d$ *denote the angular coordinates of the roots of* $\tilde{P}(z)$. *For any* $0 \leq \underline{\alpha} < \bar{\alpha} \leq 2\pi$,

$$\left| \# \left\{ k \in \{1, ..., d\} | \underline{\alpha} \leq \varphi_k < \bar{\alpha} \right\} - \left(\frac{\bar{\alpha} - \underline{\alpha}}{2\pi} \right) d \right| \leq 16 \sqrt{d \log \left(\frac{1}{\sqrt{|\tilde{p}_0 \tilde{p}_d|}} \sum_{k=0}^{d} |\tilde{p}_k| \right)}.$$

Proof: See Erdős and Turán (1950). ■

I apply this theorem to $\tilde{P}(z) = P(z)$. For $\tilde{P}(z) = P(z)$ and $h \leq -1$, we have

$$\frac{1}{\sqrt{|\tilde{p}_0 \tilde{p}_d|}} \sum_{k=0}^{d} |\tilde{p}_k| = \left[1 + |\phi| + 2 (1 + \beta) \frac{\sigma}{\kappa} \right] \sqrt{\frac{\kappa}{\beta \sigma |\phi|}}.$$

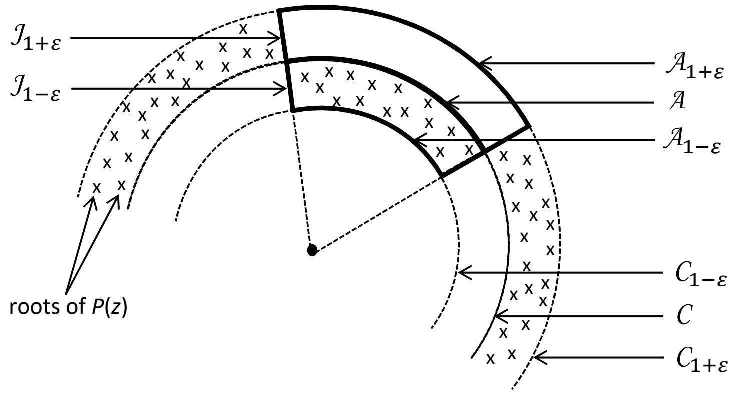
So, the Erdős-Turán theorem, combined with the result of the previous step, straightforwardly implies that all but one root of $P(z)$ uniformly converge in distribution to the uniform distribution on $\mathcal{C}$ as $h \to -\infty$, for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$.

In the third step, I show that the number of roots of $P(z)$ inside $\mathcal{C}$ grows unboundedly as $h \to -\infty$, for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$. Since $|\phi| > \underline{\phi}$, there exists an arc $\mathcal{A}$ of $\mathcal{C}$ such that $\forall z \in \mathcal{A}$, $|\phi| > |Q(z)|$. For any $r \in \mathbb{R}_+$, let $\mathcal{A}_r$ denote the image of $\mathcal{A}$ under the homothety whose center is the origin of the complex plane and whose ratio is r (so that in particular $\mathcal{A}_1 = \mathcal{A}$). By continuity, there exists $\varepsilon \in (0, 1)$ such that $|\phi| > |Q(z)|$ for all z on the Jordan curve $\mathcal{J}_{1+\varepsilon}$ made of $\mathcal{A}$, $\mathcal{A}_{1+\varepsilon}$, and the two radial line segments joining the endpoints of $\mathcal{A}$ and $\mathcal{A}_{1+\varepsilon}$ (see Figure 9). I apply Rouché's theorem to $\mathcal{J} = \mathcal{J}_{1+\varepsilon}$, $P_b(z) = \phi z^{2-h}$, and $P_s(z) = Q(z)$. For any $h \leq 2$ and any $z \in \mathcal{J}_{1+\varepsilon}$, we have

$$\left| \phi z^{2-h} \right| \geq |\phi| > |Q(z)|.$$

So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{J}_{1+\varepsilon}$ as ϕz^{2-h}. Therefore, $P(z)$ has no roots inside $\mathcal{J}_{1+\varepsilon}$ for any $h \leq 2$. Using the results of the first two steps, we get that the number of roots of $P(z)$ inside the Jordan curve $\mathcal{J}_{1-\varepsilon}$ made of $\mathcal{A}_{1-\varepsilon}$, $\mathcal{A}$, and the two radial line segments joining the endpoints of $\mathcal{A}_{1-\varepsilon}$ and $\mathcal{A}$, grows unboundedly as $h \to -\infty$. As a result, p grows unboundedly as $h \to -\infty$. Thus, there exists $\underline{h}(|\phi|)$ such that $p > 2 = \nu$ and $S(\phi, h) = E$ for all $h \leq \underline{h}(|\phi|)$.

Figure 9: Roots of $P(z)$ as $h \to -\infty$



In the fourth step, I just note that for any $\varepsilon \in \left(0, \bar{\phi} - \underline{\phi}\right)$ and any $|\phi| \in (\underline{\phi} + \varepsilon, \bar{\phi})$, there exists, by continuity, $\ell(\varepsilon) > 0$ such that the arc $\mathcal{A}$ can be chosen of length higher than $\ell(\varepsilon)$. As a result, $\underline{h}(|\phi|)$ can be chosen a bounded function of $|\phi|$ for $|\phi| \in (\underline{\phi} + \varepsilon, \bar{\phi})$.

Determination of $\underline{\phi}$ and $\bar{\phi}$. For any $z \in \mathcal{C}$, we have

$$|Q(z)| = \frac{\sigma}{\kappa}\left|\beta - \left(1 + \beta + \frac{\kappa}{\sigma}\right) z + z^2\right| \leq \frac{\sigma}{\kappa}\left[\beta + \left(1 + \beta + \frac{\kappa}{\sigma}\right)|z| + |z|^2\right] = 1 + 2\left(1 + \beta\right)\frac{\sigma}{\kappa},$$

with equality only for $z = -1$. Therefore, $\operatorname{argmax}_{z \in \mathcal{C}} |Q(z)| = \{-1\}$ and

$$\bar{\phi} := \max_{z \in \mathcal{C}} |Q(z)| = |Q(-1)| = 1 + 2\left(1 + \beta\right)\frac{\sigma}{\kappa}.$$

For any $z = a + ib \in \mathcal{C}$, where $(a, b) \in [-1, 1]^2$ and $a^2 + b^2 = 1$, some simple algebra leads to $|Q(z)|^2 = (\sigma/\kappa)^2 T(a)$, where

$$T(a) := 4\beta a^2 - 2\left(1 + \beta\right)\left(1 + \beta + \frac{\kappa}{\sigma}\right)a + \left[\left(1 - \beta\right)^2 + \left(1 + \beta + \frac{\kappa}{\sigma}\right)^2\right].$$

For any $a \in [-1, 1]$, we have $T'(a) \leq T'(1) = -2(1 - \beta)^2 - 2(1 + \beta)\kappa/\sigma < 0$. So, $T(a)$ is strictly decreasing in a over $[-1, 1]$. Therefore, $\operatorname{argmin}_{a \in [-1,1]} T(a) = \{1\}$, $\operatorname{argmin}_{z \in \mathcal{C}} |Q(z)| = \{1\}$, and

$$\underline{\phi} := \min_{z \in \mathcal{C}} |Q(z)| = |Q(1)| = 1.$$

Online Appendix

This Online Appendix contains the proofs of Propositions 2-8 and Lemma 1 (these proofs are not in the Appendix of the paper because they do not bring any substantial new insight, compared to their brief discussion in the main text and/or to the proof of Proposition 1 in the Appendix of the paper). This Online Appendix also contains some figures that are briefly discussed but not shown in the main text (mostly because they are very similar to other figures shown in the main text). Finally, it contains a detailed description of the methodology I use to apply my general results to MMB models.

A.1 Proof of Proposition 2

Consider an arbitrary, fixed $h \in \mathbb{Z}$, and the corresponding fixed ν. Recall that determinacy obtains if and only if $p = \nu$. The proof of Point (a) of Proposition 1 in the Appendix of the paper establishes that $\forall \phi \in (-\underline{\phi}, \underline{\phi})$, $p = \nu - 1$. So, as ϕ decreases from $-\underline{\phi}$ to $-\bar{\phi}$, we need p to change by an *odd* number in order to get determinacy, i.e. we need a *real* root of $P(z)$ to cross the unit circle, which can only happen if ϕ reaches a value such that $P(1) = 0$ or $P(-1) = 0$. However, it is easy to check that $P(1) = 0$ if and only if $\phi = \underline{\phi}$, and that $P(-1) = 0$ only if $|\phi| = \bar{\phi}$. So, $\forall \phi \in (-\bar{\phi}, -\underline{\phi})$, $P(1) \neq 0$ and $P(-1) \neq 0$, and hence $S(\phi, h) \neq D$.

A.2 Proof of Proposition 3

Proposition 1 straightforwardly implies that the Taylor principle is *necessary* for determinacy for any $h \in \mathbb{Z}$. To show that it is *locally sufficient* for determinacy if and only if $h < h_1$, I rewrite $P(z)$ as a function of two variables: $\tilde{P}(\phi, z) := Q(z) z^{\max(0,h-2)} + \phi z^{\max(0,2-h)}$, where $(\phi, z) \in \mathbb{R} \times \mathbb{C}$. Simple algebra leads to $\tilde{P}(1,1) = 0$ and

$$\frac{\partial \tilde{P}}{\partial z}(1,1) = -\left(h - h_1\right),$$

where $h_1 := 2 - Q'(1)/Q(1) = 1 + (1-\beta)\sigma/\kappa$. The expression obtained for $\partial \tilde{P}/\partial z(1,1)$ is generically non-zero (it can be zero only if h_1 is an integer, and I ignore this zero-measure case). So, one root of the polynomial $\tilde{P}(1, z)$ is 1, and this root is of multiplicity one. The implicit-function theorem implies the existence of a continuously differentiable function $\phi \mapsto Z(\phi)$ such that one real root of $P(z)$ can be written as $Z(\phi)$ in the neighborhood of

$\phi = 1$, with $Z(1) = 1$ and

$$Z'(1) = \frac{-\frac{\partial \tilde{P}}{\partial \phi}(1,1)}{\frac{\partial \tilde{P}}{\partial z}(1,1)} = \frac{1}{h - h_1}.$$

This root of $P(z)$ crosses $\mathcal{C}$ at point 1 as ϕ goes through 1. It is the only root that crosses $\mathcal{C}$ as ϕ goes through 1. Indeed, any root $z \in \mathbb{C}$ having this property must satisfy $\tilde{P}(1,z) = 0$, which implies $|Q(z)| = 1$ and hence $z = 1$ (since $\min_{\tilde{z} \in \mathcal{C}} |Q(\tilde{z})| = 1$ and $\mathrm{argmin}_{\tilde{z} \in \mathcal{C}} |Q(\tilde{z})| = \{1\}$, as shown in the Appendix of the paper).

For any $h < h_1$, we have $Z'(1) < 0$, and therefore the root of $P(z)$ goes from outside to inside $\mathcal{C}$ as ϕ goes from below 1 to above 1. So, the number p of roots of $P(z)$ inside $\mathcal{C}$ increases by exactly one as ϕ goes from below 1 to above 1. We know from the Appendix of the paper that this number is $p = \max(1, h-1) = \nu - 1$ for ϕ just below $\underline{\phi} = 1$. Therefore, we have $p = \max(2, h) = \nu$ for ϕ just above 1. As a result, $S(\phi, h)$ changes from M to D as ϕ goes from below 1 to above 1. Thus, the Taylor principle is locally sufficient for determinacy for any $h < h_1$.

Alternatively, for any $h > h_1$, we have $Z'(1) > 0$, and therefore the root of $P(z)$ goes this time from inside to outside $\mathcal{C}$ as ϕ goes from below 1 to above 1. So, p decreases by exactly one as ϕ goes from below 1 to above 1. Again, we know from the Appendix of the paper that $p = \nu - 1$ for ϕ just below $\underline{\phi} = 1$. Therefore, we have $p = \nu - 2$ for ϕ just above 1. As a result, $S(\phi, h)$ remains equal to M as ϕ goes from below 1 to above 1. Thus, for any $h > h_1$, the Taylor principle is not locally sufficient for determinacy.

A.3 Proof of Proposition 4

Under Rule 1 with $\phi \neq 0$, we have $\nu = \max(2, h)$ and $P(z) = Q(z) z^{\max(0,h-2)} + \phi z^{\max(0,2-h)}$. Under Rule 2 with $\phi \neq 0$, the number of non-predetermined variables is still $\nu = \max(2, h)$, since the new terms in the rule are past (as opposed to expected future) values of the interest rate. The characteristic polynomial of the dynamic system is the same as the characteristic polynomial of the corresponding perfect-foresight system

$$\begin{bmatrix} \sigma(1-L) & 1 & -L \\ L & \frac{\beta - L}{\kappa} & 0 \\ 0 & -\phi L^{-h} & \frac{1-\rho L}{1-\rho} \end{bmatrix} \begin{bmatrix} y_t \\ \pi_t \\ i_t \end{bmatrix} = \mathbf{0}.$$

Using a standard result in time-series analysis (see, e.g., Hamilton, 1994, Proposition 10.1, Page 259), I get that there exists $k \in \mathbb{Z}$ such that $P(z)$, the reciprocal polynomial of this characteristic polynomial, is

$$P(z) = z^k \det \begin{bmatrix} \sigma(1-z) & 1 & -z \\ z & \frac{\beta - z}{\kappa} & 0 \\ 0 & -\phi z^{-h} & \frac{1-\rho z}{1-\rho} \end{bmatrix} = z^k \left[Q(z) \frac{1-\rho z}{1-\rho} + \phi z^{2-h} \right].$$

As a reciprocal polynomial, $P(z)$ is such that $P(0) \neq 0$; therefore, we get $k = \max(0, h-2)$, and thus $P(z) = [Q(z)(1-\rho z)/(1-\rho)] z^{\max(0,h-2)} + \phi z^{\max(0,2-h)}$ (except in the zero-measure case in which $(\phi, h) = (-\beta\sigma/\kappa, 2)$). So, Propositions 1-3 still hold for Rule 2 instead of Rule 1, if $Q(z)$ is replaced by $\tilde{Q}(z) := Q(z)(1-\rho z)/(1-\rho)$ in these propositions.

Since $\text{argmin}_{z \in \mathcal{C}} |Q(z)| = \{1\}$ (as shown in the Appendix of the paper) and $\rho \in (0,1)$, we have $\text{argmin}_{z \in \mathcal{C}} |\tilde{Q}(z)| = \{1\}$; therefore, $\underline{\tilde{\phi}} := \min_{z \in \mathcal{C}} |\tilde{Q}(z)| = |\tilde{Q}(1)| = |Q(1)| = \underline{\phi}$. Similarly, since $\text{argmax}_{z \in \mathcal{C}} |Q(z)| = \{-1\}$ (as shown in the Appendix of the paper) and $\rho \in (0,1)$, we have $\text{argmax}_{z \in \mathcal{C}} |\tilde{Q}(z)| = \{-1\}$; therefore, $\bar{\tilde{\phi}} := \max_{z \in \mathcal{C}} |\tilde{Q}(z)| = |\tilde{Q}(-1)| = [(1+\rho)/(1-\rho)]|Q(-1)| = [(1+\rho)/(1-\rho)]\bar{\phi}$. Finally, $\tilde{h}_1 := 2 - \tilde{Q}'(1)/\tilde{Q}(1) = 2 - Q'(1)/Q(1) + \rho/(1-\rho) = h_1 + \rho/(1-\rho)$.

A.4 Proof of Lemma 1

I start with the case of a peg ($\phi = 0$). In this case, the dynamic system boils down to $\mathbb{E}_t\{\mathbf{\Delta}(L^{-1})\mathbf{A}(L)\mathbf{X}_t\} = \mathbf{0}$. The characteristic polynomial of this system is the same as the characteristic polynomial of the corresponding perfect-foresight system. The latter system is $\mathbf{A}(L)\mathbf{X}_t = \mathbf{0}$. Since $\det[\mathbf{A}(0)] \neq 0$, I can use the same standard result in time-series analysis as in Online Appendix A.3, and I get that $P(z)$, the reciprocal polynomial of this characteristic polynomial, is equal to $Q(z) := \det[\mathbf{A}(z)]$.

Since $\det[\mathbf{A}(0)] \neq 0$, the dynamic system can be rewritten as $\mathbb{E}_t\{\mathbf{\Delta}(L^{-1})\tilde{\mathbf{A}}(L)\tilde{\mathbf{X}}_t\} = \mathbf{0}$, where $\tilde{\mathbf{A}}(z) := \mathbf{A}(z)[\mathbf{A}(0)]^{-1}$ and $\tilde{\mathbf{X}}_t := \mathbf{A}(0)\mathbf{X}_t$. Let $\tilde{X}_{j,t}$ denote the j^{th} element of $\tilde{\mathbf{X}}_t$ for $j \in \{1, ..., n\}$. The non-predetermined variables of the system are the variables $\mathbb{E}_t\{\tilde{X}_{j,t+k_j}\}$ for all $j \in \{1, ..., n\}$ such that $\delta_j \geq 1$ and all $k_j \in \{0, ..., \delta_j - 1\}$. Their number, ν, is equal to $\delta := \sum_{j=1}^n \delta_j$.

I now turn to the case in which $\phi \neq 0$. In this case, the characteristic polynomial of the dynamic system is still the same as the characteristic polynomial of the corresponding perfect-foresight system, but the latter system is now

$$\begin{bmatrix} \mathbf{A}(L) & L^{-\gamma}\mathbf{B}(L) \\ -\phi L^{-h}\mathbf{V}(L) & 1 \end{bmatrix} \begin{bmatrix} \mathbf{X}_t \\ i_t \end{bmatrix} = \mathbf{0}.$$

Using the same standard result in time-series analysis as in Online Appendix A.3, I get that there exists $k \in \mathbb{Z}$ such that $P(z)$, the reciprocal polynomial of the characteristic polynomial, is

$$P(z) = z^k \det \begin{bmatrix} \mathbf{A}(z) & z^{-\gamma}\mathbf{B}(z) \\ -\phi z^{-h}\mathbf{V}(z) & 1 \end{bmatrix}.$$

Using the Laplace expansion and the notations introduced in the main text, I rewrite $P(z)$ as $P(z) = z^k\{\det[\mathbf{A}(z)] - \phi z^{-\gamma-h}W(z)\} = z^k[Q(z) + \phi z^{m-h}R(z)]$. As a reciprocal polynomial, $P(z)$ is such that $P(0) \neq 0$; moreover, we have $Q(0) \neq 0$ and $R(0) \neq 0$; as a consequence, we get $k = \max(0, h-m)$, and thus $P(z) = Q(z)z^{\max(0,h-m)} + \phi R(z)z^{\max(0,m-h)}$ (except in the zero-measure case in which $(\phi, h) = (-Q(0)/R(0), m)$).

The number of non-predetermined variables, ν, is equal to δ when h is lower than or equal to a certain threshold, and it increases one-for-one with h when h is higher than this threshold. This threshold is equal to the highest value of h for which $P(0)$ depends on $Q(0)$, i.e. for which the most forward variable in the dynamic system is the same as under a peg (except in the zero-measure case in which $\phi = -Q(0)/R(0)$). This value is m, and thus $\nu = \delta + \max(0, h-m)$.

A.5 Proof of Proposition 5

The proof of Proposition 5 is essentially a generalization of the proof of Proposition 1, using this time $P(z) = Q(z)z^{\max(0,h-m)} + \phi R(z)z^{\max(0,m-h)}$ and $\nu = \delta + \max(0, h-m)$ (as stated in Lemma 1).

Point (a). I apply Rouché's theorem to $\mathcal{J} = \mathcal{C}$, $P_b(z) = Q(z)z^{\max(0,h-m)}$, and $P_s(z) = \phi R(z)z^{\max(0,m-h)}$. For any $|\phi| < \underline{\phi}$ and any $z \in \mathcal{C}$, we have

$$\left|Q(z)z^{\max(0,h-m)}\right| = |Q(z)| \geq \min_{\tilde{z}\in\mathcal{C}} \left|\frac{Q(\tilde{z})}{R(\tilde{z})}\right| |R(z)| = \underline{\phi}\,|R(z)| > |\phi R(z)| = \left|\phi R(z)z^{\max(0,m-h)}\right|.$$

So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{C}$ as $Q(z)z^{\max(0,h-m)}$, i.e. that $p = q + \max(0, h-m)$. Since $\nu = \delta + \max(0, h-m)$, we get $\nu - p = \delta - q = d_{peg}$, and hence $S(\phi, h) = S_{peg}$, for any $|\phi| < \underline{\phi}$ and any $h \in \mathbb{Z}$.

Point (b). I apply Rouché's theorem to $\mathcal{J} = \mathcal{C}$, $P_b(z) = \phi R(z)z^{\max(0,m-h)}$, and $P_s(z) = Q(z)z^{\max(0,h-m)}$. For any $|\phi| > \bar{\phi}$ and any $z \in \mathcal{C}$, we have

$$\left|\phi R(z)z^{\max(0,m-h)}\right| = |\phi R(z)| > \bar{\phi}\,|R(z)| = \max_{\tilde{z}\in\mathcal{C}} \left|\frac{Q(\tilde{z})}{R(\tilde{z})}\right| |R(z)| \geq |Q(z)| = \left|Q(z)z^{\max(0,h-m)}\right|.$$

So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{C}$ as $\phi R(z)z^{\max(0,m-h)}$, i.e. that $p = r + \max(0, m-h)$. Since $\nu = \delta + \max(0, h-m)$, we get, for any $|\phi| > \bar{\phi}$: (i) if $h \leq h^* - 1$, then $p > \nu$ and $S(\phi, h) = E$; (ii) if $h = h^*$, then $p = \nu$ and $S(\phi, h) = D$; and (iii) if $h \geq h^* + 1$, then $p < \nu$ and $S(\phi, h) = M$.

Points (d)(i) and (d)(ii). For $h \leq m$, we have $\nu = \delta$ and $P(z) = Q(z) + \phi R(z) z^{m-h}$. I proceed in four steps.

In the first step, I show that for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$, all but $q + \deg(R) - r$ roots of $P(z)$ converge uniformly to $\mathcal{C}$ as $h \to -\infty$.[1] I get this result by applying Rouché's theorem twice. Since $Q(z)$ and $R(z)$ have no roots *exactly* on $\mathcal{C}$, I can consider an arbitrary $\epsilon \in (0,1)$ such that neither $Q(z)$ nor $R(z)$ has any root inside the annulus whose borders are $\mathcal{C}_{1-\epsilon}$ and $\mathcal{C}_{1+\epsilon}$ (where again, for any $r \in \mathbb{R}_+$, $\mathcal{C}_r$ denotes the circle of radius r centered at the origin of the complex plane).

I first apply Rouché's theorem to $\mathcal{J} = \mathcal{C}_{1-\epsilon}$, $P_b(z) = Q(z)$, and $P_s(z) = \phi R(z) z^{m-h}$. For any $|\phi| \in (\underline{\phi}, \bar{\phi})$, any

$$h \leq \underline{h}_{1-\epsilon} := m + \min\left\{0, \left\lfloor \frac{\log\left(\min_{\tilde{z} \in \mathcal{C}_{1-\epsilon}} |Q(\tilde{z})|\right) - \log\left(\bar{\phi} \max_{\tilde{z} \in \mathcal{C}_{1-\epsilon}} |R(\tilde{z})|\right)}{-\log(1-\epsilon)} \right\rfloor\right\},$$

and any $z \in \mathcal{C}_{1-\epsilon}$, we have

$$|Q(z)| \geq \min_{\tilde{z} \in \mathcal{C}_{1-\epsilon}} |Q(\tilde{z})| \geq \bar{\phi} \max_{\tilde{z} \in \mathcal{C}_{1-\epsilon}} |R(\tilde{z})| (1-\epsilon)^{m-h} \geq \bar{\phi} \left| R(z) z^{m-h} \right| > \left| \phi R(z) z^{m-h} \right|,$$

where the second inequality follows from the definition of $\underline{h}_{1-\epsilon}$. So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{C}_{1-\epsilon}$ as $Q(z)$. Therefore, $P(z)$ has also exactly q roots inside $\mathcal{C}_{1-\epsilon}$ for any $|\phi| \in (\underline{\phi}, \bar{\phi})$ and any $h \leq \underline{h}_{1-\epsilon}$.

I then apply Rouché's theorem to $\mathcal{J} = \mathcal{C}_{1+\epsilon}$, $P_b(z) = \phi R(z) z^{m-h}$, and $P_s(z) = Q(z)$. For any $|\phi| \in (\underline{\phi}, \bar{\phi})$, any

$$h \leq \underline{h}_{1+\epsilon} := m + \min\left\{0, \left\lfloor \frac{\log\left(\underline{\phi} \min_{\tilde{z} \in \mathcal{C}_{1+\epsilon}} |R(\tilde{z})|\right) - \log\left(\max_{\tilde{z} \in \mathcal{C}_{1+\epsilon}} |Q(\tilde{z})|\right)}{\log(1+\epsilon)} \right\rfloor\right\},$$

and any $z \in \mathcal{C}_{1+\epsilon}$, we have

$$\left| \phi R(z) z^{m-h} \right| = |\phi R(z)| (1+\epsilon)^{m-h} > \underline{\phi} \min_{\tilde{z} \in \mathcal{C}_{1+\epsilon}} |R(\tilde{z})| (1+\epsilon)^{m-h} \geq \max_{\tilde{z} \in \mathcal{C}_{1+\epsilon}} |Q(\tilde{z})| \geq |Q(z)|,$$

where the last but one inequality follows from the definition of $\underline{h}_{1+\epsilon}$. So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{C}_{1+\epsilon}$ as $\phi R(z) z^{m-h}$. Therefore, $P(z)$ has exactly $r + m - h$ roots inside $\mathcal{C}_{1+\epsilon}$ for any $h \leq \underline{h}_{1+\epsilon}$. As a consequence, for any $|\phi| \in (\underline{\phi}, \bar{\phi})$ and any $h \leq \min\left(\underline{h}_{1-\epsilon}, \underline{h}_{1+\epsilon}\right)$, $P(z)$ has exactly $r + m - h - q$ roots inside the annulus whose borders are $\mathcal{C}_{1-\epsilon}$ and $\mathcal{C}_{1+\epsilon}$. Now, the degree of $P(z)$ is $\deg(R) + m - h$ when $h \leq m + \deg(R) - \deg(Q)$. So, we eventually get that for any $|\phi| \in (\underline{\phi}, \bar{\phi})$ and any $h \leq \min\left[\underline{h}_{1-\epsilon}, \underline{h}_{1+\epsilon}, m + \deg(R) - \deg(Q)\right]$, all but $q + \deg(R) - r$ roots of $P(z)$ lie

[1] Throughout the Online Appendix, for any $T(z) \in \mathbb{R}[z]$, $\deg(T)$ denotes the degree of $T(z)$.

between $\mathcal{C}_{1-\epsilon}$ and $\mathcal{C}_{1+\epsilon}$. We conclude that for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$, all but $q+\deg(R)-r$ roots of $P(z)$ converge uniformly to $\mathcal{C}$ as $h \to -\infty$.

In the second step, I show that for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$, the roots of $P(z)$ uniformly converging to $\mathcal{C}$ as $h \to -\infty$ converge in distribution to the uniform distribution on $\mathcal{C}$. This result is a direct consequence of the Erdős-Turán theorem (stated in the Appendix of the paper). Applying this theorem to $\tilde{P}(z) = P(z)$, and using the result of the previous step, I thus get that all but $q+\deg(R)-r$ roots of $P(z)$ uniformly converge in distribution to the uniform distribution on $\mathcal{C}$ as $h \to -\infty$, for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$.

In the third step, I show that the number of roots of $P(z)$ inside $\mathcal{C}$ grows unboundedly as $h \to -\infty$, for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$. Since $|\phi| > \underline{\phi}$, there exists an arc $\mathcal{A}$ of $\mathcal{C}$ such that $\forall z \in \mathcal{A}$, $|\phi R(z)| > |Q(z)|$. For any $r \in \mathbb{R}_+$, let $\mathcal{A}_r$ denote the image of $\mathcal{A}$ under the homothety whose center is the origin of the complex plane and whose ratio is r (so that in particular $\mathcal{A}_1 = \mathcal{A}$). By continuity, there exists $\varepsilon \in (0,1)$ such that $|\phi R(z)| > |Q(z)|$ for all z on the Jordan curve $\mathcal{J}_{1+\varepsilon}$ made of $\mathcal{A}$, $\mathcal{A}_{1+\varepsilon}$, and the two radial line segments joining the endpoints of $\mathcal{A}$ and $\mathcal{A}_{1+\varepsilon}$ (see Figure 9 in the Appendix of the paper). I apply Rouché's theorem to $\mathcal{J} = \mathcal{J}_{1+\varepsilon}$, $P_b(z) = \phi R(z) z^{m-h}$, and $P_s(z) = Q(z)$. For any $h \leq m$ and any $z \in \mathcal{J}_{1+\varepsilon}$, we have

$$\left|\phi R(z) z^{m-h}\right| \geq |\phi R(z)| > |Q(z)| \, .$$

So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{J}_{1+\varepsilon}$ as $\phi R(z) z^{m-h}$. Therefore, $P(z)$ has at most $\deg(R)$ roots inside $\mathcal{J}_{1+\varepsilon}$ for any $h \leq m$. (Figure 9 in the Appendix of the paper represents the case in which $P(z)$ has no roots inside $\mathcal{J}_{1+\varepsilon}$; we necessarily get this case if ε is sufficiently small.) Using the results of the first two steps, we get that the number of roots of $P(z)$ inside the Jordan curve $\mathcal{J}_{1-\varepsilon}$ made of $\mathcal{A}_{1-\varepsilon}$, $\mathcal{A}$, and the two radial line segments joining the endpoints of $\mathcal{A}_{1-\varepsilon}$ and $\mathcal{A}$, grows unboundedly as $h \to -\infty$. As a result, p grows unboundedly as $h \to -\infty$. Thus, there exists $\underline{h}\left(|\phi|\right)$ such that $p > \delta = \nu$ and $S(\phi, h) = E$ for all $h \leq \underline{h}\left(|\phi|\right)$.

In the fourth step, I just note that for any $\varepsilon \in \left(0, \bar{\phi} - \underline{\phi}\right)$ and any $|\phi| \in (\underline{\phi} + \varepsilon, \bar{\phi})$, there exists, by continuity, $\ell(\varepsilon) > 0$ such that the arc $\mathcal{A}$ can be chosen of length higher than $\ell(\varepsilon)$. As a result, $\underline{h}\left(|\phi|\right)$ can be chosen a bounded function of $|\phi|$ for $|\phi| \in (\underline{\phi} + \varepsilon, \bar{\phi})$.

Points (c)(i) and (c)(ii). For $h \geq m$, we have $\nu = \delta + h - m$ and $P(z) = Q(z) z^{h-m} + \phi R(z)$. I follow the same four steps as in the proof of Points (d)(i) and (d)(ii) above, with some variants.

In the first step, I show that for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$, all but $r+\deg(Q)-q$ roots of $P(z)$

converge uniformly to $\mathcal{C}$ as $h \to +\infty$. I get this result by applying Rouché's theorem twice. Since $Q(z)$ and $R(z)$ have no roots *exactly* on $\mathcal{C}$, I can consider an arbitrary $\epsilon \in (0,1)$ such that neither $Q(z)$ nor $R(z)$ has any root inside the annulus whose borders are $\mathcal{C}_{1-\epsilon}$ and $\mathcal{C}_{1+\epsilon}$.

I first apply Rouché's theorem to $\mathcal{J} = \mathcal{C}_{1-\epsilon}$, $P_b(z) = \phi R(z)$, and $P_s(z) = Q(z)z^{h-m}$. For any $|\phi| \in (\underline{\phi}, \bar{\phi})$, any

$$h \geq \bar{h}_{1-\epsilon} := m + \max\left\{0, \left\lceil \frac{\log\left(\max_{\tilde{z}\in\mathcal{C}_{1-\epsilon}} |Q(\tilde{z})|\right) - \log\left(\underline{\phi} \min_{\tilde{z}\in\mathcal{C}_{1-\epsilon}} |R(\tilde{z})|\right)}{-\log(1-\epsilon)} \right\rceil\right\},$$

and any $z \in \mathcal{C}_{1-\epsilon}$, we have

$$|\phi R(z)| > \underline{\phi} \min_{\tilde{z}\in\mathcal{C}_{1-\epsilon}} |R(\tilde{z})| \geq \max_{\tilde{z}\in\mathcal{C}_{1-\epsilon}} |Q(\tilde{z})| (1-\epsilon)^{h-m} \geq \left|Q(z)z^{h-m}\right|,$$

where the second inequality follows from the definition of $\bar{h}_{1-\epsilon}$. So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{C}_{1-\epsilon}$ as $\phi R(z)$. Therefore, $P(z)$ has also exactly r roots inside $\mathcal{C}_{1-\epsilon}$ for any $|\phi| \in (\underline{\phi}, \bar{\phi})$ and any $h \geq \bar{h}_{1-\epsilon}$.

I then apply Rouché's theorem to $\mathcal{J} = \mathcal{C}_{1+\epsilon}$, $P_b(z) = Q(z)z^{h-m}$, and $P_s(z) = \phi R(z)$. For any $|\phi| \in (\underline{\phi}, \bar{\phi})$, any

$$h \geq \bar{h}_{1+\epsilon} := m + \max\left\{0, \left\lceil \frac{\log\left(\bar{\phi} \max_{\tilde{z}\in\mathcal{C}_{1+\epsilon}} |R(\tilde{z})|\right) - \log\left(\min_{\tilde{z}\in\mathcal{C}_{1+\epsilon}} |Q(\tilde{z})|\right)}{\log(1+\epsilon)} \right\rceil\right\},$$

and any $z \in \mathcal{C}_{1+\epsilon}$, we have

$$\left|Q(z)z^{h-m}\right| = |Q(z)| (1+\epsilon)^{h-m} \geq \min_{\tilde{z}\in\mathcal{C}_{1+\epsilon}} |Q(\tilde{z})| (1+\epsilon)^{h-m} \geq \bar{\phi} \max_{\tilde{z}\in\mathcal{C}_{1+\epsilon}} |R(\tilde{z})| > |\phi R(z)|,$$

where the last but one inequality follows from the definition of $\bar{h}_{1+\epsilon}$. So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{C}_{1+\epsilon}$ as $Q(z)z^{h-m}$. Therefore, $P(z)$ has exactly $q + h - m$ roots inside $\mathcal{C}_{1+\epsilon}$ for any $h \geq \bar{h}_{1+\epsilon}$. As a consequence, for any $|\phi| \in (\underline{\phi}, \bar{\phi})$ and any $h \geq \max\left(\bar{h}_{1-\epsilon}, \bar{h}_{1+\epsilon}\right)$, $P(z)$ has exactly $q + h - m - r$ roots inside the annulus whose borders are $\mathcal{C}_{1-\epsilon}$ and $\mathcal{C}_{1+\epsilon}$. Now, the degree of $P(z)$ is $\deg(Q) + h - m$ when $h \geq m + \deg(R) - \deg(Q)$. So, we eventually get that for any $|\phi| \in (\underline{\phi}, \bar{\phi})$ and any $h \geq \max\left[\bar{h}_{1-\epsilon}, \bar{h}_{1+\epsilon}, m + \deg(R) - \deg(Q)\right]$, all but $r + \deg(Q) - q$ roots of $P(z)$ lie between $\mathcal{C}_{1-\epsilon}$ and $\mathcal{C}_{1+\epsilon}$. We conclude that for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$, all but $r + \deg(Q) - q$ roots of $P(z)$ converge uniformly to $\mathcal{C}$ as $h \to +\infty$.

In the second step, I show that for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$, the roots of $P(z)$ uniformly converging to $\mathcal{C}$ as $h \to +\infty$ converge in distribution to the uniform distribution on $\mathcal{C}$. This result is, again, a direct consequence of the Erdős-Turán theorem (stated in the

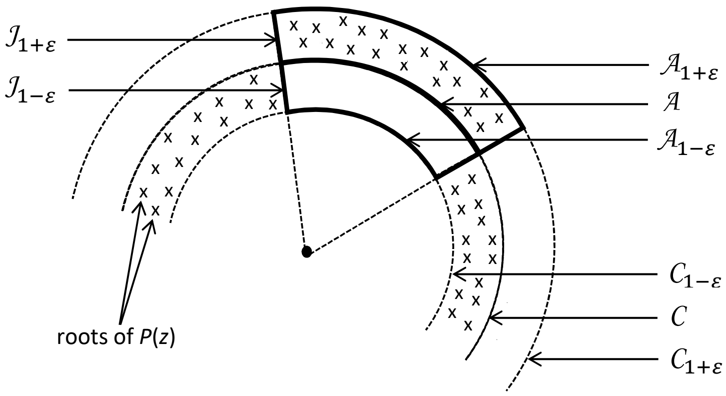
Appendix of the paper). Applying this theorem to $\tilde{P}(z) = P(z)$, and using the result of the previous step, I thus get that all but $r + \deg(Q) - q$ roots of $P(z)$ uniformly converge in distribution to the uniform distribution on $\mathcal{C}$ as $h \to +\infty$, for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$.

In the third step, I show that the ratio p/ν is lower than 1 as $h \to +\infty$, for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$. Since $|\phi| > \underline{\phi}$, there exists an arc $\mathcal{A}$ of $\mathcal{C}$ such that $\forall z \in \mathcal{A}$, $|\phi R(z)| > |Q(z)|$. By continuity, there exists $\varepsilon \in (0,1)$ such that $|\phi R(z)| > |Q(z)|$ for all z on the Jordan curve $\mathcal{J}_{1-\varepsilon}$ (defined above and represented in Figure A.1). I apply Rouché's theorem to $\mathcal{J} = \mathcal{J}_{1-\varepsilon}$, $P_b(z) = \phi R(z)$, and $P_s(z) = Q(z)z^{h-m}$. For any $h \geq m$ and any $z \in \mathcal{J}_{1-\varepsilon}$, we have

$$|\phi R(z)| > |Q(z)| \geq \left|Q(z)z^{h-m}\right|.$$

So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{J}_{1-\varepsilon}$ as $\phi R(z)$. Therefore, $P(z)$ has at most $\deg(R)$ roots inside $\mathcal{J}_{1-\varepsilon}$ for any $h \geq m$. (Figure A.1 represents the case in which $P(z)$ has no roots inside $\mathcal{J}_{1-\varepsilon}$; we necessarily get this case if ε is sufficiently small.)

Figure A.1: Roots of $P(z)$ as $h \to +\infty$



Using the results of the first two steps, we get that the ratio of the number of roots of $P(z)$ inside the Jordan curve $\mathcal{J}_{1+\varepsilon}$ (defined above and represented in Figure A.1) to the total number of roots of $P(z)$ converges to $\ell(\mathcal{A})/(2\pi)$ as $h \to +\infty$, where again $\ell(.)$ denotes the standard length operator (i.e., the Lebesgue measure on $\mathcal{C}$). So, as $h \to +\infty$, the ratio of the number of roots of $P(z)$ outside $\mathcal{C}$ to the total number of roots of $P(z)$ is bounded away from 0; or, equivalently, the ratio of the number of roots of $P(z)$ inside $\mathcal{C}$ to the total number of roots of $P(z)$, i.e. the ratio $p/\deg(P)$, is bounded away from 1. Since the ratio of the number of non-predetermined variables to the total number of roots of $P(z)$, i.e. the ratio $\nu/\deg(P)$, converges to 1 as $h \to +\infty$ (given that both ν and $\deg(P)$ increase one-for-one with h), we eventually get that the ratio p/ν is lower than

1 as $h \to +\infty$. Thus, for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$, there exists $\bar{h}(|\phi|)$ such that $p < \nu$ and $S(\phi, h) = M$ for all $h \geq \bar{h}(|\phi|)$.

In the fourth step, I just note that for any $\varepsilon \in \left(0, \bar{\phi} - \underline{\phi}\right)$ and any $|\phi| \in (\underline{\phi} + \varepsilon, \bar{\phi})$, there exists, by continuity, $\ell(\varepsilon) > 0$ such that the arc $\mathcal{A}$ can be chosen of length higher than $\ell(\varepsilon)$. As a result, $\bar{h}(|\phi|)$ can be chosen a bounded function of $|\phi|$ for $|\phi| \in (\underline{\phi} + \varepsilon, \bar{\phi})$.

Point (c)(iii). For $h \geq m + \max[0, \deg(R) - \deg(Q)]$, we have $\nu = \delta + h - m$, $P(z) = Q(z)z^{h-m} + \phi R(z)$, and $\deg(P) = \deg(Q) + h - m$. Consider a Jordan curve $\mathcal{J}_o$ (where the subscript "o" stands for "outside $\mathcal{C}$") that: (i) lies entirely outside $\mathcal{C}$, (ii) surrounds the $\deg(Q) - q$ roots of $Q(z)$ outside $\mathcal{C}$ (if $\deg(Q) - q \geq 1$), and (iii) does not surround $\mathcal{C}$. I apply Rouché's theorem to $\mathcal{J} = \mathcal{J}_o$, $P_b(z) = Q(z)z^{h-m}$, and $P_s(z) = \phi R(z)$. For any $|\phi| \in (\underline{\phi}, \bar{\phi})$, any

$$h \geq \bar{h} := m + \max\left\{0, \deg(R) - \deg(Q), \left\lceil \frac{\log\left(\bar{\phi} \max_{\tilde{z} \in \mathcal{J}_o} |R(\tilde{z})|\right) - \log\left(\min_{\tilde{z} \in \mathcal{J}_o} |Q(\tilde{z})|\right)}{\log\left(\min_{\tilde{z} \in \mathcal{J}_o} |\tilde{z}|\right)} \right\rceil \right\},$$

and any $z \in \mathcal{J}_o$, we have

$$\left|Q(z)z^{h-m}\right| \geq \min_{\tilde{z} \in \mathcal{J}_o} \left|Q(\tilde{z})\, \tilde{z}^{h-m}\right| \geq \left(\min_{\tilde{z} \in \mathcal{J}_o} |Q(\tilde{z})|\right) \left(\min_{\tilde{z} \in \mathcal{J}_o} |\tilde{z}|\right)^{h-m} \geq \bar{\phi} \max_{\tilde{z} \in \mathcal{J}_o} |R(\tilde{z})| > |\phi R(z)|,$$

where the last but one inequality follows from the definition of $\bar{h}$. So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{J}_o$ as $Q(z)z^{h-m}$. Therefore, $P(z)$ has exactly $\deg(Q) - q$ roots inside $\mathcal{J}_o$, and hence at least $\deg(Q) - q$ roots outside $\mathcal{C}$. We thus get $p \leq \deg(P) - [\deg(Q) - q] = h - m + q = \nu - (\delta - q) = \nu - d_{peg}$. Therefore, if $d_{peg} \geq 1$ (or equivalently if $S_{peg} = M$), then $p < \nu$ and consequently $S(\phi, h) = M$ for any $|\phi| \in (\underline{\phi}, \bar{\phi})$ and any $h \geq \bar{h}$.

Point (d)(iii). For $h \leq m$, we have $\nu = \delta$ and $P(z) = Q(z) + \phi R(z)z^{m-h}$. Consider a Jordan curve $\mathcal{J}_i$ (where the subscript "i" stands for "inside $\mathcal{C}$") that: (i) lies entirely inside $\mathcal{C}$, and (ii) surrounds the q roots of $Q(z)$ inside $\mathcal{C}$ (if $q \geq 1$). I apply Rouché's theorem to $\mathcal{J} = \mathcal{J}_i$, $P_b(z) = Q(z)$, and $P_s(z) = \phi R(z)z^{m-h}$. For any $|\phi| \in (\underline{\phi}, \bar{\phi})$, any

$$h \leq \underline{h} := m + \min\left\{0, \left\lfloor \frac{\log\left(\min_{\tilde{z} \in \mathcal{J}_i} |Q(\tilde{z})|\right) - \log\left(\bar{\phi} \max_{\tilde{z} \in \mathcal{J}_i} |R(\tilde{z})|\right)}{-\log\left(\max_{\tilde{z} \in \mathcal{J}_i} |\tilde{z}|\right)} \right\rfloor \right\},$$

and any $z \in \mathcal{J}_i$, we have

$$|Q(z)| \geq \min_{\tilde{z} \in \mathcal{J}_i} |Q(\tilde{z})| \geq \bar{\phi} \max_{\tilde{z} \in \mathcal{J}_i} |R(\tilde{z})| \left(\max_{\tilde{z} \in \mathcal{J}_i} |\tilde{z}|\right)^{m-h} \geq \bar{\phi} \max_{\tilde{z} \in \mathcal{J}_i} \left|R(\tilde{z})\, \tilde{z}^{m-h}\right| > \left|\phi R(z) z^{m-h}\right|,$$

where the second inequality follows from the definition of $\underline{h}$. So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{J}_i$ as $Q(z)$. Therefore, $P(z)$ has exactly q roots inside $\mathcal{J}_i$, and hence at least q roots inside $\mathcal{C}$. We thus get $p \geq q = \nu - (\delta - q) = \nu - d_{peg}$. Therefore, if $d_{peg} \leq -1$ (or equivalently if $S_{peg} = E$), then $p > \nu$ and consequently $S(\phi, h) = E$ for any $|\phi| \in (\underline{\phi}, \bar{\phi})$ and any $h \leq \underline{h}$.

A.6 Proof of Proposition 6

The proof of Proposition 6 is essentially a generalization of the proof of Proposition 2. Consider an arbitrary, fixed $h \in \mathbb{Z}$, and the corresponding fixed ν. Recall that determinacy obtains if and only if $p = \nu$, and hence only if $p - \nu$ is even. The proof of Point (a) in Online Appendix A.5 establishes that $\forall \phi \in \left(-\underline{\phi}, \underline{\phi}\right)$, $p - \nu = d_{peg}$; therefore, $\forall \phi \in \left(-\underline{\phi}, \underline{\phi}\right)$, $p - \nu$ has the same parity as d_{peg}. As ϕ moves outside $\left(-\underline{\phi}, \underline{\phi}\right)$, the parity of $p - \nu$ changes if and only if a *real* root of $P(z)$ crosses the unit circle, that is to say if and only if ϕ goes through a value such that $P(1) = 0$ or $P(-1) = 0$. It is easy to check that $P(1) = 0$ if and only if $\phi = \phi_1$, and that $P(-1) = 0$ only if $|\phi| = \phi_{-1}$.

Point (a). Suppose that d_{peg} is even and $|\phi_1| < \phi_{-1}$ (as stated in this point). If $\phi_1 > 0$, then $p - \nu$ is even for all $\phi \in \left(-\underline{\phi}, \phi_1\right)$ and odd for all $\phi \in (\phi_1, \phi_{-1})$; therefore, $\forall \phi \in (\phi_1, \phi_{-1})$, $S(\phi, h) \neq D$. Alternatively, if $\phi_1 < 0$, then $p - \nu$ is even for all $\phi \in \left(\phi_1, \underline{\phi}\right)$ and odd for all $\phi \in (-\phi_{-1}, \phi_1)$; therefore, $\forall \phi \in (-\phi_{-1}, \phi_1)$, $S(\phi, h) \neq D$.

Point (b). Suppose that d_{peg} is odd and $|\phi_1| < \phi_{-1}$ (as stated in this point). If $\phi_1 > 0$, then $\forall \phi \in (-\phi_{-1}, \phi_1)$, $p - \nu$ is odd and hence $S(\phi, h) \neq D$. Alternatively, if $\phi_1 < 0$, then $\forall \phi \in (\phi_1, \phi_{-1})$, $p - \nu$ is odd and hence $S(\phi, h) \neq D$.

Point (c). Suppose that d_{peg} is odd and $|\phi_1| > \phi_{-1}$ (as stated in this point). Then $\forall \phi \in (-\phi_{-1}, \phi_{-1})$, $p - \nu$ is odd and hence $S(\phi, h) \neq D$.

A.7 Proof of Proposition 7

Points (a) and (b). Points (a) and (b)(i) straightforwardly follow from Proposition 5 and the associated Figure 4. Points (b)(ii) and (b)(iii) straightforwardly follow from Points (a) and (b) of Proposition 6 (with $\phi_1 > 0$).

Point (c). The proof of this point is essentially a generalization of the proof of Proposition 3. Suppose that $\phi_1 = \underline{\phi}$ (as stated in this point). Proposition 5 straightforwardly implies that the Taylor principle is *necessary* for determinacy for any $d_{peg} \in \mathbb{Z} \setminus \{0\}$ and any $h \in \mathbb{Z}$. To show that it is *locally sufficient* for determinacy if and only if ($d_{peg} = 1$ and $h < h_1$) or ($d_{peg} = -1$ and $h > h_1$), I rewrite $P(z)$ as a function of two variables:

$\tilde{P}(\phi, z) := Q(z)z^{\max(0,h-m)} + \phi R(z)z^{\max(0,m-h)}$, where $(\phi, z) \in \mathbb{R} \times \mathbb{C}$. Simple algebra leads to $\tilde{P}(\phi_1, 1) = 0$ and

$$\frac{\partial \tilde{P}}{\partial z}(\phi_1, 1) = Q(1)\,(h - h_1)\,,$$

where $h_1 := m + R'(1)/R(1) - Q'(1)/Q(1)$. The expression obtained for $\partial\tilde{P}/\partial z(\phi_1, 1)$ is generically non-zero (it can be zero only if h_1 is an integer, and I ignore this zero-measure case). So, one root of the polynomial $\tilde{P}(\phi_1, z)$ is 1, and this root is of multiplicity one. The implicit-function theorem implies the existence of a continuously differentiable function $\phi \mapsto Z(\phi)$ such that one real root of $P(z)$ can be written as $Z(\phi)$ in the neighborhood of $\phi = \phi_1$, with $Z(\phi_1) = 1$ and

$$Z'(\phi_1) = \frac{-\frac{\partial \tilde{P}}{\partial \phi}(\phi_1, 1)}{\frac{\partial \tilde{P}}{\partial z}(\phi_1, 1)} = \frac{1}{\phi_1\,(h - h_1)}.$$

This root of $P(z)$ crosses $\mathcal{C}$ at point 1 as ϕ goes through ϕ_1. It is the only root that crosses $\mathcal{C}$ as ϕ goes through ϕ_1. Indeed, any root $z \in \mathbb{C}$ having this property must satisfy $\tilde{P}(\phi_1, z) = 0$, which implies $|Q(z)/R(z)| = \phi_1$ and hence $z = 1$ (since $\min_{\tilde{z}\in\mathcal{C}} |Q(\tilde{z})/R(\tilde{z})| = \phi_1$ and $\operatorname{argmin}_{\tilde{z}\in\mathcal{C}} |Q(\tilde{z})/R(\tilde{z})| = \{1\}$).

For any $h < h_1$, we have $Z'(\phi_1) < 0$, and therefore the root of $P(z)$ goes from outside to inside $\mathcal{C}$ as ϕ goes from below ϕ_1 to above ϕ_1. So, the number p of roots of $P(z)$ inside $\mathcal{C}$ increases by exactly one as ϕ goes from below ϕ_1 to above ϕ_1. We know from Online Appendix A.5 that this number is $p = q + \max(0, h - m)$ for ϕ just below $\underline{\phi} = \phi_1$. We also know from Lemma 1 that $\nu = \delta + \max(0, h - m)$. So, $\nu - p = \delta - q = d_{peg}$ for ϕ just below ϕ_1, and $\nu - p = d_{peg} - 1$ for ϕ just above ϕ_1. Therefore, if $d_{peg} = 1$, then we move from $p < \nu$ to $p = \nu$, and hence from $S(\phi, h) = M$ to $S(\phi, h) = D$, as ϕ goes from below ϕ_1 to above ϕ_1; in this case, the Taylor principle is locally sufficient for determinacy. Alternatively, if $d_{peg} \geq 2$ (resp. $d_{peg} \leq 0$), then we get $p < \nu$ and $S(\phi, h) = M$ (resp. $p > \nu$ and $S(\phi, h) = E$) for ϕ just above ϕ_1, and the Taylor principle is not locally sufficient for determinacy.

For any $h > h_1$, we have $Z'(\phi_1) > 0$, and therefore the root of $P(z)$ goes this time from inside to outside $\mathcal{C}$ as ϕ goes from below ϕ_1 to above ϕ_1. So, p decreases by exactly one as ϕ goes from below ϕ_1 to above ϕ_1. Again, we know from Online Appendix A.5 that $p = q + \max(0, h - m)$ for ϕ just below $\underline{\phi} = \phi_1$, and we know from Lemma 1 that $\nu = \delta + \max(0, h - m)$. So, we still have $\nu - p = d_{peg}$ for ϕ just below ϕ_1, but we now have $\nu - p = d_{peg} + 1$ for ϕ just above ϕ_1. Therefore, if $d_{peg} = -1$, then we move from $p > \nu$ to $p = \nu$, and hence from $S(\phi, h) = E$ to $S(\phi, h) = D$, as ϕ goes from below ϕ_1 to above ϕ_1; in this case, the Taylor principle is locally sufficient for determinacy. Alternatively, if $d_{peg} \geq 0$ (resp. $d_{peg} \leq -2$), then we get $p < \nu$ and $S(\phi, h) = M$ (resp. $p > \nu$ and $S(\phi, h) = E$) for ϕ just above ϕ_1, and the Taylor principle is not locally sufficient for determinacy.

A.8 Proof of Proposition 8

The proof of Proposition 8 is essentially a generalization of the proof of Proposition 4. Under Rule (6) with $\phi \neq 0$, as stated in Lemma 1, we have $\nu = \delta + \max(0, h-m)$ and $P(z) = Q(z)z^{\max(0,h-m)} + \phi R(z)z^{\max(0,m-h)}$. Under Rule (10) (which requires $\phi \neq 0$), the number of non-predetermined variables is still $\nu = \delta + \max(0, h-m)$, since the new terms in the rule are past (as opposed to expected future) values of the policy instrument. The characteristic polynomial of the dynamic system is the same as the characteristic polynomial of the corresponding perfect-foresight system

$$\begin{bmatrix} \mathbf{A}(L) & L^{-\gamma}\mathbf{B}(L) \\ -\phi L^{-h}\mathbf{V}(L) & \rho(L)/\rho(1) \end{bmatrix} \begin{bmatrix} \mathbf{X}_t \\ i_t \end{bmatrix} = \mathbf{0}.$$

Using the same standard result in time-series analysis as in Online Appendix A.3, I get that there exists $k \in \mathbb{Z}$ such that $P(z)$, the reciprocal polynomial of the characteristic polynomial, is

$$P(z) = z^k \det \begin{bmatrix} \mathbf{A}(z) & z^{-\gamma}\mathbf{B}(z) \\ -\phi z^{-h}\mathbf{V}(z) & \rho(z)/\rho(1) \end{bmatrix}.$$

Using the Laplace expansion and the notations introduced in the main text, I rewrite $P(z)$ as $P(z) = z^k\{\det[\mathbf{A}(z)]\rho(z)/\rho(1) - \phi z^{-\gamma-h}W(z)\} = z^k[Q(z)\rho(z)/\rho(1) + \phi z^{m-h}R(z)]$. As a reciprocal polynomial, $P(z)$ is such that $P(0) \neq 0$; moreover, we have $Q(0) \neq 0$, $\rho(0) \neq 0$, and $R(0) \neq 0$; as a consequence, we get $k = \max(0, h-m)$, and thus $P(z) = [Q(z)\rho(z)/\rho(1)]z^{\max(0,h-m)} + \phi R(z)z^{\max(0,m-h)}$ (except in the zero-measure case in which $(\phi, h) = (-Q(0)\rho(0)/[R(0)\rho(1)], m)$). So, Lemma 1 still holds for Rule (10) instead of Rule (6), if $Q(z)$ is replaced by $Q(z)\rho(z)/\rho(1)$ in this lemma. As a consequence, Propositions 5-7 still hold for Rule (10) instead of Rule (6), if in these propositions $Q(z)$ is replaced by $\tilde{Q}(z) := Q(z)\rho(z)/\rho(1)$ and, accordingly, d_{peg} is replaced by $d_{peg} - d_{sup}$ and S_{peg} by $\tilde{S}(d_{peg} - d_{sup})$.

The replacement of $Q(z)$ by $\tilde{Q}(z)$ leaves $h^* := m + r - \delta$ unchanged. Moreover, $\tilde{\phi}_1 := -\tilde{Q}(1)/R(1) = -Q(1)/R(1) = \phi_1$ and $\tilde{\phi}_{-1} := \left|\tilde{Q}(-1)/R(-1)\right| = |Q(-1)\rho(-1)/[R(-1)\rho(1)]| = |\rho(-1)/\rho(1)|\,\phi_{-1}$. Finally, $\tilde{h}_1 := m + R'(1)/R(1) - \tilde{Q}'(1)/\tilde{Q}(1) = m + R'(1)/R(1) - Q'(1)/Q(1) - \rho'(1)/\rho(1) = h_1 - \rho'(1)/\rho(1)$.

A.9 Counterparts of Figures 5 and 6 for $\phi < 0$

In the main text, Figures 5 and 6 show the determinacy status for some models and some rules, focusing on $\phi > 0$. In the present Online Appendix, Figures A.2 and A.3 show the determinacy status for the same models and the same rules, but this time for $\phi < 0$.

Figure A.2: Determinacy status for Models 2-5 and Rule 1 with $\phi < 0$

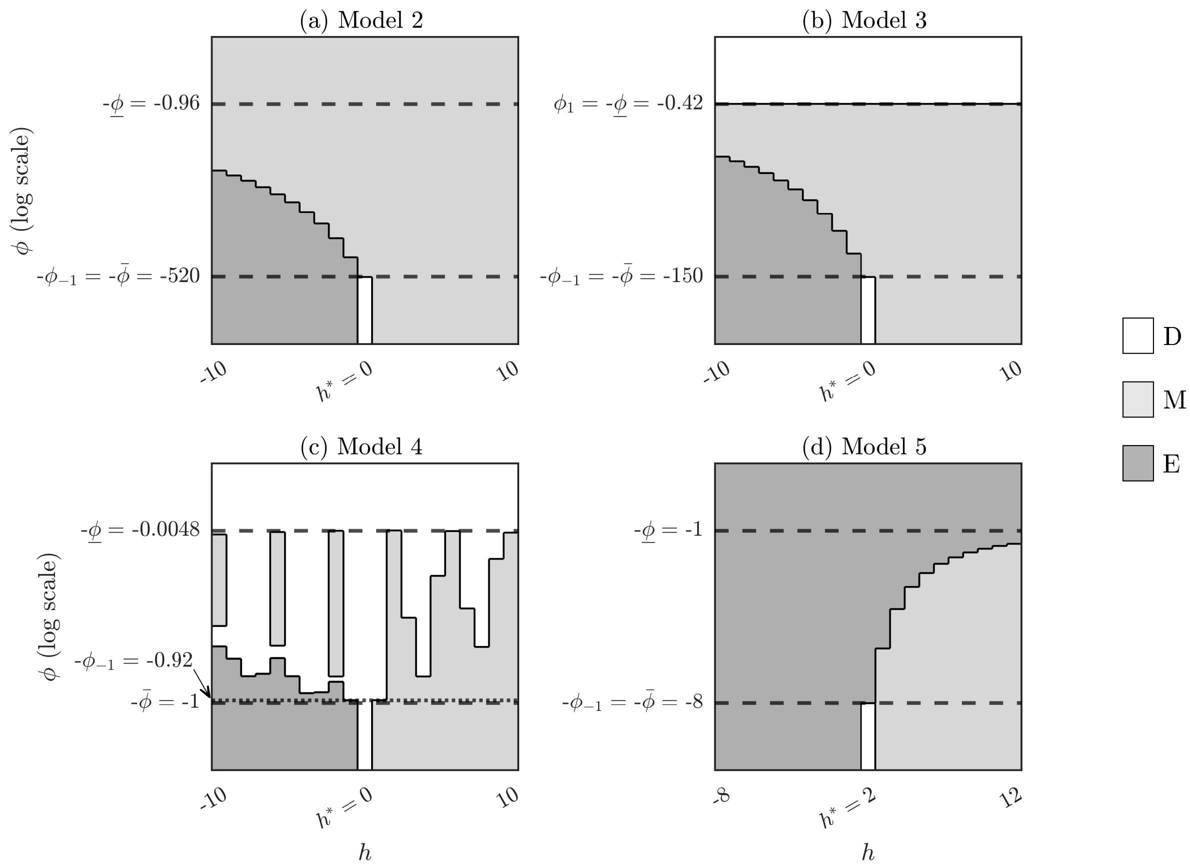
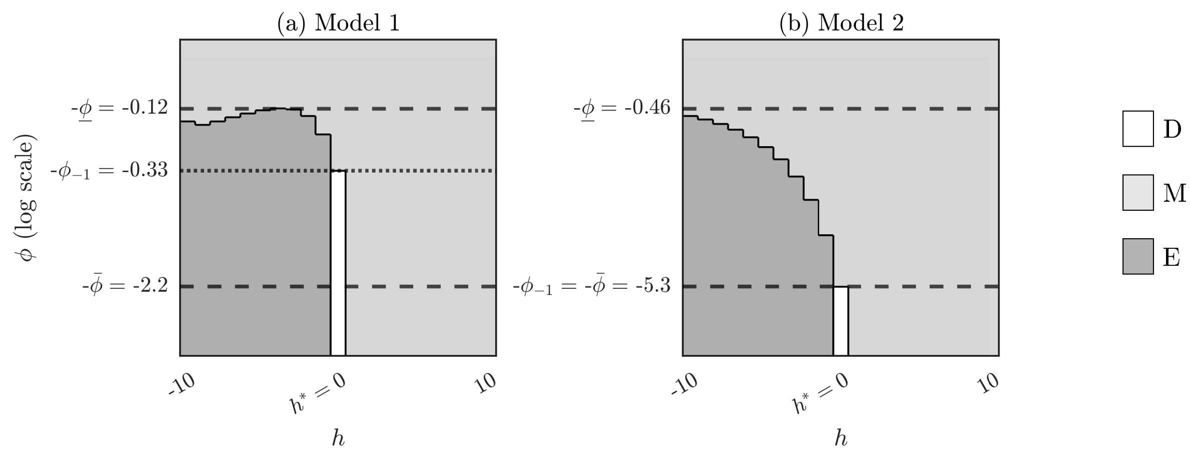


Figure A.3: Determinacy status for Models 1-2 and Rule (9) with $\phi < 0$



In Figures A.2a, A.2b, A.2d and A.3b, we have no determinacy for all $(\phi, h) \in (-\bar{\phi}, -\underline{\phi}) \times \mathbb{Z}$ as a consequence of Points (a) and (b) of Proposition 6 (as Table 2 in the main text explains). In Figure A.3a, we have no determinacy for all $(\phi, h) \in (-\phi_{-1}, -\underline{\phi}) \times \mathbb{Z}$ as a consequence of Point (c) of Proposition 6 (again, as Table 2 in the main text explains).

A.10 Application to MMB models: Non-regular cases

I make three regularity assumptions in the main text: (i) $Q(z)$ has no roots *exactly* on the unit circle $\mathcal{C}$; (ii) similarly, $R(z)$ has no roots *exactly* on $\mathcal{C}$; and (iii) if $1 \in \operatorname{argmin}_{z\in\mathcal{C}} |Q(z)/R(z)|$, then $\operatorname{argmin}_{z\in\mathcal{C}} |Q(z)/R(z)| = \{1\}$.

In my application to MMB models, I dismiss all model-and-rule combinations such that either (a) Assumption (i) is not met, or (b) Assumption (ii) is not met, or (c) Assumption (iii) is not met, or (d) the model has an exogenous monetary policy and is thus inconsistent with a rule.[2] Table A.1 reports which model-and-rule combinations I consider, which ones I dismiss, and for which reason (a, b, c or d) I dismiss the latter.

Table A.1: Model-and-rule combinations

		Rules			
		1	2	3 & 5	4 & 6
Models	NK_GS14	✗ (a)	✗ (a)	✗ (a)	✓
	US_MR07	✗ (b)	✗ (b)	✓	✗ (b)
	US_OW98	✗ (b)	✗ (b)	✗ (b)	✗ (b)
	US_RE09	✗ (c)	✓	✗ (a)	✗ (b)
	US_RS99	✗ (b)	✗ (b)	✗ (b)	✗ (b)
	EA_CW05fm	✗ (b)	✗ (b)	✗ (b)	✗ (b)
	EA_CW05ta	✗ (b)	✗ (b)	✗ (b)	✗ (b)
	EA_PV17	✗ (b)	✗ (b)	✗ (b)	✗ (b)
	EACZ_GEM03	✗ (a)	✗ (a)	✗ (a)	✓
	FI_AINO16	✗ (d)	✗ (d)	✗ (d)	✗ (d)
	All 130 other models	✓	✓	✓	✓
Total number of models		140	140	140	140
Number of models dismissed (✗)		10	9	9	8
Number of models considered (✓)		130	131	131	132

Note: the table reports which model-and-rule combinations I consider (check mark), which ones I dismiss (cross mark), and for which reason (a, b, c or d) I dismiss the latter. The parameter ρ in the inertial Rules 2, 5 and 6 is set to 0.8.

In practice, the polynomials $Q(z)$ and $R(z)$ that I obtain may have no roots *exactly* on $\mathcal{C}$, even when it can be shown that the true polynomials $Q(z)$ and $R(z)$ do have a root *exactly* on $\mathcal{C}$. So, I use, as a practical proxy for Case (a) (resp. Case (b)), the case in which the polynomial $Q(z)$ (resp. $R(z)$) that I obtain has a root whose modulus is

[2] Case (d) concerns only one model; this model has an exogenous monetary policy because it is a model of the Finnish economy within the euro area.

between $1-10^{-7}$ and $1+10^{-7}$. Similarly, I use, as a practical proxy for Case (c), the case in which there exists $\tilde{z} \in \mathcal{C}$ such that I obtain $|Q(1)/R(1)| - \min_{z\in\mathcal{C}} |Q(z)/R(z)| < 10^{-7}$, $|Q(\tilde{z})/R(\tilde{z})| - \min_{z\in\mathcal{C}} |Q(z)/R(z)| < 10^{-7}$ and $|\tilde{z}-1| > 10^{-1}$.

A.11 Application to MMB models: Methodology

In this Online Appendix, I outline the methodology I use to apply my general results to MMB models. The MMB models are coded in Dynare. I write a single Matlab program that: (i) calls all these Dynare files under alternative interest-rate rules; (ii) computes the coefficient and horizon thresholds $\underline{\phi}$, $\bar{\phi}$, ϕ_1, ϕ_{-1}, h^* and h_1 for each model-and-rule combination; (iii) checks in different ways that the threshold values obtained are the correct ones; and (iv) produces the figures displayed in the main text and in the Online Appendix.

I start by making a few changes to some of the Dynare files. These changes serve two main purposes: to facilitate the search for the steady state under various interest-rate rules, and to ensure that the foreign interest-rate rule remains invariant in open-economy models as I modify the domestic interest-rate rule. I provide a detailed description of these few changes in my Matlab code (in comments).

The Dynare files include a "user-specified interest-rate rule" of type

$$i_t = \sum_{k=1}^{4} \rho_k i_{t-k} + \sum_{h=-4}^{4} \phi_h^{\pi} \mathbb{E}_t\{\pi_{t+h}\} + \sum_{h=-4}^{4} \phi_h^{y} \mathbb{E}_t\{y_{t+h}\},$$

where i_t, π_t and y_t denote the interest rate, the inflation rate and the output level at date t, and all coefficients ρ_k, ϕ_h^{π} and ϕ_h^{y} are real numbers to be chosen by the user.[3] For any value of these coefficients, Dynare gives the eigenvalues of the dynamic system composed of the structural equations and the interest-rate rule; i.e., it gives the characteristic polynomial of this system, and hence its reciprocal polynomial $P(z)$. It also gives the number of non-predetermined variables ν.

For each MMB model, I compute $Q(z)$ and δ by setting all the rule's coefficients to zero: under an interest-rate peg ($i_t = 0$), we have $P(z) = Q(z)$ and $\nu = \delta$.

The computation of $R(z)$ and m depends on the rule considered. For Rule 1, I set all the rule's coefficients to zero except ϕ_{-4}^{π}, for which I consider several alternative values. For

[3] In addition to inflation and the output level, the user-specified rule also involves the output gap, which is defined as the output level minus an exogenous term. I abstract from the output gap, as it affects the determinacy status in exactly the same way as the output level.

each value of ϕ^{π}_{-4}, Dynare gives me

$$P(z) = Q(z)z^{\max(0,-m-4)} + \phi^{\pi}_{-4}R(z)z^{\max(0,m+4)}.$$

For all MMB models, I find that $P(0)$ does not depend on ϕ^{π}_{-4} (when $P(z)$ is normalized to be monic); so, $m + 4 > 0$ and

$$P(z) = Q(z) + \phi^{\pi}_{-4}R(z)z^{m+4}.$$

Using two alternative values ϕ_a and ϕ_b for ϕ^{π}_{-4}, and the corresponding polynomials $P_a(z)$ and $P_b(z)$, I then compute

$$R(z)z^{m+4} = \frac{P_a(z) - P_b(z)}{\phi_a - \phi_b},$$

from which I get $R(z)$ and m. With $Q(z)$, δ, $R(z)$ and m, I then compute the coefficient and horizon thresholds $\underline{\phi}$, $\bar{\phi}$, ϕ_1, ϕ_{-1}, h^* and h_1.

For Rules 3 and 4, I proceed in the same way as for Rule 1 except that I also set ϕ^{y}_{-4} to $\phi^{\pi}_{-4}/3$ (for Rule 3) or ϕ^{y}_{0} to 0.5 (for Rule 4). For the inertial Rules 2, 5 and 6 with $\rho = 0.8$, I proceed in the same way as for the non-inertial Rules 1, 3 and 4 respectively, except that I also set ρ_1 to ρ and I multiply ϕ^{π}_{-4}, ϕ^{y}_{-4} and ϕ^{y}_{0} by $1 - \rho$.[4]

In the process, there are two difficulties to overcome. The first difficulty is that most, if not all MMB models have autoregressive (AR) exogenous shocks, unlike my generic model (5). So, the eigenvalues given by Dynare include those coming from the shocks' AR processes, and the polynomial $P(z)$ given by Dynare is actually my polynomial $P(z)$ times another polynomial reflecting these AR processes. This difficulty is easily overcome, however, because the latter polynomial is in factor of both $Q(z)$ and $R(z)$. So, it does not affect the ratio $Q(z)/R(z)$, nor the difference $R'(1)/R(1) - Q'(1)/Q(1)$. As a consequence, it does not affect the coefficient and horizon thresholds $\underline{\phi}$, $\bar{\phi}$, ϕ_1, ϕ_{-1} and h_1. Since the eigenvalues coming from the shocks' AR processes are "stable," it does not affect r either, and hence it does not affect the horizon threshold h^*. So, to sum up, I do not need to remove the eigenvalues coming from the shocks' AR processes: I can just ignore them.

The second difficulty is that Dynare computes *generalized* eigenvalues. The true generalized eigenvalues typically include infinite eigenvalues (due to singular state-space representations, as static equations are included among dynamic equations). In practice, however, Dynare often gets finite and very large eigenvalues, instead of infinite ones. I

[4] I get $m > -4$ for all model-and-rule combinations expect the combinations (Model GMP_IMF13, Rule 3) and (Model GMP_IMF13, Rule 5), for which I get $m = -4$. For these two combinations, I consider alternative values for ϕ^{π}_{-3} and ϕ^{y}_{-3}, rather than ϕ^{π}_{-4} and ϕ^{y}_{-4}.

need to identify these eigenvalues, in order to remove them. In many cases, they are easily identified, as the order of magnitude of eigenvalue moduli, sorted in ascending order, jumps from 0 to 15 or more. In some cases, however, especially for large-scale models with several hundreds of eigenvalues, their identification is more difficult. I then choose a cutoff value for the moduli (above which I ignore the eigenvalues) such that the results are stable and the checks are passed.

I conduct two kinds of checks. First, I get eight alternative measures of $Q(z)$ and $R(z)$, and I check that the results do not depend on the measure used. More specifically, I consider four alternative pairs of values (ϕ_a, ϕ_b) for ϕ_{-4}^{π}, and I thus get four measures of $R(z)$. For each of these values of $R(z)$, I use two different methods to get $Q(z)$: the direct method described above, and the indirect method computing $Q(z)$ as

$$Q(z) = P_a(z) - \phi_a R(z) z^{m+4},$$

using a given value ϕ_a of ϕ_{-4}^{π} and the corresponding polynomial $P_a(z)$.

The second kind of check consists in running Dynare for ϕ just below and just above the computed values of the thresholds $-\bar{\phi}$, $-\underline{\phi}$, $\underline{\phi}$, $\bar{\phi}$, ϕ_1, $-\phi_{-1}$ and ϕ_{-1}, and for h ranging from -4 to 4. I check that the degree of indeterminacy $\nu - p$ obtained, and the change in the degree of indeterminacy as ϕ crosses the threshold considered, are consistent with my analytical results. The Dynare runs for ϕ just above $\bar{\phi}$ (resp. just above ϕ_1) also serve to check the value of h^* (resp. the value of h_1).

A.12 Application to MMB models: Additional distributions

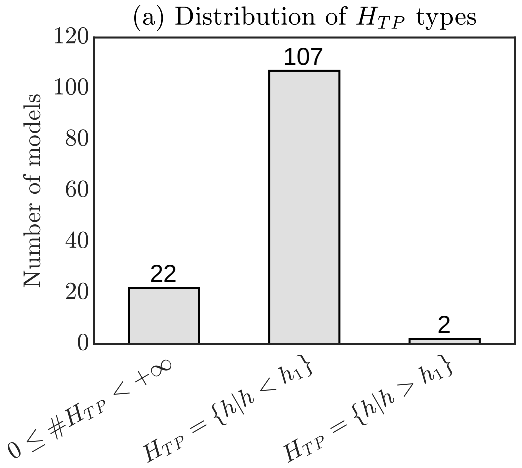
In Section 4 of the paper, which applies my general results to MMB models, the results essentially take the form of distributions of coefficient and horizon thresholds across MMB models. In this Online Appendix, I show some distributions that are briefly discussed but not shown in the main text (mostly because they are very similar to other distributions shown in the main text).

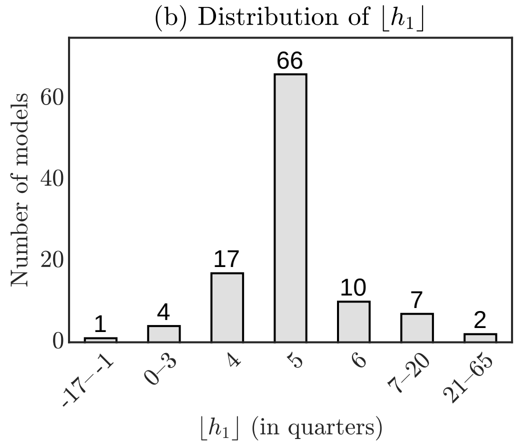
Figure A.4a shows the distribution of H_{TP} types under the inertial Rule 2 (with $\rho = 0.8$). As I explain in the main text, if H_{TP} is of type $\{h | h < h_1\}$ under Rule 1 (as is the case for 96 models), then H_{TP} remains of this type under Rule 2. For the same reason, if H_{TP} is of type $\{h | h > h_1\}$ under Rule 1 (as is the case for only 2 models), then H_{TP} remains of this type under Rule 2.[5] So, the distribution of H_{TP} types is broadly similar across

[5] As we move from Rule 1 to Rule 2, the increase in h_1 thus widens H_{TP} in the former (96) models, and narrows H_{TP} in the latter (2) models.

the two rules. The main change in the distribution of H_{TP} types, as we move from Rule 1 to Rule 2 (i.e. from Figure 7e to Figure A.4a), is that for 10 models H_{TP} is finite under Rule 1 and infinite, of type $\{h|h < h_1\}$, under Rule 2.[6]

Figure A.4: Some distributions across MMB models under Rule 2 *(131 models)*





(for the 107 models such that $H_{TP} = \{h|h < h_1\}$)

Figure A.4b shows the distribution of $\lfloor h_1 \rfloor$ under Rule 2, across the 107 models such that H_{TP} is of type $\{h|h < h_1\}$. For the 96 models such that H_{TP} is of type $\{h|h < h_1\}$ also under Rule 1, Proposition 8 implies that $\lfloor h_1 \rfloor$ increases by exactly 4 quarters – typically from 1 to 5 quarters – as we move from Rule 1 to Rule 2. For the 10 models such that H_{TP} is of type $\{h|h < h_1\}$ only under Rule 2, the mean and median values of $\lfloor h_1 \rfloor$ under Rule 2 are 4.1 and 4 respectively. So, the distribution of $\lfloor h_1 \rfloor$ across the 107 models under Rule 2 (Figure A.4b) looks essentially like the distribution of $\lfloor h_1 \rfloor$ across the 96 models under Rule 1 (Figure 7f) shifted to the right by 4 quarters.[7]

Before I turn to the results obtained under Rules 3-6, let me point out that Rules 4 and 6 are not of type (6). However, as I explain in Subsection 3.8, Propositions 5-7 still hold for Rules 4 and 6, if S_{peg} and d_{peg} in these propositions are replaced by the determinacy status S_0 and the degree of indeterminacy d_0 obtained for $\phi = 0$, i.e. obtained under the rule $i_t = (1/2)y_t$ (for Rule 4) or $i_t = \rho i_{t-1} + (1-\rho)(1/2)y_t$ (for Rule 6). Table A.2 reports the distribution of d_0 across MMB models for Rules 4 and 6: most of the models are such that $d_0 = 1$ (and hence $S_0 = M$).

The results obtained under Rules 3-4 (resp. Rules 5-6) are similar to the results obtained

[6] These 10 models are such that $\text{argmin}_{z\in\mathcal{C}} |Q(z)/R(z)| \neq \{1\}$ and hence $\phi_1 \neq \underline{\phi}$ under Rule 1, but $\text{argmin}_{z\in\mathcal{C}} |(1-\rho z)Q(z)/[(1-\rho)R(z)]| = \{1\}$ and hence $\phi_1 = \underline{\phi}$ under Rule 2.

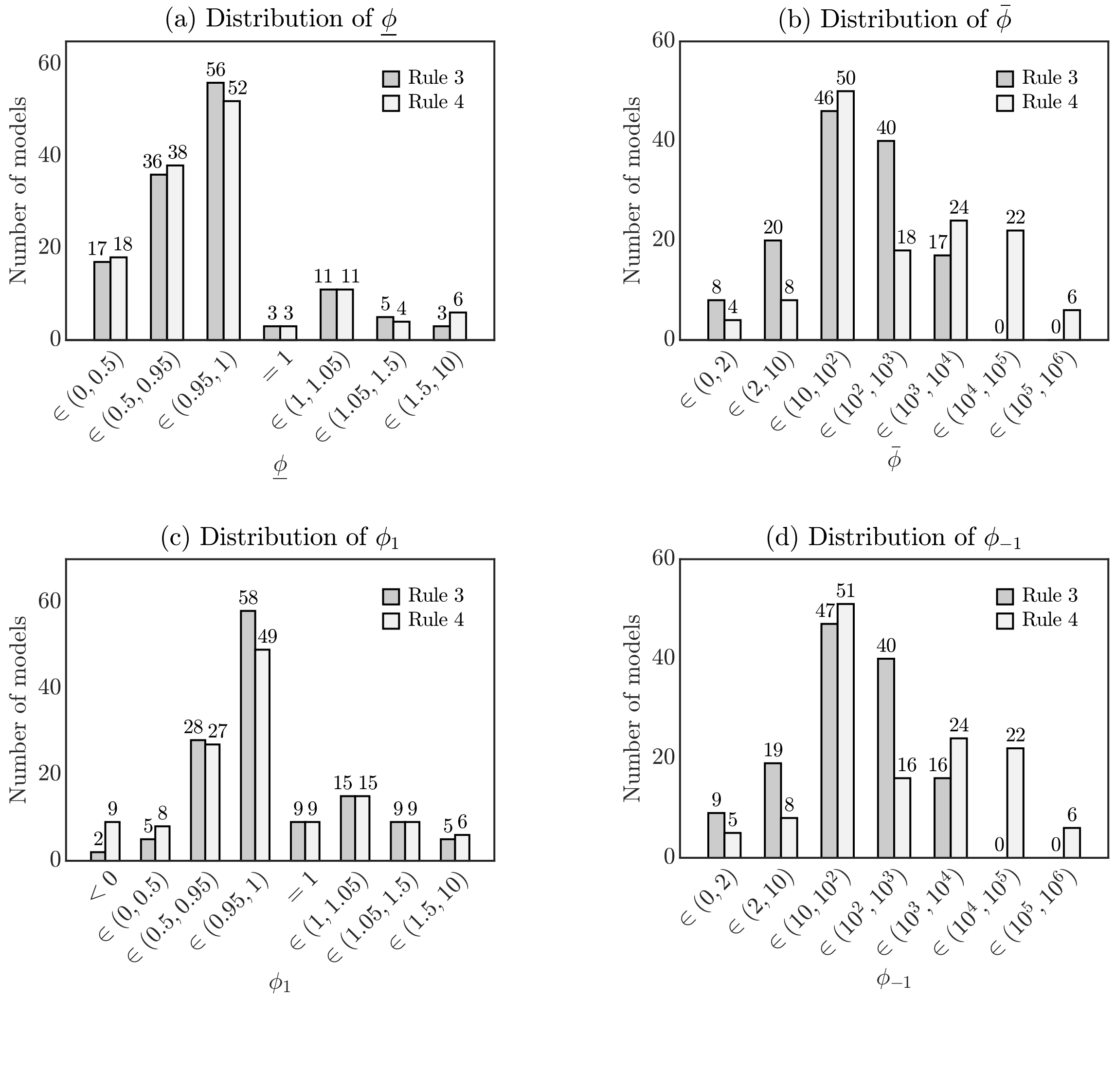
[7] There are $107 - 96 = 11$ more models with $H_{TP} = \{h|h < \overline{h}_1\}$ under Rule 2 than under Rule 1: the ten models that I discuss here, plus one model that was not considered under Rule 1 (because it did not satisfy a regularity condition).

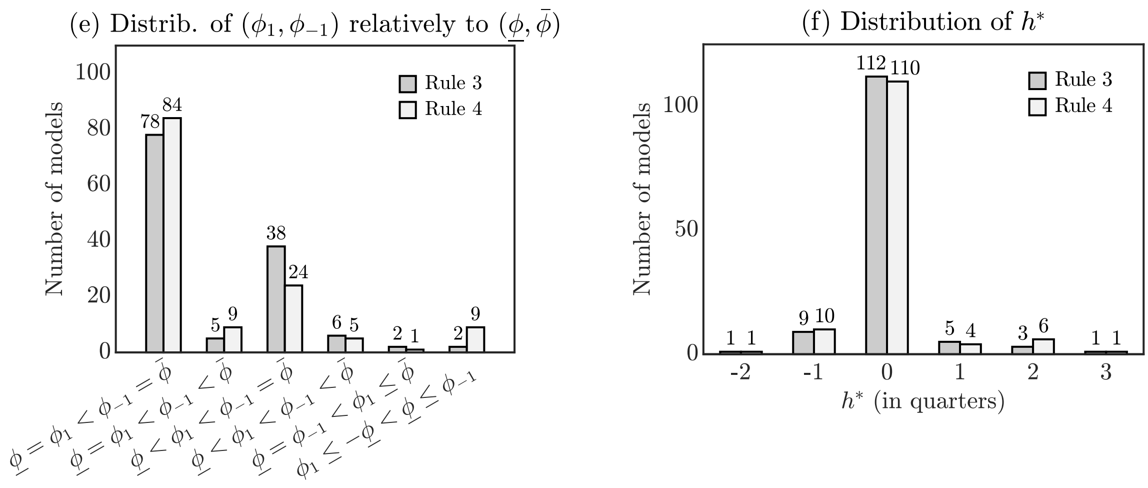
Table A.2: Distribution of d_0 across MMB models for Rules 4 and 6 *(140 models)*

Value of d_0	-1	0	1
Number of models	5	14	121

under Rule 1 (resp. Rule 2), except for h_1 (as discussed in the main text). Under Rules 3-4 (Figure A.5), compared to Rule 1 (Figure 7), $\underline{\phi}$ and ϕ_1 are still predominantly close to 1; $\bar{\phi}$ and ϕ_{-1} are still typically very large (although they are low, between 0 and 2, in a few models again); and h^* is still 0 for a large majority of models.

Figure A.5: Some distributions across MMB models under Rules 3-4 *(131-132 models)*





The distribution of H_{TP} types is broadly the same under Rules 5-6 (Figure A.6a) as under Rules 3-4 (Figure 8a), although for 17 models (resp. 10 models) H_{TP} is finite under Rule 3 (resp. Rule 4) and infinite, of type $\{h|h < h_1\}$, under Rule 5 (resp. Rule 6). For models with $H_{TP} = \{h|h < h_1\}$ under Rule 3, replacing the non-inertial Rule 3 with the inertial Rule 5 increases h_1 by $\rho/(1-\rho)$ (as a consequence of Proposition 8), just like for the basic NK model. So, for $\rho = 0.8$, $\lfloor h_1 \rfloor$ increases by exactly 4 quarters as we move from Rule 3 (Figure 8b) to Rule 5 (Figure A.6b). We observe a similar increase in $\lfloor h_1 \rfloor$ as we move from Rule 4 (Figure 8b) to Rule 6 (Figure A.6b), although the latter increase is not a consequence of Proposition 8.

Figure A.6: Some distributions across MMB models under Rules 5-6 *(131-132 models)*