Online Appendix to
"A Model of Post-2008 Monetary Policy"

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In this Online Appendix, to lighten up the notation, we sometimes omit function arguments when no ambiguity results.

Appendix A: Non-Linear Benchmark Model

In this appendix, we prove the existence of the function $\mathcal{M}$, and we establish the properties of the functions f^b, g^b, Γ, $\mathcal{M}$, and $\mathcal{F}$ (which play a role in our benchmark model). We also show how our steady-state targets pin down the parameters in the calibration of our benchmark model under iso-elastic production and utility functions. We omit time subscripts to lighten up the notation, thus writing h^b, ℓ, and m instead of h^b_t, ℓ_t, and m_t.

A.1 Concavity of the Function f^b

Since f^b is homogeneous of degree d, we have $\forall x \in \mathbb{R}_{\geq 0}$, $f^b(xh^b, xm) = x^d f^b(h^b, m)$. Computing the first derivative of the left- and right-hand sides of this equation with respect to x at $x = 1$ leads to

$$df^b = h^b f^b_h + m f^b_m. \tag{A.1}$$

In turn, computing the first derivative of the left- and right-hand sides of the last equation with respect to h^b and m leads to

$$f^b_{hh} = -\frac{(1-d)\, f^b_h + m f^b_{hm}}{h^b} \quad \text{and} \quad f^b_{mm} = -\frac{(1-d)\, f^b_m + h^b f^b_{hm}}{m}.$$

Using these expression for f^b_{hh} and f^b_{hm}, as well as (A.1), we get

$$\begin{aligned} f^b_{hh} f^b_{mm} - \left(f^b_{hm}\right)^2 &= \frac{1-d}{h^b m}\left[(1-d)\, f^b_h f^b_m + f^b_{hm}\left(h^b f^b_h + m f^b_m\right)\right] \\ &= \frac{1-d}{h^b m}\left[(1-d)\, f^b_h f^b_m + df^b f^b_{hm}\right] \\ &\geq 0, \end{aligned}$$

which implies (together with $f^b_{hh} \leq 0$ and $f^b_{mm} \leq 0$) that the function f^b is (weakly) concave.

A.2 Properties of the Function g^b

Computing the first and second derivatives of the left- and right-hand sides of $\ell = f^b[g^b(\ell,m),m]$ with respect to ℓ and m gives

$$1 = f_h^b g_\ell^b, \quad 0 = f_h^b g_m^b + f_m^b, \quad 0 = f_{hh}^b \left(g_\ell^b\right)^2 + f_h^b g_{\ell\ell}^b,$$
$$0 = f_{hh}^b g_\ell^b g_m^b + f_{hm}^b g_\ell^b + f_h^b g_{\ell m}^b, \quad \text{and} \quad 0 = f_{hh}^b \left(g_m^b\right)^2 + 2 f_{hm}^b g_m^b + f_h^b g_{mm}^b + f_{mm}^b.$$

Using these equations and $f_h^b > 0$, $f_m^b > 0$, $f_{hh}^b < 0$, $f_{hm}^b \geq 0$, and $f_{mm}^b < 0$, we sequentially get

$$g_\ell^b = \frac{1}{f_h^b} > 0, \quad g_m^b = \frac{-f_m^b}{f_h^b} < 0, \quad g_{\ell\ell}^b = \frac{-f_{hh}^b}{\left(f_h^b\right)^3} > 0,$$
$$g_{\ell m}^b = \frac{f_m^b f_{hh}^b}{\left(f_h^b\right)^3} - \frac{f_{hm}^b}{\left(f_h^b\right)^2} < 0, \quad \text{and} \quad g_{mm}^b = \frac{-f_{hh}^b \left(f_m^b\right)^2}{\left(f_h^b\right)^3} + 2\frac{f_m^b f_{hm}^b}{\left(f_h^b\right)^2} - \frac{f_{mm}^b}{f_h^b} > 0.$$

Then, using these expressions for $g_{\ell\ell}^b$, g_{mm}^b, $g_{\ell m}^b$, and the concavity of f^b, we easily get

$$g_{\ell\ell}^b g_{mm}^b - \left(g_{\ell m}^b\right)^2 = \frac{f_{hh}^b f_{mm}^b - \left(f_{hm}^b\right)^2}{\left(f_h^b\right)^4} \geq 0,$$

which implies (together with $g_{\ell\ell}^b > 0$ and $g_{mm}^b > 0$) that the function g^b is (weakly) convex.

Moreover, since f^b is homogeneous of degree d, we have $\forall x \in \mathbb{R}_{\geq 0}$, $g^b(x^d \ell, xm) = x g^b(\ell,m)$. Computing the first derivative of the left- and right-hand sides of this equation with respect to x at $x = 1$ leads to

$$g^b = d\ell g_\ell^b + m g_m^b. \tag{A.2}$$

In turn, computing the first derivative of the left- and right-hand sides of the last equation with respect to ℓ and m leads to

$$g_{\ell\ell}^b = \frac{(1-d)\, g_\ell^b - m g_{\ell m}^b}{d\ell}, \tag{A.3}$$
$$g_{mm}^b = \frac{-d\ell g_{\ell m}^b}{m}. \tag{A.4}$$

Finally, as a direct consequence of $\lim_{m\to+\infty} f_m^b(h^b,m) = 0$ and $\lim_{m\to 0} f_h^b(h^b,m) = 0$, we get $\lim_{m\to+\infty} g_m^b(\ell,m) = 0$ and $\lim_{m\to 0} g_\ell^b(\ell,m) = +\infty$ for all $\ell \in \mathbb{R}_{\geq 0}$.

A.3 Properties of the Function Γ

Computing the first and second derivatives of the left- and right-hand sides of $\Gamma(\ell,m) \equiv v^b[g^b(\ell,m)]$ with respect to ℓ and m gives

$$\Gamma_\ell = v^{b\prime} g_\ell^b > 0, \quad \Gamma_m = v^{b\prime} g_m^b < 0, \quad \Gamma_{\ell\ell} = v^{b\prime\prime}\left(g_\ell^b\right)^2 + v^{b\prime} g_{\ell\ell}^b > 0,$$
$$\Gamma_{\ell m} = v^{b\prime\prime} g_\ell^b g_m^b + v^{b\prime} g_{\ell m}^b < 0, \quad \text{and} \quad \Gamma_{mm} = v^{b\prime\prime}\left(g_m^b\right)^2 + v^{b\prime} g_{mm}^b > 0,$$

where the inequalities follow from $v^{b\prime} > 0$, $v^{b\prime\prime} \geq 0$, $g^b_\ell > 0$, $g^b_m < 0$, $g^b_{\ell\ell} > 0$, $g^b_{mm} > 0$, and $g^b_{\ell m} < 0$. In addition, using first (A.3)-(A.4) and then (A.2), we easily get

$$\begin{aligned}
\Gamma_{\ell\ell}\Gamma_{mm} - (\Gamma_{\ell m})^2 &= \left(v^{b\prime}\right)^2 \left[g^b_{\ell\ell} g^b_{mm} - \left(g^b_{\ell m}\right)^2\right] \\
&\quad + v^{b\prime} v^{b\prime\prime} \left[\left(g^b_\ell\right)^2 g^b_{mm} + \left(g^b_m\right)^2 g^b_{\ell\ell} - 2 g^b_\ell g^b_m g^b_{\ell m}\right] \\
&= \frac{-(1-d)\left(v^{b\prime}\right)^2 g^b_\ell g^b_{\ell m}}{m} \\
&\quad + \frac{v^{b\prime} v^{b\prime\prime}}{d\ell m} \left[-g^b_{\ell m}\left(d\ell g^b_\ell + m g^b_m\right)^2 + (1-d)\, m g^b_\ell \left(g^b_m\right)^2\right] \\
&= \frac{-(1-d)\left(v^{b\prime}\right)^2 g^b_\ell g^b_{\ell m}}{m} \\
&\quad + \frac{v^{b\prime} v^{b\prime\prime}}{d\ell m} \left[-\left(g^b\right)^2 g^b_{\ell m} + (1-d)\, m g^b_\ell \left(g^b_m\right)^2\right] \\
&\geq 0,
\end{aligned} \tag{A.5}$$

which implies (together with $\Gamma_{\ell\ell} > 0$ and $\Gamma_{mm} > 0$) that the function Γ is (weakly) convex. Finally, as a direct consequence of $\lim_{m\to+\infty} g^b_m(\ell, m) = 0$ and $\lim_{m\to 0} g^b_\ell(\ell, m) = +\infty$, we get $\lim_{m\to+\infty} \Gamma_m(\ell, m) = 0$ and $\lim_{m\to 0} \Gamma_\ell(\ell, m) = +\infty$ for all $\ell \in \mathbb{R}_{\geq 0}$.

A.4 Existence and Properties of the Function $\mathcal{M}$

Using (3), (5), (9), (13), (17), and (18), we can rewrite households' first-order condition for loans (6), at the steady state, as a relationship between reserves m and employment h:

$$\Gamma_\ell\left[\mathcal{L}(h), m\right] = \mathcal{A}(h) \equiv \frac{u'\left[f(h)\right]}{\phi}\left\{\left(\frac{\varepsilon - 1}{\varepsilon}\right)\frac{u'\left[f(h)\right] f'(h)}{v'(h)} - 1\right\}. \tag{A.6}$$

Because the left-hand side of (A.6) is positive, we restrict the domain of the function $\mathcal{A}$ to $(0, h^\star)$, where $h^\star > 0$ is implicitly and uniquely defined by $u'[f(h^\star)]f'(h^\star)/v'(h^\star) = \varepsilon/(\varepsilon - 1)$. The value $h^\star$ is the value that h would take in the absence of financial frictions, i.e. if the marginal banking cost Γ_ℓ were zero. The function $\mathcal{A}$ is strictly decreasing ($\mathcal{A}' < 0$), with $\lim_{h\to 0} \mathcal{A}(h) = +\infty$ and $\lim_{h\to h^\star} \mathcal{A}(h) = 0$.

Since $\Gamma_{\ell\ell} > 0$, $\mathcal{L}' > 0$, $\Gamma_{\ell m} < 0$, and $\mathcal{A}' < 0$, Equation (A.6) implicitly and uniquely defines a function $\mathcal{M}$ such that

$$m = \mathcal{M}(h).$$

This function is strictly increasing ($\mathcal{M}' > 0$). Moreover, since $\lim_{m\to 0} \Gamma_\ell(\ell, m) = +\infty$, $\mathcal{M}$ is defined over $(0, \bar{h})$, where $\bar{h} \in (0, h^\star]$ is implicitly and uniquely defined by $\lim_{m\to+\infty} \Gamma_\ell[\mathcal{L}(\bar{h}), m] =$

$\mathcal{A}(\bar{h})$.[1] Finally, this last equation straightforwardly implies

$$\lim_{h\to\bar{h}} \mathcal{M}(h) = +\infty. \tag{A.7}$$

A.5 Properties of the Function $\mathcal{F}$

Using (A.6), we can rewrite $\mathcal{F}(h)$ as

$$\mathcal{F}(h) = \frac{1}{\phi}\mathcal{F}_1(h)\,\mathcal{F}_2(h),$$

where the functions $\mathcal{F}_1$ and $\mathcal{F}_2$ are defined over $(0,\bar{h})$ by

$$\begin{aligned}
\mathcal{F}_1(h) &\equiv \frac{\Gamma_m[\mathcal{L}(h),\mathcal{M}(h)]}{\Gamma_\ell[\mathcal{L}(h),\mathcal{M}(h)]} = \frac{g_m^b[\mathcal{L}(h),\mathcal{M}(h)]}{g_\ell^b[\mathcal{L}(h),\mathcal{M}(h)]},\\
\mathcal{F}_2(h) &\equiv \left(\frac{\varepsilon-1}{\varepsilon}\right)\frac{u'[f(h)]\,f'(h)}{v'(h)} - 1.
\end{aligned}$$

We have

$$\begin{aligned}
\left(g_\ell^b\right)^2 \mathcal{F}_1' &= g_\ell^b\left(g_{\ell m}^b\mathcal{L}' + g_{mm}^b\mathcal{M}'\right) - g_m^b\left(g_{\ell\ell}^b\mathcal{L}' + g_{\ell m}^b\mathcal{M}'\right)\\
&= -g_{\ell m}^b\left(d\mathcal{L}g_\ell^b + \mathcal{M}g_m^b\right)\left(\frac{\mathcal{M}'}{\mathcal{M}} - \frac{\mathcal{L}'}{d\mathcal{L}}\right) - (1-d)\,g_\ell^b g_m^b\frac{\mathcal{L}'}{d\mathcal{L}}\\
&= -g^b g_{\ell m}^b\left(\frac{\mathcal{M}'}{\mathcal{M}} - \frac{\mathcal{L}'}{d\mathcal{L}}\right) - (1-d)\,g_\ell^b g_m^b\frac{\mathcal{L}'}{d\mathcal{L}},
\end{aligned}$$

where the second equality follows from (A.3)-(A.4), and the third equality from (A.2). Now, deriving the left- and right-hand sides of (A.6) with respect to h gives

$$\Gamma_{\ell m}\mathcal{M}' + \Gamma_{\ell\ell}\mathcal{L}' = \mathcal{A}' < 0.$$

Moreover, using (A.2) and (A.3), we get

$$\begin{aligned}
d\mathcal{L}\Gamma_{\ell\ell} + \mathcal{M}\Gamma_{\ell m} &= d\mathcal{L}\left[v^{b\prime\prime}\left(g_\ell^b\right)^2 + v^{b\prime}g_{\ell\ell}^b\right] + \mathcal{M}\left(v^{b\prime\prime}g_\ell^b g_m^b + v^{b\prime}g_{\ell m}^b\right)\\
&= v^{b\prime\prime}g_\ell^b\left(d\mathcal{L}g_\ell^b + \mathcal{M}g_m^b\right) + v^{b\prime}\left(d\mathcal{L}g_{\ell\ell}^b + \mathcal{M}g_{\ell m}^b\right)\\
&= v^{b\prime\prime}g^b g_\ell^b + (1-d)\,v^{b\prime}g_\ell^b\\
&\geq 0.
\end{aligned}$$

The last two inequalities together imply

$$\frac{\mathcal{M}'}{\mathcal{M}} > \frac{\mathcal{L}'}{d\mathcal{L}}, \tag{A.8}$$

from which we conclude that $\mathcal{F}_1' > 0$. Then, using $\mathcal{F}_1' > 0$, $\mathcal{F}_1 < 0$, $\mathcal{F}_2' < 0$, and $\mathcal{F}_2 > 0$, we get that the function $\mathcal{F}$ is strictly increasing ($\mathcal{F}' > 0$).

[1] The upper bound of employment $\bar{h}$ coincides with the frictionless employment level $h^\star$ in the case where the marginal banking cost Γ_ℓ converges to zero as real reserves tend to infinity. In general, however, we allow the marginal banking cost to converge to a positive value – in which case we have $\bar{h} < h^\star$, and our economy with the financial friction cannot attain the employment level of the frictionless economy.

Moreover, $\mathcal{F}_1' > 0$ and $\mathcal{F}_1 < 0$ imply that $\lim_{h \to 0} \mathcal{F}_1(h) < 0$, while the Inada condition $\lim_{c \to 0} u'(c) = +\infty$ implies that $\lim_{h \to 0} \mathcal{F}_2(h) = +\infty$, so that

$$\lim_{h \to 0} \mathcal{F}(h) = -\infty.$$

Finally, both $\lim_{h \to \bar{h}} \mathcal{F}_1(h)$ and $\lim_{h \to \bar{h}} \mathcal{F}_2(h)$ are finite, since $\mathcal{F}_1$ is increasing and negative, and $\mathcal{F}_2$ decreasing and positive. If $\bar{h} < h^\star$, then (A.7) and the Inada condition $\lim_{m \to +\infty} \Gamma_m(\ell, m) = 0$ implies $\lim_{h \to \bar{h}} \mathcal{F}_1(h) = 0$. Alternatively, if $\bar{h} = h^\star$, then $\lim_{h \to \bar{h}} \mathcal{F}_2(h) = 0$. We conclude that, in both cases,

$$\lim_{h \to \bar{h}} \mathcal{F}(h) = 0.$$

A.6 Calibration Under Iso-Elastic Production and Utility Functions

To see how our targets for I^ℓ and m/ℓ pin down the parameters ς and V_b, we first rewrite households' first-order conditions (6) and (7) as

$$\beta I^\ell = 1 + \frac{V_b}{(1-\varsigma)\lambda} A_b^{\frac{-(1+\eta)}{1-\varsigma}} \ell^{\frac{\eta+\varsigma}{1-\varsigma}} m^{\frac{-\varsigma(1+\eta)}{1-\varsigma}} \tag{A.9}$$

and

$$\beta I^m = 1 - \frac{\varsigma V_b}{(1-\varsigma)\lambda} A_b^{\frac{-(1+\eta)}{1-\varsigma}} \ell^{\frac{1+\eta}{1-\varsigma}} m^{\frac{-(1+\varsigma\eta)}{1-\varsigma}} \tag{A.10}$$

in the steady state. Equations (A.9) and (A.10) give parameter ς as a function of I^ℓ, m/ℓ, and already calibrated parameters:

$$\varsigma = \left(\frac{m}{\ell}\right)\left(\frac{1-\beta I^m}{\beta I^\ell - 1}\right).$$

Thus, the targets for I^ℓ and m/ℓ pin down ς; we get $\varsigma = 0.0039$.

Next, we rewrite firms' first-order condition under flexible prices (13) as

$$w = \alpha A \left(\frac{\varepsilon - 1}{\varepsilon}\right)\left[\phi \frac{I^\ell}{I} + (1-\phi)\right]^{-1} h^{-(1-\alpha)} \tag{A.11}$$

in the steady state, and we use (3), (9), and (17) to rewrite households' intra-temporal first-order condition (5) as

$$w = V A^\sigma h^{\eta + \alpha\sigma} \tag{A.12}$$

in the steady state. Equations (A.11) and (A.12) give the steady-state value of hours worked h as a function of I^ℓ and already calibrated parameters:

$$h = \left\{\frac{\alpha A^{1-\sigma}}{V}\left(\frac{\varepsilon - 1}{\varepsilon}\right)\left[\phi \frac{I^\ell}{I} + (1-\phi)\right]^{-1}\right\}^{\frac{1}{\eta + \alpha\sigma + (1-\alpha)}}.$$

Thus, the target for I^ℓ pins down h. We plug the value obtained for h into either (A.11) or (A.12) to get the steady-state real wage w. Using the borrowing constraint (10) holding with equality, we then get the steady-state value of real loans $\ell = \phi w h$, from which we get in turn the steady-state value of real reserves $m = (m/\ell)\ell$. The value that we have obtained for h also gives us the steady-state value of consumption $c = y = Ah^{\alpha}$, from which we get in turn the steady-state value of the marginal utility of consumption $\lambda = c^{-\sigma}$. By plugging these values of ℓ, m, and λ, as well as the value that we have obtained for ς, into either (A.9) or (A.10), we recover the implied value of V_b; we get $V_b = 0.019$.

Appendix B: Log-Linearized Benchmark Model

In this appendix, we log-linearize our benchmark model around its unique steady state; we show that the model has a unique local equilibrium under exogenous monetary-policy instruments; we characterize this equilibrium; and we derive the necessary and sufficient condition for local-equilibrium determinacy under a floor system.

B.1 Log-Linearization

The IS equation (21), with $\sigma \equiv -u''(c)c/u'(c)$ (where c denotes steady-state consumption), is straightforwardly obtained by log-linearizing households' first-order conditions (3)-(4) and the goods-market-clearing condition (17).

To derive the Phillips curve (22), we log-linearize firms' first-order condition (12), and use the definition of the real wage $w_t \equiv W_t/P_t$, to get

$$\hat{P}_t^* = (1-\beta\theta)\,\mathbb{E}_t\left\{\sum_{k=0}^{+\infty}(\beta\theta)^k\left[\alpha_\phi\left(i_{t+k}^\ell - i_{t+k}\right) + \hat{w}_{t+k} + \hat{P}_{t+k} - \hat{mp}_{t+k|t}\right]\right\}, \tag{B.1}$$

where $\alpha_\phi \equiv \phi I^\ell / [\phi I^\ell + (1-\phi)I] \in (0,1]$, variables with hats denote log deviations from steady-state values, $i_t^\ell \equiv \hat{I}_t^\ell$, $i_t \equiv \hat{I}_t$, and $mp_{t+k|t}$ denotes the marginal productivity in period $t+k$ for a firm whose price was last set in period t. Log-linearizing the production function (9) gives

$$\hat{h}_t = \frac{f}{f'h}\hat{y}_t, \tag{B.2}$$

so that we can rewrite $\hat{mp}_{t+k|t}$ as

$$\begin{aligned}\hat{mp}_{t+k|t} &= \frac{f''h}{f'}\hat{h}_{t+k|t} = \hat{mp}_{t+k} + \frac{f''h}{f'}\left(\hat{h}_{t+k|t} - \hat{h}_{t+k}\right) \\ &= \hat{mp}_{t+k} + \frac{ff''}{(f')^2}\left(\hat{y}_{t+k|t} - \hat{y}_{t+k}\right) = \hat{mp}_{t+k} - \frac{\varepsilon ff''}{(f')^2}\left(\hat{P}_t^* - \hat{P}_{t+k}\right),\end{aligned} \tag{B.3}$$

where mp_{t+k} denotes the average marginal productivity in period $t+k$. Using this result and

$$\pi_t \equiv \log(\Pi_t) = (1-\theta)\left(\hat{P}_t^* - \hat{P}_{t-1}\right),$$

and following the same steps as in, e.g., Galí (2015, Chapter 3), we can rewrite (B.1) as

$$\pi_t = \beta\mathbb{E}_t\{\pi_{t+1}\} + \frac{(1-\theta)(1-\beta\theta)}{\theta\left[1-\frac{\varepsilon ff''}{(f')^2}\right]}\left[\alpha_\phi\left(i_t^\ell - i_t\right) + \hat{w}_t - \hat{mp}_t\right]. \tag{B.4}$$

Now, log-linearizing the goods-market-clearing condition (17) gives

$$\hat{c}_t = \hat{y}_t. \tag{B.5}$$

Log-linearizing the first-order condition (6), and using (B.5), gives

$$i_t^{\ell} - i_t = \alpha_{\ell}\frac{\Gamma_{\ell\ell}\ell}{\Gamma_{\ell}}\hat{\ell}_t + \alpha_{\ell}\frac{\Gamma_{\ell m}m}{\Gamma_{\ell}}\hat{m}_t + \alpha_{\ell}\sigma\hat{y}_t, \tag{B.6}$$

where $\alpha_{\ell} \equiv (I^{\ell} - I)/I^{\ell} \in (0,1)$. Log-linearizing the first-order condition (5), and using (B.2) and (B.5), gives

$$\hat{w}_t = \left(\sigma + \frac{v''h}{v'}\frac{f}{f'h}\right)\hat{y}_t. \tag{B.7}$$

Log-linearizing the constraint (10) holding with equality, and using (B.2) and (B.7), gives

$$\hat{\ell}_t = \left(\sigma + \frac{v''h}{v'}\frac{f}{f'h} + \frac{f}{f'h}\right)\hat{y}_t. \tag{B.8}$$

Moreover, we have

$$\hat{mp}_t = \frac{ff''}{(f')^2}(\hat{y}_t). \tag{B.9}$$

Using (B.6), (B.7), (B.8), and (B.9), we can then rewrite (B.4) as the Phillips curve (22):

$$\pi_t = \beta\mathbb{E}_t\{\pi_{t+1}\} + \kappa(\hat{y}_t - \delta_m\hat{m}_t)$$

with

$$\begin{aligned}
\kappa &\equiv \left[\sigma + \frac{v''h}{v'}\frac{f}{f'h} - \frac{ff''}{(f')^2} + \alpha_{\ell}\alpha_{\phi}\sigma + \alpha_{\ell}\alpha_{\phi}\frac{\Gamma_{\ell\ell}\ell}{\Gamma_{\ell}}\left(\sigma + \frac{v''h}{v'}\frac{f}{f'h} + \frac{f}{f'h}\right)\right]\Psi > 0,\\
\delta_m &\equiv -\alpha_{\ell}\alpha_{\phi}\left(\frac{\Gamma_{\ell m}m}{\Gamma_{\ell}}\right)\frac{\Psi}{\kappa} > 0,
\end{aligned}$$

where in turn

$$\Psi \equiv \frac{(1-\theta)(1-\beta\theta)}{\theta\left[1 - \frac{\varepsilon ff''}{(f')^2}\right]}.$$

To derive the reserves-demand equation (23), we log-linearize the first-order condition (7) and use (B.5) to get

$$i_t - i_t^m = \alpha_m\frac{\Gamma_{\ell m}\ell}{\Gamma_m}\hat{\ell}_t + \alpha_m\frac{\Gamma_{mm}m}{\Gamma_m}\hat{m}_t + \alpha_m\sigma\hat{y}_t, \tag{B.10}$$

where $i_t^m \equiv \hat{I}_t^m$ and $\alpha_m \equiv (I - I^m)/I^m > 0$. Using (B.8), we can rewrite (B.10) as the reserves-demand equation (23):

$$\hat{m}_t = \chi_y\hat{y}_t - \chi_i(i_t - i_t^m)$$

with

$$\begin{aligned}
\chi_y &\equiv -\left[\sigma + \frac{\Gamma_{\ell m}\ell}{\Gamma_m}\left(\sigma + \frac{v''h}{v'}\frac{f}{f'h} + \frac{f}{f'h}\right)\right]\left(\frac{\Gamma_{mm}m}{\Gamma_m}\right)^{-1} > 0,\\
\chi_i &\equiv \left(-\alpha_m\frac{\Gamma_{mm}m}{\Gamma_m}\right)^{-1} > 0.
\end{aligned}$$

B.2 Unique Local Equilibrium Under Exogenous Instruments

Under permanently exogenous monetary-policy instruments i_t^m and $\hat{M}_t$, the IS equation (21), the Phillips curve (22), the reserves-demand equation (23), and the identities $\hat{m}_t = \hat{M}_t - \hat{P}_t$ and $\pi_t = \hat{P}_t - \hat{P}_{t-1}$ lead to the following dynamic equation relating $\hat{P}_t$ to $\mathbb{E}_t\{\hat{P}_{t+2}\}$, $\mathbb{E}_t\{\hat{P}_{t+1}\}$, $\hat{P}_{t-1}$, and exogenous terms:

$$\mathbb{E}_t\left\{L\mathcal{P}\left(L^{-1}\right)\hat{P}_t\right\} = Z_t,$$

where

$$\begin{aligned}
\mathcal{P}(z) \quad &\equiv \quad z^3 - \left[2 + \frac{1}{\beta} + \frac{\chi_y}{\sigma\chi_i} + \left(\frac{1}{\sigma} - \delta_m\right)\frac{\kappa}{\beta}\right]z^2 + \left[1 + \frac{2}{\beta} + \left(1 + \frac{1}{\beta}\right)\frac{\chi_y}{\sigma\chi_i}\right. \\
&\qquad \left. + \left(\frac{1}{\sigma} - \delta_m\right)\frac{\kappa}{\beta} + (1 - \delta_m\chi_y)\frac{\kappa}{\beta\sigma\chi_i}\right]z - \left(\frac{1}{\beta} + \frac{\chi_y}{\beta\sigma\chi_i}\right),
\end{aligned}$$

$$Z_t \equiv \frac{-\kappa}{\beta\sigma}\left(i_t^m - r_t\right) + \left[\frac{1}{\sigma\chi_i} - \left(1 + \frac{\chi_y}{\sigma\chi_i}\right)\delta_m\right]\frac{\kappa}{\beta}\hat{M}_t + \frac{\delta_m\kappa}{\beta}\mathbb{E}_t\left\{\hat{M}_{t+1}\right\}.$$

Now, our model, given its structure, implies that

$$\sigma < \chi_y < \frac{1}{\delta_m}, \tag{B.11}$$

as we show Appendix B.3. The first inequality in (B.11) arises from the fact that bank loans serve to finance the wage bill (or some fraction of it). If output $\hat{y}_t$ increases by 1%, the marginal utility of consumption decreases by σ%; so, the wage, the wage bill, and loans all increase by more than σ%; and, in turn, so does the demand for reserves $\hat{m}_t$ for a given spread $i_t - i_t^m$ (i.e., $\chi_y > \sigma$). The second inequality in (B.11) reflects how holding reserves mitigates the costs of banking. For a given spread $i_t - i_t^m$, a rise in output $\hat{y}_t$ has two opposite effects on firms' marginal cost of production (i.e., on the term in factor of κ in the Phillips curve): a standard positive direct effect (with elasticity 1), and a negative indirect effect via the implied rise in reserves $\hat{m}_t$ (with elasticity $\chi_y\delta_m$). The inequality states that the direct effect dominates the indirect one (i.e., $\chi_y\delta_m < 1$).

The polynomial $\mathcal{P}(z)$ can be rewritten as

$$\begin{aligned}
\mathcal{P}(z) \quad &= \quad z^3 - \left(\frac{1 + 2\beta + \beta\Theta_1 + \Theta_2}{\beta}\right)z^2 + \left[\frac{2 + \beta + (1+\beta)\Theta_1 + \Theta_2 + \Theta_3}{\beta}\right]z - \left(\frac{1 + \Theta_1}{\beta}\right) \\
&= \quad (z - 1 - \Theta_1)\left[z^2 - \left(\frac{1 + \beta + \Theta_2}{\beta}\right)z + \frac{1}{\beta}\right] - \left(\frac{\Theta_1\Theta_2 - \Theta_3}{\beta}\right)z,
\end{aligned}$$

where $\Theta_1 \equiv \chi_y/(\sigma\chi_i) > 0$, $\Theta_2 \equiv (1/\sigma - \delta_m)\kappa$, and $\Theta_3 \equiv (1 - \delta_m\chi_y)\kappa/(\sigma\chi_i)$. The double inequality (B.11) implies $\Theta_2 > 0$, $\Theta_3 > 0$, and $\Theta_1\Theta_2 - \Theta_3 = (\chi_y - \sigma)\kappa/(\sigma^2\chi_i) > 0$. Therefore, we get $\mathcal{P}(0) = -(1+\Theta_1)/\beta < 0$, $\mathcal{P}(1) = \Theta_3/\beta > 0$, $\mathcal{P}(1+\Theta_1) = -(\Theta_1\Theta_2 - \Theta_3)(1+\Theta_1)/\beta < 0$, and $\lim_{z\in\mathbb{R}, z\to+\infty}\mathcal{P}(z) = +\infty > 0$. As a consequence, the roots of $\mathcal{P}(z)$ are three real numbers ρ, ω_1, and ω_2 such that $0 < \rho < 1 < \omega_1 < 1 + \Theta_1 < \omega_2$.

With one eigenvalue inside the unit circle (ρ) for one predetermined variable ($\hat{P}_{t-1}$), thus, our model satisfies Blanchard and Kahn's (1980) conditions and has a unique bounded solution. We rewrite the dynamic equation as

$$\mathbb{E}_t \left\{ \left(L^{-1} - \omega_1\right) \left(L^{-1} - \omega_2\right) (1 - \rho L) \hat{P}_t \right\} = Z_t$$

and use the method of partial fractions to solve this equation forward and get the unique bounded solution for $\hat{P}_t - \rho \hat{P}_{t-1}$:

$$\begin{aligned}
\hat{P}_t - \rho \hat{P}_{t-1} &= \mathbb{E}_t \left\{ \frac{Z_t}{\left(L^{-1} - \omega_1\right)\left(L^{-1} - \omega_2\right)} \right\} = \frac{\mathbb{E}_t}{\omega_2 - \omega_1} \left\{ \frac{\omega_1^{-1} Z_t}{1 - (\omega_1 L)^{-1}} - \frac{\omega_2^{-1} Z_t}{1 - (\omega_2 L)^{-1}} \right\} \\
&= \frac{\mathbb{E}_t}{\omega_2 - \omega_1} \left\{ \sum_{k=0}^{+\infty} \left(\omega_1^{-k-1} - \omega_2^{-k-1} \right) Z_{t+k} \right\}. \qquad \text{(B.12)}
\end{aligned}$$

Using the price-level solution (B.12), the Phillips curve (22), and the identities $\hat{m}_t = \hat{M}_t - \hat{P}_t$ and $\pi_t = \hat{P}_t - \hat{P}_{t-1}$, we then get

$$\begin{aligned}
\pi_t &= -(1-\rho) \hat{P}_{t-1} + \frac{\mathbb{E}_t}{\omega_2 - \omega_1} \left\{ \sum_{k=0}^{+\infty} \left(\omega_1^{-k-1} - \omega_2^{-k-1} \right) Z_{t+k} \right\}, \\
\hat{y}_t &= -\vartheta \hat{P}_{t-1} + \delta_m \hat{M}_t - \frac{\mathbb{E}_t}{(\omega_2 - \omega_1)\kappa} \left\{ \sum_{k=0}^{+\infty} \left(\xi_1 \omega_1^{-k-1} - \xi_2 \omega_2^{-k-1} \right) Z_{t+k} \right\},
\end{aligned}$$

where $\vartheta \equiv (1-\rho)(1-\beta\rho)/\kappa + \delta_m \rho$ and $\xi_j \equiv \beta(\omega_j + \rho - 1) + \kappa \delta_m - 1$ for $j \in \{1, 2\}$.

B.3 Proof of the Double Inequality (B.11)

To show that $\chi_y < 1/\delta_m$, we define $\Omega \equiv \delta_m \kappa / (\alpha_m \chi_i \Psi) > 0$ and we write

$$\begin{aligned}
\Omega \left(\frac{1}{\delta_m} - \chi_y \right) &= \frac{-\Gamma_{mm} m}{\Gamma_m} \left[\alpha_\ell \alpha_\phi \frac{\Gamma_{\ell\ell} \ell}{\Gamma_\ell} \left(\sigma + \frac{v'' h}{v'} \frac{f}{f' h} + \frac{f}{f' h} \right) \right. \\
&\quad \left. + (1 + \alpha_\ell \alpha_\phi) \sigma + \frac{v'' h}{v'} \frac{f}{f' h} - \frac{f f''}{(f')^2} \right] \\
&\quad + \alpha_\ell \alpha_\phi \frac{\Gamma_{\ell m} m}{\Gamma_\ell} \left[\sigma + \frac{\Gamma_{\ell m} \ell}{\Gamma_m} \left(\sigma + \frac{v'' h}{v'} \frac{f}{f' h} + \frac{f}{f' h} \right) \right] \\
&= \frac{-\Gamma_{mm} m}{\Gamma_m} \left[\sigma + \frac{v'' h}{v'} \frac{f}{f' h} - \frac{f f''}{(f')^2} \right] \\
&\quad - \frac{\alpha_\ell \alpha_\phi \ell m}{\Gamma_\ell \Gamma_m} \left(\sigma + \frac{v'' h}{v'} \frac{f}{f' h} + \frac{f}{f' h} \right) \left[\Gamma_{\ell\ell} \Gamma_{mm} - (\Gamma_{\ell m})^2 \right] \\
&\quad + \alpha_\ell \alpha_\phi \sigma m \left(\frac{\Gamma_{\ell m}}{\Gamma_\ell} - \frac{\Gamma_{mm}}{\Gamma_m} \right).
\end{aligned}$$

The last expression is the sum of three terms (one per line). The first term is positive. So is the second one, given (A.5). And so is the third one, given that

$$\frac{\Gamma_{\ell m}}{\Gamma_\ell} - \frac{\Gamma_{mm}}{\Gamma_m} = \frac{\left(v^{b\prime}\right)^2 \left(g_m^b g_{\ell m}^b - g_\ell^b g_{mm}^b\right)}{\Gamma_\ell \Gamma_m} = \frac{\left(v^{b\prime}\right)^2 \left(f_h^b f_{mm}^b - f_m^b f_{hm}^b\right)}{\Gamma_\ell \Gamma_m \left(f_h^b\right)^3} > 0. \qquad \text{(B.13)}$$

Therefore, the whole expression is positive, which implies that $\chi_y < 1/\delta_m$.

To show that $\sigma < \chi_y$, we write

$$\begin{aligned}\frac{1}{\alpha_m \sigma \chi_i}(\chi_y - \sigma) &= \frac{\Gamma_{mm} m}{\Gamma_m} + 1 + \frac{1}{\sigma}\frac{\Gamma_{\ell m}\ell}{\Gamma_m}\left(\sigma + \frac{v''h}{v'}\frac{f}{f'h} + \frac{f}{f'h}\right) \\ &= 1 + \frac{1}{\sigma}\frac{\Gamma_{\ell m}\ell}{\Gamma_m}\left(\frac{v''h}{v'}\frac{f}{f'h} + \frac{f}{f'h}\right) + \frac{1}{\Gamma_m}(\ell\Gamma_{\ell m} + m\Gamma_{mm}).\end{aligned}$$

The last expression is the sum of three terms. The first two terms are positive. And so is the third one, given that

$$\begin{aligned}\ell\Gamma_{\ell m} + m\Gamma_{mm} &= (1-d)\,\ell\Gamma_{\ell m} + d\ell\Gamma_{\ell m} + m\Gamma_{mm} \\ &= (1-d)\,\ell\Gamma_{\ell m} + d\ell\left[v^{b\prime\prime} g_\ell^b g_m^b + v^{b\prime} g_{\ell m}^b\right] + m\left[v^{b\prime\prime}\left(g_m^b\right)^2 + v^{b\prime} g_{mm}^b\right] \\ &= (1-d)\,\ell\Gamma_{\ell m} + v^{b\prime\prime} g_m^b\left(d\ell g_\ell^b + m g_m^b\right) + v^{b\prime}\left(d\ell g_{\ell m}^b + m g_{mm}^b\right) \\ &= (1-d)\,\ell\Gamma_{\ell m} + v^{b\prime\prime} g^b g_m^b \\ &\leq 0, \end{aligned} \tag{B.14}$$

where the last equality follows from (A.2) and (A.4). Therefore, the whole expression is positive, which implies that $\sigma < \chi_y$.

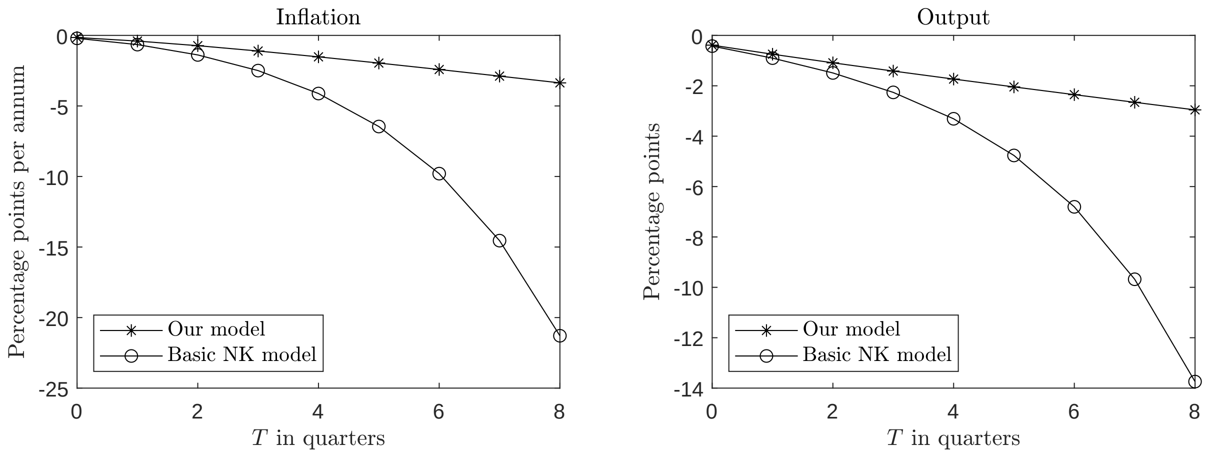
B.4 Comparison With the Basic NK Model

In Subsection 2.3 of the main text, we contrast the implications of our model for inflation during a temporary ZLB episode with the implications of the basic NK model. More specifically, we show that the deflation rate at the start of the ZLB episode grows exponentially with the expected duration of the ZLB episode in the basic NK model, while in our model it converges to a finite value as the expected duration of the ZLB episode goes to infinity.

In this appendix, we illustrate these results numerically. We consider the same calibration of our model as in Subsection 3.1, and we consider the corresponding calibration of the basic NK model, i.e. we set the structural parameters of the basic NK model to the values that they take in our calibrated model (given that all the structural parameters of the basic NK model are also structural parameters of our model). In both models, we set the value of the discount-factor shock (leading to the ZLB) based on Cúrdia's (2015) average estimate of the natural rate of interest over the 2009Q1-2015Q4 ZLB episode. This average estimate is -3.1% per year; average CPI inflation over the same period is 1.4% per year; so, we get an estimate of Cúrdia's "interest-rate gap" at -3.1+1.4=-1.7% per year, and we thus set $i_t^m - r_t$ to 1.7% per year during our ZLB episode (i.e. $z^* = 1.7\%$ in the notation of Subsection 2.3).

Figure B.1 shows the value of inflation and output at the start of the ZLB episode, depending on the expected duration T of the ZLB episode, in the basic NK model and in our model. In the basic NK model (but not in our model), inflation and output decrease exponentially with

Figure B.1 – Effects of a ZLB episode of expected duration T on inflation and output at the start of the episode



T, and their fall quickly reaches substantial values: -21% per year for inflation and -14% for output for an expected ZLB duration of 8 quarters (i.e. 2 years).

Eight quarters for the expected ZLB duration does not seem an unrealistic figure to us. The Survey of Primary Dealers of the Federal Reserve Bank of New York indicates that in September 2012, primary dealers were expecting the ZLB episode to last for the next 11 quarters (until mid-2015), as emphasized by Yellen (2012). In October and December 2012, as well as in January 2013, they were still expecting the ZLB episode to last for the next 10 quarters.[2]

B.5 Determinacy Condition Under a Floor System

Using the IS equation (21), the Phillips curve (22), the reserves-demand equation (23), the Taylor rule (26), and the identities $\hat{m}_t = \hat{M}_t - \hat{P}_t$ and $\pi_t = \hat{P}_t - \hat{P}_{t-1}$, we get a dynamic equation relating $\hat{P}_t$ to $\mathbb{E}_t\{\hat{P}_{t+2}\}$, $\mathbb{E}_t\{\hat{P}_{t+1}\}$, $\hat{P}_{t-1}$, and exogenous terms, whose characteristic polynomial is

$$\mathcal{P}_r(z) \equiv z^3 - a_2 z^2 + a_1 z - a_0$$

with

$$\begin{aligned}
a_2 &\equiv 2 + \frac{1}{\beta} + \frac{(1-\sigma\delta_m)\,\kappa}{\beta\sigma} + \frac{\chi_y}{\sigma\chi_i} + \frac{r_y}{\sigma} > 0, \\
a_1 &\equiv 1 + \frac{2}{\beta} + \frac{(1-\sigma\delta_m)\,\kappa}{\beta\sigma} + \frac{(1+\beta)\,\chi_y}{\beta\sigma\chi_i} + \frac{(1-\delta_m\chi_y)\,\kappa}{\beta\sigma\chi_i} + \frac{\kappa r_\pi}{\beta\sigma} + \frac{(1+\beta-\delta_m\kappa)\,r_y}{\beta\sigma}, \\
a_0 &\equiv \frac{1}{\beta} + \frac{\chi_y}{\beta\sigma\chi_i} + \frac{\kappa r_\pi}{\beta\sigma} + \frac{r_y}{\beta\sigma} > 0,
\end{aligned}$$

where the first inequality follows from the double inequality (B.11). Given that there is exactly one predetermined variable ($\hat{P}_{t-1}$), the necessary and sufficient condition for local-equilibrium

[2] Similarly, the Blue-Chip Survey of Forecasters indicates that in several months of 2011-2013, the Fed Funds rate was expected to remain constant for "7 or more" quarters. And the Philadelphia Fed's Survey of Professional Forecasters indicates that in 2012Q1, the 3-month T-bill rate was expected to remain at 0.10% for at least 7 quarters (i.e. at least throughout both 2012 and 2013).

determinacy is that $\mathcal{P}_r(z)$ have exactly one root inside the unit circle. This root must be a real number (indeed, if it were a complex number, its conjugate would be another root inside the unit circle). We have $\mathcal{P}_r(0) = -a_0 < 0$ and $\mathcal{P}_r(1) = (1 - \delta_m \chi_y - \delta_m \chi_i r_y)\kappa/(\beta\sigma\chi_i)$. In the following, we consider two alternative cases in turn, depending on the sign of $\mathcal{P}_r(1)$.

We first consider the case in which $\mathcal{P}_r(1) > 0$, that is to say equivalently the case in which

$$r_y < \zeta_1, \tag{B.15}$$

where

$$\zeta_1 \equiv \frac{1 - \delta_m \chi_y}{\delta_m \chi_i} > 0,$$

where in turn the last inequality follows from the second inequality in (B.11). In this case, $\mathcal{P}_r(0)$ and $\mathcal{P}_r(1)$ are of opposite signs, so that $\mathcal{P}_r(z)$ has either one or three real roots inside $(0, 1)$. Moreover, in this case, we have

$$a_1 = 1 + \frac{2}{\beta} + \frac{(1 - \sigma\delta_m)\,\kappa}{\beta\sigma} + \frac{(1 + \beta)\,\chi_y}{\beta\sigma\chi_i} + \frac{\kappa r_\pi}{\beta\sigma} + \frac{(1 + \beta)\,r_y}{\beta\sigma} + \frac{(\zeta_1 - r_y)\,\delta_m \kappa}{\beta\sigma} > 0,$$

where the inequality comes from (B.11) and (B.15). In turn, $a_1 > 0$, together with $a_0 > 0$ and $a_2 > 0$, implies that $\mathcal{P}_r(z) < 0$ for all $z < 0$, and hence that $\mathcal{P}_r(z)$ has no negative real roots. So, $\mathcal{P}_r(z)$ has at least one real root inside $(0, 1)$, which we denote by ρ, and its other two roots, which we denote by ω_1 and ω_2 with $|\omega_1| \leq |\omega_2|$, are either (i) both real and inside $(0, 1)$, or (ii) both real and higher than 1, or (iii) both complex and conjugates of each other. Now, we have $\rho + \omega_1 + \omega_2 = a_2 > 3$, where the inequality follows from the double inequality (B.11) and from $\beta < 1$. Therefore, Case (i) is impossible, and in Case (iii) the common real part of ω_1 and ω_2 is higher than 1. So, in the remaining two possible cases, namely Cases (ii) and (iii), ω_1 and ω_2 lie outside the unit circle. As a consequence, we get local-equilibrium determinacy.

We now turn to the alternative case in which $\mathcal{P}_r(1) < 0$, that is to say equivalently the case in which Condition (B.15) is not met. In this case, $\mathcal{P}_r(0)$ and $\mathcal{P}_r(1)$ have the same sign, so that $\mathcal{P}_r(z)$ has either zero or two real roots inside $(0, 1)$. Therefore, a necessary condition for local-equilibrium determinacy is then that $\mathcal{P}_r(-1)$ be of the opposite sign, i.e. $\mathcal{P}_r(-1) > 0$, so that $\mathcal{P}_r(z)$ can have either one or three real roots inside $(-1, 0)$. This necessary condition for determinacy can be written as

$$[\delta_m \kappa - 2\,(1 + \beta)]\,r_y > 4\,(1 + \beta)\,\sigma + \frac{2\,(1 + \beta)\,\chi_y}{\chi_i} + 2\,(1 - \sigma\delta_m)\,\kappa + \frac{(1 - \delta_m \chi_y)\,\kappa}{\chi_i} + 2\kappa r_\pi.$$

The right-hand side of this inequality is positive, given the double inequality (B.11). Therefore, the necessary condition for determinacy can be equivalently rewritten as

$$\delta_m \kappa > 2\,(1 + \beta) \quad \text{and} \quad r_y > \zeta_2 + \zeta_3 r_\pi, \tag{B.16}$$

where

$$\begin{aligned}
\zeta_2 &\equiv \frac{4\,(1 + \beta)\,\sigma\chi_i + 2\,(1 + \beta)\,\chi_y + 2\,(1 - \sigma\delta_m)\,\kappa\chi_i + (1 - \delta_m \chi_y)\,\kappa}{[\delta_m \kappa - 2\,(1 + \beta)]\,\chi_i} > \zeta_1, \\
\zeta_3 &\equiv \frac{2\kappa}{\delta_m \kappa - 2\,(1 + \beta)} > 0,
\end{aligned}$$

where in turn the last two inequalities follow from the first inequality in (B.16). We now show that Condition (B.16) is not only necessary, but also sufficient for local-equilibrium determinacy in that case. To that aim, assume that this condition is met. Then, $\mathcal{P}_r(-1)$ and $\mathcal{P}_r(0)$ are of opposite signs, so that $\mathcal{P}_r(z)$ has either one or three real roots inside $(-1, 0)$. Let ρ denote one root of $\mathcal{P}_r(z)$ inside $(-1, 0)$. The other two roots of $\mathcal{P}_r(z)$, which we denote by ω_1 and ω_2 with $|\omega_1| \leq |\omega_2|$, can be either (i) both real and inside $(-1, 0)$, or (ii) both real and inside $(0, 1)$, or (iii) both real and outside $(-1, 1)$, or (iv) both complex and conjugates of each other. Since $\rho + \omega_1 + \omega_2 = a_2 > 3$, however, Cases (i) and (ii) are impossible, and in Case (iv) the common real part of ω_1 and ω_2 is higher than 1. Therefore, in the remaining two possible cases, namely Cases (iii) and (iv), ω_1 and ω_2 lie outside the unit circle. As a consequence, Condition (B.16) is, indeed, sufficient for local-equilibrium determinacy in that case.

From the results obtained in the two alternative cases considered, we get that there is local-equilibrium determinacy if and only if either Condition (B.15) is met, or Condition (B.15) is not met and Condition (B.16) is met. Now, Conditions (B.15) and (B.16) are mutually exclusive, given that $\zeta_2 > \zeta_1$. We conclude that there is local-equilibrium determinacy if and only if either Condition (B.15) or Condition (B.16) is met.

B.6 Floor system and supply shock

In this appendix, motivated in part by the surge in US inflation in 2021-2022, we introduce a supply shock A_t into our model and compare the performance of alternative rules that link the policy rate to inflation. We rewrite the production function as

$$y_t(i) = A_t f[h_t(i)].$$

We log-linearize the model in the same way as in Appendix B.1. The IS equation (21) is unchanged, but a new term in $\hat{A}_t$ appears in the Phillips curve and the reserves-market-clearing condition:

$$\begin{aligned}
\pi_t &= \beta \mathbb{E}_t\{\pi_{t+1}\} + \kappa\left(\hat{y}_t - \delta_m \hat{m}_t - \delta_a \hat{A}_t\right), \\
\hat{m}_t &= \chi_y \hat{y}_t - \chi_i(i_t - i_t^m) - \chi_a \hat{A}_t,
\end{aligned}$$

where

$$\begin{aligned}
\delta_a &\equiv \left[1 + \frac{v''h}{v'}\frac{f}{f'h} - \frac{ff''}{(f')^2} + \alpha_\ell \alpha_\phi \frac{\Gamma_{\ell\ell}\ell}{\Gamma_\ell}\left(\frac{v''h}{v'}\frac{f}{f'h} + \frac{f}{f'h}\right)\right]\frac{\Psi}{\kappa} > 0 \\
\chi_a &\equiv \alpha_m \frac{\Gamma_{\ell m}\ell}{\Gamma_m}\left(\frac{v''h}{v'}\frac{f}{f'h} + \frac{f}{f'h}\right)\chi_i > 0.
\end{aligned}$$

A positive supply shock ($\hat{A}_t > 0$) reduces the marginal cost of production for a given output level and a given level of real reserve balances (hence the negative term $-\delta_a \hat{A}_t$ in the Phillips curve), and it also reduces the demand for reserves for a given output level and a given spread (hence the negative term $-\chi_a \hat{A}_t$ in the reserves-market-clearing condition).

We consider a floor system in which M_t is set exogenously and i_t^m is set according to the simple rule

$$i_t^m = r_\pi \pi_t,$$

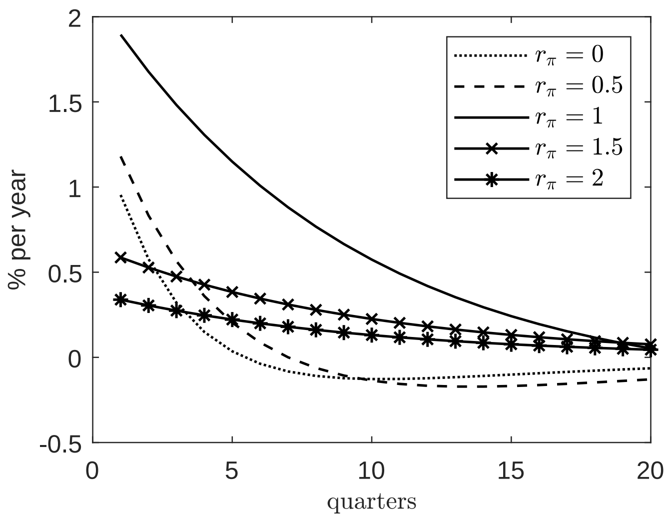
where $r_\pi \geq 0$. This floor system delivers local-equilibrium determinacy, as we have shown in Section 5.2 and Appendix B.5. Finally, we assume that $\hat{A}_t$ follows an AR(1) process:

$$\hat{A}_t = \rho_a \hat{A}_{t-1},$$

where $\rho_a \in [0, 1]$.

For the standard value $\rho_a = 0.9$, Figure B.2 shows the response of inflation to a 1% negative supply shock under alternative policies with different values of r_π. Interestingly, the performance of the policy rule deteriorates – in terms of stabilizing inflation over horizons up to 10 quarters – as we go from $r_\pi = 0$ to $r_\pi = 0.5$, and then to $r_\pi = 1$.

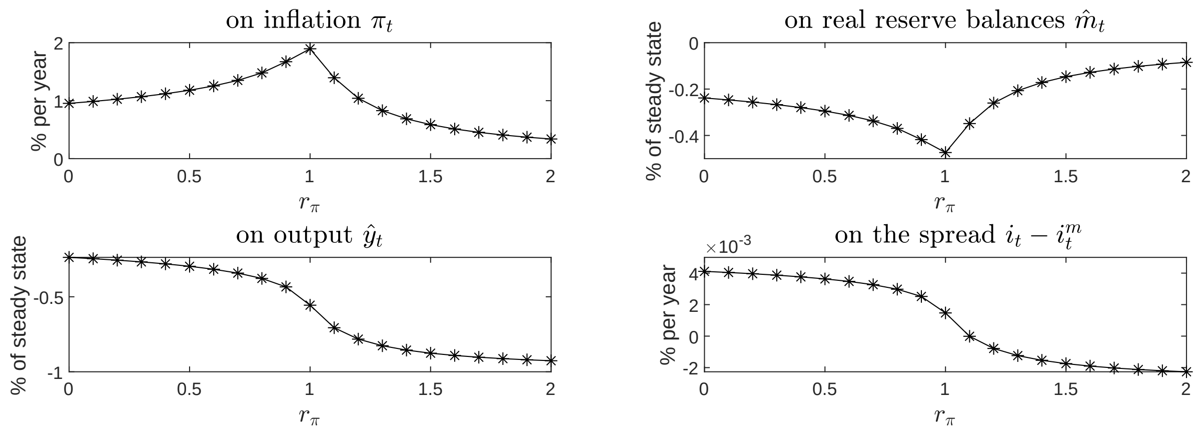
Figure B.2 – Response of inflation to a 1% negative supply shock, depending on r_π



Changing r_π affects inflation through two channels in our model. One channel is the familiar one: for $r_\pi > 1$, an inflationary shock leads to an increase in the *real* policy rate. The second channel is a cost channel of monetary policy reflected in Equations (22) and (23). A stronger response to inflation leads to a sharper drop in output, which leads to a sharper drop in real money balances in (23). The sharper drop in real money balances, in turn, raises banking costs, which raises inflation in (22).

To be more precise about how the cost channel works in our model, we focus on the *impact* effects of the 1% adverse supply shock under alternative rules. Figure B.3 shows the contemporaneous impact of the shock on inflation, real reserve balances, output and the spread, depending on r_π. Since nominal reserve balances are constant, the impact on inflation (top left panel of Figure B.3) is the opposite of the impact on real reserve balances (top right panel of Figure B.3), up to a factor 4 (as inflation is expressed in % per year, not in % per quarter). The inflationary impact of the adverse supply shock is non-monotonic in the rule's coefficient r_π: it first increases with r_π for r_π below one, and then decreases with r_π for r_π above one (top left panel of Figure B.3). This non-monotonicity can be understood as the result of two opposite effects: an output effect, and a spread effect.

Figure B.3 – Impact of a 1% negative supply shock, depending on r_π



The output effect is the following: for a given increase in inflation (triggered by the adverse supply shock), as r_π increases, I_t^m rises further, and so does I_t. The increase in I_t has a contractionary effect on output (bottom left panel of Figure B.3). For a given spread $I_t - I_t^m$, this output contraction makes the demand for real reserve balances fall in the reserves-market-clearing condition (23). Since nominal reserve balances are constant, prices rise. So, the output effect makes inflation increase with r_π, for a given spread.

The spread effect is that the increase in r_π directly raises I_t^m, which in turn indirectly raises I_t, but I_t rises by less than I_t^m; so, the spread $I_t - I_t^m$ is compressed (bottom right panel of Figure B.3). For a given output y_t, this spread compression makes the demand for real reserve balances increase in the reserves-market-clearing condition (23). Since nominal reserve balances are constant, prices fall. So, the spread effect makes inflation decrease with r_π, for a given output.

The response of inflation to the adverse supply shock can thus increase or decrease with the rule's coefficient r_π, depending on which of the two (output and spread) effects dominates the other. To further understand why inflation increases (resp. decreases) with r_π for small (resp. large) values of r_π, we can solve for the unique local equilibrium of the model in the same way as in Appendix B.2; we then get

$$\begin{aligned}
\pi_t &= -(1-\rho)\hat{P}_{t-1} + \frac{(1-\delta_m\chi_y)\kappa}{\beta\sigma\chi_i(\omega_1-1)(\omega_2-1)}\hat{M}_{t-1} \\
&+\frac{\kappa}{\beta(\omega_2-\omega_1)}\mathbb{E}_t\left\{\frac{1}{\sigma}\sum_{k=0}^{+\infty}\left(\omega_1^{-k-1}-\omega_2^{-k-1}\right)r_{t+k}\right. \\
&+\sum_{k=0}^{+\infty}\left[\left(\frac{1-\delta_m\chi_y}{\sigma\chi_i}\right)\left(\frac{\omega_1^{-k}}{\omega_1-1}-\frac{\omega_2^{-k}}{\omega_2-1}\right)+\delta_m\left(\omega_1^{-k}-\omega_2^{-k}\right)\right]\hat{\mu}_{t+k} \\
&\left.+\sum_{k=0}^{+\infty}\left[\left(\frac{\chi_a-\delta_a\chi_y}{\sigma\chi_i}\right)\left(\frac{\omega_1^{-k}}{\omega_1-1}-\frac{\omega_2^{-k}}{\omega_2-1}\right)+\delta_a\left(\omega_1^{-k}-\omega_2^{-k}\right)\right]\Delta\hat{A}_{t+k}\right\}, \qquad \text{(B.17)}
\end{aligned}$$

where ρ, ω_1 and ω_2 denote again the three roots of the characteristic polynomial, with $|\rho| < 1 < |\omega_1| \leq |\omega_2|$. There are two changes compared to Equation (25). First, inflation now responds to the (newly introduced) supply shock $\hat{A}_t$. Second, the roots ρ, ω_1 and ω_2, which characterize the dynamic responses of inflation to all shocks, now depend on the coefficient r_π of the IOR-rate rule.

Focusing on the supply shock (i.e. setting $\hat{M}_{t-1} = 0$ and $r_{t+k} = \hat{\mu}_{t+k} = 0$ for all $k \geq 0$) and using the AR(1) assumption for this shock (i.e. using $\mathbb{E}_t\{\Delta\hat{A}_{t+k}\} = -(1-\rho_a)\rho_a^{k-1}\hat{A}_t$ for all $k \geq 1$), we can rewrite (B.17) as

$$\begin{aligned}
\pi_t &= -(1-\rho)\hat{P}_{t-1} + \frac{(\delta_a\chi_y-\chi_a)\kappa}{\beta\sigma\chi_i(\omega_1-1)(\omega_2-1)}\hat{A}_{t-1} \\
&-\frac{\kappa}{\beta(\omega_1-\rho_a)(\omega_2-\rho_a)}\left[\frac{\delta_a\chi_y-\chi_a}{\sigma\chi_i}+\delta_a(1-\rho_a)\right]\hat{A}_t. \qquad \text{(B.18)}
\end{aligned}$$

It is easy to show that $\delta_a\chi_y - \chi_a > 0$ for iso-elastic production and utility functions. So, the coefficient of $\hat{A}_t$ in (B.18) is always negative:

$$\frac{\partial\pi_t}{\partial\hat{A}_t} = -\frac{\kappa}{\beta(\omega_1-\rho_a)(\omega_2-\rho_a)}\left[\frac{\delta_a\chi_y-\chi_a}{\sigma\chi_i}+\delta_a(1-\rho_a)\right] < 0,$$

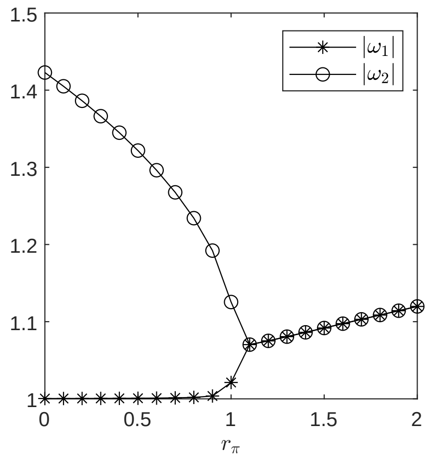
that is to say that a negative supply shock ($\hat{A}_t < 0$) always raises inflation ($\pi_t > 0$).

The only way in which r_π affects $\partial\pi_t/\partial\hat{A}_t$ is via ω_1 and ω_2. We still have $1 < |\omega_1| \leq |\omega_2|$ (as we had for $r_\pi = 0$), but now ω_1 and ω_2 may be complex conjugates. For the calibration that we use in the paper, as r_π changes, $|\omega_1|$ and $|\omega_2|$ move as described in Figure B.4.

For r_π between 0 and 1, ω_1 and ω_2 are real numbers, with $1 < \omega_1 < \omega_2$; ω_1 varies very little (remains very close to 1), while ω_2 decreases with r_π; so, $\partial\pi_t/\partial\hat{A}_t$ increases with r_π. Alternatively, for r_π between 1.1 and 2, ω_1 and ω_2 are complex conjugates, and their common modulus $|\omega_1| = |\omega_2|$ increases with r_π; so, $\partial\pi_t/\partial\hat{A}_t$ decreases with r_π.

The result that we have emphasized – that the inflationary impact of the adverse supply shock is maximal when the rule's coefficient is around one – is robust to alternative specifications for

Figure B.4 – $|\omega_1|$ and $|\omega_2|$, depending on r_π

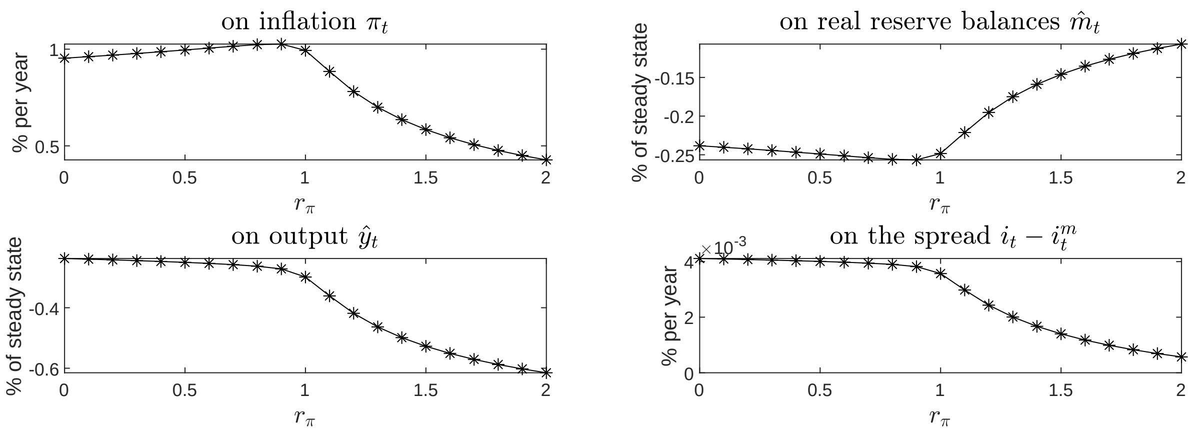


the IOR-rate rule and alternative model calibrations. We consider two other rules:

$$i_t^m = r_\pi \pi_{t-4},$$
$$i_t^m = 0.8 i_{t-1}^m + 0.2 r_\pi \pi_t,$$

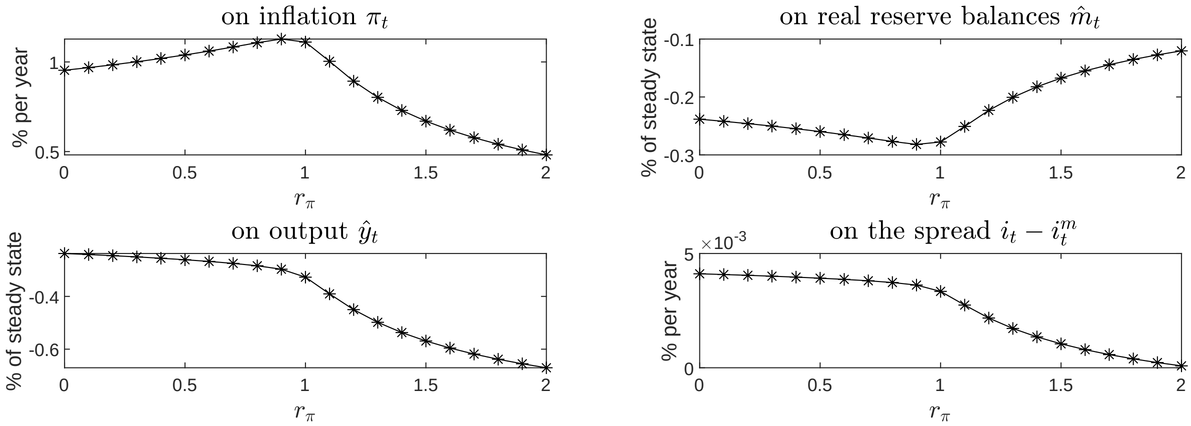
where $r_\pi \geq 0$. The first (lagged) rule is motivated by the fact that central banks started to raise rates in 2022Q2, about four quarters after inflation started to surge (2021Q2). The second (inertial) rule is commonly considered in the literature. Figures B.5 and B.6 are the counterparts of Figure B.3 for these alternative rules. The three figures are broadly similar, although the rise of the inflationary impact of the shock as a function of r_π for $0 < r_\pi < 1$ is less pronounced in Figures B.5 and B.6 than in Figure B.3.

Figure B.5 – Impact of a 1% negative supply shock (lagged rule), depending on r_π



We also consider a range of alternative values for the persistence parameter ρ_a of the supply shock and for the steady-state spread $I - I^m$ (which measures how far we are from satiation at the steady state). Values of ρ_a ranging from 0 to 0.99 (benchmark: 0.9) may change the results

Figure B.6 – Impact of a 1% negative supply shock (inertial rule), depending on r_π



quantitatively, but not qualitatively. Values of $I - I^m$ ranging from 5 to 50 basis points per annum (benchmark: 10 basis points per annum) do not change the results, neither quantitatively nor qualitatively.

Appendix C: Benchmark Model With Reserves-Supply Rule

In the main text (Section 2), we show that our benchmark model delivers local-equilibrium determinacy under exogenous monetary-policy instruments, and we use this determinacy result to explain the low volatility of inflation and the absence of significant deflation at the ZLB. Our (simplifying) assumption of exogenous nominal reserves does not seem to us like a bad approximation of reality, given how the Fed has announced in advance a path for its balance sheet. Nonetheless, the alternative assumption of endogenous nominal reserves (i.e. a reserves-supply rule) seems also relevant. In the present appendix, we show that our determinacy result is essentially robust to the endogenization of nominal reserves. More specifically, we consider a reserves-supply rule in our benchmark model; we derive a sufficient condition for local-equilibrium determinacy under this rule and an exogenous IOR rate; and we argue that this sufficient determinacy condition is likely to be met.

C.1 Reserves-Supply Rule, Steady State, and Log-Linearization

We assume that the central bank sets the stock of nominal reserves according to the rule

$$M_t = P_t \mathcal{Q}\left(P_t, y_t\right), \tag{C.1}$$

where the function $\mathcal{Q}$, from $\mathbb{R}^2_{>0}$ to $\mathbb{R}_{>0}$, is differentiable, decreasing in P_t ($\mathcal{Q}_P < 0$), and non-increasing in y_t ($\mathcal{Q}_y \leq 0$). This assumption ensures that real reserve balances respond negatively to the price level for a given output level, and non-positively to the output level for a given price level. This specification nests, in particular, the case of (constant) exogenous nominal reserves considered in the rest of the paper, which corresponds to $\mathcal{Q}(P_t, y_t) = M/P_t$ with $M > 0$.

This reserves-supply rule does not change any of the steady-state-equilibrium conditions stated in Subsection 2.1. In particular, in any steady state, because real money balances and real output are constant over time (by definition of a steady state), the reserves-supply rule (C.1) implies that the price level is constant over time as well, like previously. Therefore, the necessary and sufficient condition on I^m for existence and uniqueness of a steady state is still $I^m < 1/\beta$.

Log-linearizing the model around its unique steady state, we get the same IS equation (21), Phillips curve (22), and reserves-demand equation (23) as previously, plus now the rule

$$\hat{m}_t = -q_P \hat{P}_t - q_y \hat{y}_t, \tag{C.2}$$

where $q_P \equiv -P\mathcal{Q}_P(P,y)/\mathcal{Q}(P,y) > 0$ and $q_y \equiv -y\mathcal{Q}_y(P,y)/\mathcal{Q}(P,y) \geq 0$ (variables without time subscript denote steady-state values).

C.2 Derivation of a Sufficient Determinacy Condition

Using the four log-linearized equations just mentioned and the identity $\pi_t = \hat{P}_t - \hat{P}_{t-1}$, we get the following dynamic system in $\hat{m}_t$ and $\hat{y}_t$:

$$\mathbb{E}_t \left\{ \begin{bmatrix} \hat{m}_{t+1} \\ \hat{m}_t \\ \hat{y}_{t+1} \\ \hat{y}_t \end{bmatrix} \right\} = \mathbf{A} \begin{bmatrix} \hat{m}_t \\ \hat{m}_{t-1} \\ \hat{y}_t \\ \hat{y}_{t-1} \end{bmatrix} + \mathbf{B} \begin{bmatrix} i_t^m \\ r_t \end{bmatrix}, \tag{C.3}$$

where

$$\begin{aligned}
\mathbf{A} \equiv & \begin{bmatrix} \frac{1+\beta-\delta_m\kappa}{\beta} & \frac{-1}{\beta} & \frac{\kappa}{\beta} & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1-\delta_m\kappa}{\beta\sigma} - \frac{1}{\sigma\chi_i} & \frac{-1}{\beta\sigma} & 1 + \frac{\chi_y}{\sigma\chi_i} + \frac{\kappa}{\beta\sigma} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} + (q_P - 1) \begin{bmatrix} \frac{-\delta_m\kappa}{\beta} & 0 & \frac{\kappa}{\beta} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
& + \frac{q_P - 1}{q_P} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{-1}{\beta\sigma} & \frac{1}{\beta\sigma} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + q_y \begin{bmatrix} \frac{\delta_m\kappa}{\beta\sigma} + \frac{1}{\sigma\chi_i} & 0 & \frac{1}{\beta} - \frac{\kappa}{\beta\sigma} - \frac{\chi_y}{\sigma\chi_i} & \frac{-1}{\beta} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
& + \frac{q_y}{q_P} \begin{bmatrix} \frac{-1}{\beta\sigma} & \frac{1}{\beta\sigma} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\beta\sigma} & \frac{-1}{\beta\sigma} \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{q_y^2}{q_P} \begin{bmatrix} 0 & 0 & \frac{-1}{\beta\sigma} & \frac{1}{\beta\sigma} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

$$\text{and} \quad \mathbf{B} \equiv \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{\sigma} & \frac{-1}{\sigma} \\ 0 & 0 \end{bmatrix} + q_y \begin{bmatrix} \frac{-1}{\sigma} & \frac{1}{\sigma} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Since this system has two predetermined variables ($\hat{m}_{t-1}$ and $\hat{y}_{t-1}$) and two non-predetermined variables ($\hat{m}_{t+1}$ and $\hat{y}_{t+1}$), the necessary and sufficient condition for local-equilibrium determinacy is that the matrix $\mathbf{A}$ have two eigenvalues inside the unit circle and two eigenvalues outside. We write the characteristic polynomial of $\mathbf{A}$ as

$$\det\left(\mathbf{A} - z\mathbf{I}_4\right) = z\mathcal{Q}\left(z\right),$$

where $\mathbf{I}_4$ denotes the 4×4 identity matrix and

$$\begin{aligned}
\mathcal{Q}\left(z\right) \quad \equiv \quad & \mathcal{P}\left(z\right) - \left[\left(\frac{\delta_m \kappa}{\beta\sigma} + \frac{1}{\sigma\chi_i}\right) q_y - \frac{\delta_m \kappa}{\beta}\left(q_P - 1\right)\right] z^2 \\
& + \left[\left(\frac{\delta_m \kappa}{\beta\sigma} + \frac{1+\beta}{\beta\sigma\chi_i}\right) q_y + \left(\frac{-\delta_m \kappa}{\beta} + \frac{\kappa}{\beta\sigma\chi_i} - \frac{\delta_m \chi_y \kappa}{\beta\sigma\chi_i}\right)\left(q_P - 1\right)\right] z - \frac{q_y}{\beta\sigma\chi_i},
\end{aligned}$$

where in turn $\mathcal{P}\left(z\right)$ is defined in Appendix B.2. Thus, the eigenvalues of $\mathbf{A}$ are 0 and the roots of $\mathcal{Q}\left(z\right)$. We have

$$\mathcal{Q}\left(0\right) = \frac{-1}{\beta} - \frac{\chi_y}{\beta\sigma\chi_i} - \frac{q_y}{\beta\sigma\chi_i} < -1 \quad \text{and} \quad \mathcal{Q}\left(1\right) = \frac{\left(1 - \delta_m \chi_y\right)\kappa q_P}{\beta\sigma\chi_i} > 0,$$

where the second inequality follows from (B.11). Therefore, $\mathcal{Q}\left(z\right)$ has either one root or three roots in the real-number interval $(0,1)$. Now, the product of the three roots of $\mathcal{Q}\left(z\right)$ is equal to $-\mathcal{Q}\left(0\right) > 1$, so that $\mathcal{Q}\left(z\right)$ has at least one root outside the unit circle. As a consequence, $\mathcal{Q}\left(z\right)$ has exactly one root inside the real-number interval $(0,1)$.

The other roots of $\mathcal{Q}\left(z\right)$ are either two real numbers outside $[0,1]$, or two conjugate complex numbers. In the latter case, both are outside the unit circle, since $\mathcal{Q}\left(z\right)$ has at least one root outside it. Therefore, $\mathcal{Q}\left(z\right)$ has exactly two roots outside the unit circle if and only if it has no root inside the real-number interval $[-1,0)$. Since $\mathcal{Q}\left(0\right) < 0$, this condition is equivalent to $\mathcal{Q}\left(z\right) < 0$ for all $z \in [-1,0]$. Thus, the necessary and sufficient condition for determinacy is $\mathcal{Q}\left(z\right) < 0$ for all $z \in [-1,0]$.

A *sufficient* condition for determinacy is, therefore, that $\mathcal{Q}\left(z\right) < 0$ for all $z \in [-1,0]$ *and all* $\theta \in (0,1)$. To restate this sufficient condition, we rewrite $\mathcal{Q}(z)$ as

$$\mathcal{Q}\left(z\right) = \frac{\kappa}{\beta}\left[\mathcal{Q}_1\left(z\right) + \mathcal{Q}_2\left(z\right)\right],$$

where

$$\begin{aligned}
\mathcal{Q}_1\left(z\right) \quad & \equiv \quad \frac{-1}{\kappa}\left(1-z\right)\left(1-\beta z\right)\left(1 + \frac{\chi_y + q_y}{\sigma\chi_i} - z\right), \\
\mathcal{Q}_2\left(z\right) \quad & \equiv \quad \left(\frac{1 + \delta_m q_y}{\sigma} - \delta_m q_P\right) z\left(1-z\right) + \left(\frac{1 - \delta_m \chi_y}{\sigma\chi_i}\right) q_P z.
\end{aligned}$$

The only reduced-form parameter that depends on the degree of price stickiness θ is the slope of the Phillips curve κ, which is decreasing in θ. Therefore, for any $z \in [-1,0]$, $\mathcal{Q}_1\left(z\right)$ is decreasing in θ, while $\mathcal{Q}_2\left(z\right)$ does not depend on θ. As a consequence, $\mathcal{Q}\left(z\right) < 0$ for all $z \in [-1,0]$ and all $\theta \in (0,1)$ if and only if $\mathcal{Q}\left(z\right) < 0$ for all $z \in [-1,0]$ as $\theta \to 0$. In turn, this condition is equivalent to $\mathcal{Q}_2(z) < 0$ for all $z \in [-1, 0)$, since $\lim_{\theta\to 0} \kappa = +\infty$ implies $\lim_{\theta\to 0} \mathcal{Q}_1(z) = 0$. Now, we can rewrite $\mathcal{Q}_2(z)$ as

$$\mathcal{Q}_2(z) = z\left\{Kz + \left[\left(\frac{1-\delta_m\chi_y}{\sigma\chi_i}\right)q_P - K\right]\right\},$$

where $K \equiv \delta_m q_P - (1+\delta_m q_y)/\sigma$. Therefore, $\mathcal{Q}_2(z) < 0$ for all $z \in [-1, 0)$ if and only if

$$K < \left(\frac{1-\delta_m\chi_y}{2\sigma\chi_i}\right)q_P, \tag{C.4}$$

where the right-hand side is non-negative, as follows from the second inequality in (B.11).

Now consider the following condition:

$$q_P \leq \left\{\left(\frac{I^\ell - I}{I^\ell}\right)\left[\frac{-m\Gamma_{\ell m}(\ell, m)}{\Gamma_\ell(\ell, m)}\right]\right\}^{-1}. \tag{C.5}$$

This condition implies $\delta_m q_P \leq 1/\sigma$ because

$$\delta_m < \frac{\alpha_\ell\alpha_\phi}{\sigma}\left(\frac{-\Gamma_{\ell m}m}{\Gamma_\ell}\right) \leq \frac{\alpha_\ell}{\sigma}\left(\frac{-\Gamma_{\ell m}m}{\Gamma_\ell}\right) = \left(\frac{I^\ell - I}{I^\ell}\right)\left(\frac{-\Gamma_{\ell m}m}{\Gamma_\ell}\right)\frac{1}{\sigma}.$$

In turn, $\delta_m q_P \leq 1/\sigma$ implies $K < 0$, which in turn implies (C.4), which in turn implies that $\mathcal{Q}(z)$ has exactly two roots outside the unit circle, which finally implies determinacy. Therefore, Condition (C.5) is a sufficient condition for determinacy.

C.3 Assessment of the Sufficient Determinacy Condition

To assess whether Condition (C.5) is likely to be met or not, we proceed as follows. We consider, for simplicity, a Cobb-Douglas specification for the production function f^b:

$$f^b\left(h_t^b, m_t\right) \equiv A_b\left(h_t^b\right)^{1-\varsigma}(m_t)^\varsigma,$$

where $A_b > 0$ and $0 < \varsigma < 1$. This specification implies that the steady-state elasticity of marginal banking costs to reserves, which appears in Condition (C.5), can be rewritten as

$$\frac{-m\Gamma_{\ell m}(\ell, m)}{\Gamma_\ell(\ell, m)} = \left(\frac{\varsigma}{1-\varsigma}\right)\left[1 + \frac{\upsilon^{b\prime\prime}\left(h^b\right)h^b}{\upsilon^{b\prime}\left(h^b\right)}\right].$$

Our Cobb-Douglas specification for f^b also implies that households' first-order conditions (6) and (7), at the steady state, can be combined to get

$$\varsigma = \left(\frac{m}{\ell}\right)\left(\frac{I - I^m}{I^\ell - I}\right).$$

Therefore, Condition (C.5) can be rewritten as

$$q_P \leq \frac{1 - \left(\frac{m}{\ell}\right)\left(\frac{I-I^m}{I^\ell - I}\right)}{\left(\frac{m}{\ell}\right)\left(\frac{I-I^m}{I^\ell}\right)\left[1 + \frac{\upsilon^{b\prime\prime}(h^b)h^b}{\upsilon^{b\prime}(h^b)}\right]}. \tag{C.6}$$

We set the steady-state variables I^m, I^ℓ, and m/ℓ to match some features of the US economy during the 2008-2015 ZLB episode. This episode lasted from December 16, 2008, to December 16, 2015; because we use monthly data, however, we consider the period from January 2009 to November 2015. We set the net IOR rate $I^m - 1$ to 0.25% per annum (the constant value of the interest rate on excess reserves over the period); the net interest rate on bank loans $I^\ell - 1$ to 3.25% per annum (the average value of the bank prime loan rate over the period); and the ratio of bank reserves to loans m/ℓ to 0.18 (the average ratio of total reserves of depository institutions to bank credit of all commercial banks over the period).

We calibrate the parameter q_P to match the increase in the stock of nominal reserves over the period. More specifically, we rewrite (C.2) as $\hat{M}_t = (1 - q_P)\hat{P}_t - q_y \hat{y}_t$, and we assume conservatively that the Fed increased the stock of nominal reserves over the period only in response to low inflation, not in response to low output growth (i.e., $q_y = 0$). This assumption is conservative because it tends to overestimate q_P and, therefore, to make Condition (C.6) harder to satisfy. Under this assumption, we get

$$q_P = 1 - \frac{\hat{M}_t}{\hat{P}_t}. \tag{C.7}$$

Our model has constant reserves and prices at the steady state. In reality, however, reserves followed a positive trend before the crisis, and the Fed has a positive inflation target. The Fed increased the stock of nominal reserves *beyond its trend*, in response to inflation *below target*. A natural empirical counterpart of (C.7), over the January 2009-November 2015 period, is therefore

$$q_P = 1 - \frac{(\ln M_{2015:11} - \ln M_{2009:01}) - \Delta_M}{(\ln P_{2015:11} - \ln P_{2009:01}) - \Delta_P},$$

where M_t and P_t are measured by the total reserves of depository institutions and the consumer price index respectively, while Δ_M and Δ_P denote respectively the "neutral" trend growth rate in reserves and the targeted growth rate in prices over the January 2009-November 2015 period. We conservatively set Δ_M to zero, thus attributing all of the observed growth in reserves to the Fed's response to inflation below target. Like the assumption $q_y = 0$, the assumption $\Delta_M = 0$ is conservative because it tends to overestimate q_P and, therefore, to make Condition (C.6) harder to satisfy. And we set Δ_P to 14%, which corresponds to the Fed's 2% annual-inflation target over (almost) seven years. We then get $q_P = 50.0$.

Finally, we make conservative assumptions about the values of the steady-state variables $h^b v^{b\prime\prime}(h^b)/v^{b\prime}(h^b)$ and I. More specifically, we set $h^b v^{b\prime\prime}(h^b)/v^{b\prime}(h^b)$, the inverse of the steady-state Frisch elasticity of bankers' labor supply, to 5. The value 5 for the inverse of a Frisch elasticity of labor supply lies at the upper end of the range of microeconomic estimates, and is much higher than values commonly considered in macroeconomics. And we set the net interest rate $I - 1$ to 0.75% per annum, i.e. 50 basis points per annum above the net IOR rate $I^m - 1$. This value is much higher than the average value, over the January 2009-November 2015 period, of standard proxies for $I - 1$, like the 3-month T-bill rate or the 3-month AA (financial or non-financial) commercial paper rate. Our assumptions about $h^b v^{b\prime\prime}(h^b)/v^{b\prime}(h^b)$ and I are conservative because they tend to overestimate these two steady-state variables and, therefore, to make Condition (C.6) harder to satisfy.

The right-hand side of Condition (C.6) depends on the period length, through the ratio $(I - I^m)/I^\ell$. Since one-period bank loans in our model are working-capital loans, which are short-term loans in reality, we set the period length to one quarter. Thus, we express all the interest rates in the ratio $(I - I^m)/I^\ell$ as quarterly rates.

We then get the value 705.7 for the right-hand side of Condition (C.6). This value is one order of magnitude larger than the value 50.0 obtained for the left-hand side of Condition (C.6). We thus find that Condition (C.6) is met by a large margin even under our conservative assumptions. We conclude that setting exogenously the IOR rate and following the reserves-supply rule still delivers local-equilibrium determinacy, except for implausible calibrations.

Appendix D: Extended Model With Household Cash

In the main text (Section 2), we show that our benchmark model delivers local-equilibrium determinacy under exogenous monetary-policy instruments, and we use this determinacy result to explain the low volatility of inflation and the absence of significant deflation at the ZLB. Our benchmark model, however, is specific in that households hold money only in the form of reserves, in their capacity as bankers. This makes our point stark because banks cannot collectively change the aggregate nominal quantity of reserves outstanding. In reality, bank reserves can fall if households demand more cash. In the present appendix, we show that our results do not unravel when we allow for such leakages out of reserve balances. More specifically, we introduce household cash into our benchmark model through a cash-in-advance constraint; we derive a sufficient condition for local-equilibrium determinacy in the resulting model, under an exogenous IOR rate and an exogenous monetary base (made of bank reserves and household cash); and we argue that this sufficient determinacy condition is likely to be met.

D.1 Introducing Household Cash into the Benchmark Model

We assume that each period is made of a financial exchange followed by a goods exchange. Households acquire cash in the financial exchange and use it to buy goods in the goods exchange; firms receive this cash in the goods exchange and have to wait until the next period's financial exchange to spend it (repaying loans). Thus, households choose bonds b_t, consumption c_t, work hours h_t, loans ℓ_t, reserves m_t, and (now) cash m_t^c to maximize the same reduced-form utility function (1) as previously, subject to the budget constraint

$$m_t^c + b_t + \ell_t + m_t \leq \frac{m_{t-1}^c - c_{t-1}}{\Pi_t} + \frac{I_{t-1}}{\Pi_t} b_{t-1} + \frac{I_{t-1}^\ell}{\Pi_t}\ell_{t-1} + \frac{I_{t-1}^m}{\Pi_t} m_{t-1} + w_t h_t + \omega_t$$

and the cash-in-advance constraint

$$m_t^c \geq c_t, \tag{D.1}$$

taking all prices (I_t, I_t^ℓ, I_t^m, P_t, and w_t) as given. Letting λ_t and λ_t^c denote the Lagrange multipliers on these two constraints respectively, the first-order conditions of households' optimization problem are again (3), (4), (5), (6), (7), and now

$$\lambda_t^c + \frac{\beta \lambda_{t+1}}{\Pi_{t+1}} - \lambda_t = 0.$$

The objective of firm i is now to maximize

$$\mathbb{E}_t \left\{ \sum_{k=0}^{+\infty} (\beta\theta)^k \frac{\beta \lambda_{t+k+1}}{\lambda_t \Pi_{t,t+k+1}} \left[P_t^*(i)\, y_{t+k}(i) - I_{t+k}^\ell L_{t+k}(i) - \left[W_{t+k} h_{t+k}(i) - L_{t+k}(i) \right] \right] \right\},$$

since the firm has to wait until the next period to exchange its cash. The first-order condition for the firm's optimization problem is thus

$$\mathbb{E}_t \left\{ \sum_{k=0}^{+\infty} (\beta\theta)^k \frac{\beta \lambda_{t+k+1}}{\lambda_t \Pi_{t,t+k+1}} \left[P_t^*(i) - \left(\frac{\varepsilon}{\varepsilon - 1} \right) \left(\phi I_{t+k}^\ell + (1-\phi) \right) \frac{W_{t+k}}{f'\left[h_{t+k}(i) \right]} \right] y_{t+k}(i) \right\} = 0, \tag{D.2}$$

instead of (12). In the particular case of flexible prices (and in a symmetric equilibrium), this first-order condition becomes

$$P_t = \frac{\varepsilon}{\varepsilon - 1} \left[\phi I_t^\ell + (1-\phi) \right] \frac{W_t}{f'(h_t)}, \tag{D.3}$$

which replaces (13). None of the other equilibrium conditions stated in Section 1 is changed, except the reserve-market-clearing condition (16), which is replaced by the money-market-clearing condition

$$m_t + m_t^c = \frac{M_t}{P_t}, \tag{D.4}$$

since the monetary base M_t controlled by the central bank is now made not only of bank reserves, but also of household cash. As previously, the equilibrium conditions (3), (5), (9), (10) holding with equality, and (17) imply the relationship (18) between loans and employment.

D.2 Steady State and Log-Linearization

To derive the necessary and sufficient condition for steady-state existence and uniqueness under a constant IOR rate ($I_t^m = I^m \geq 1$), a constant monetary base ($M_t = M > 0$), and no discount-factor shocks ($\zeta_t = 1$), we first note that the steady-state inflation rate Π is equal to 1 under a constant monetary base. In turn, $\Pi = 1$ and (4) together imply that the steady-state interest rate on bonds I is equal to $1/\beta$, as previously. Using (3), (5), (9), (17), (18), (D.3), and $I = 1/\beta$, we can rewrite households' first-order condition for loans (6) at the steady state as a relationship between real reserves m and employment h:

$$\Gamma_\ell \left[\mathcal{L}(h), m \right] = \tilde{\mathcal{A}}(h) \equiv u'\left[f(h) \right] \left\{ \frac{\beta}{\phi} \left[\left(\frac{\varepsilon - 1}{\varepsilon} \right) \frac{u'\left[f(h) \right] f'(h)}{v'(h)} - (1-\phi) \right] - 1 \right\}. \tag{D.5}$$

Because the left-hand side of (D.5) is positive, we restrict the domain of the function $\tilde{\mathcal{A}}$ to $(0, \tilde{h})$, where $\tilde{h} > 0$ is implicitly and uniquely defined by $u'[f(\tilde{h})]f'(\tilde{h})/v'(\tilde{h}) = [\phi/\beta + (1-\phi)]\varepsilon/(\varepsilon - 1)$. The function $\tilde{\mathcal{A}}$ is strictly decreasing ($\tilde{\mathcal{A}}' < 0$), with $\lim_{h\to 0} \tilde{\mathcal{A}}(h) = +\infty$ and $\lim_{h\to \tilde{h}} \tilde{\mathcal{A}}(h) = 0$.

Since $\Gamma_{\ell\ell} > 0$, $\mathcal{L}' > 0$, $\Gamma_{\ell m} < 0$, and $\tilde{\mathcal{A}}' < 0$, Equation (D.5) implicitly and uniquely defines a function $\tilde{\mathcal{M}}$ which is strictly increasing ($\tilde{\mathcal{M}}' > 0$) and such that

$$m = \tilde{\mathcal{M}}(h). \tag{D.6}$$

Using (3), (9), (17), (18), (D.6), and $I = 1/\beta$, we can then rewrite households' first-order condition for reserves (7) at the steady state as

$$\tilde{\mathcal{F}}(h) \equiv \frac{\Gamma_m \left[\mathcal{L}(h), \tilde{\mathcal{M}}(h)\right]}{u'[f(h)]} = -(1 - \beta I^m).$$

The same reasoning as in Appendix A.5, this time applied to $\tilde{\mathcal{F}}$ instead of $\mathcal{F}$, shows that the function $\tilde{\mathcal{F}}$ is strictly increasing from $-\infty$ to 0. Therefore, the necessary and sufficient condition for existence and uniqueness of a steady state is again $I^m < 1/\beta$.

Log-linearizing the model around its unique steady state, we get the same IS equation (21) and reserves-demand equation (23) as previously. Using the goods-market-clearing condition (17) and the binding cash-in-advance constraint (D.1) to rewrite the money-market-clearing condition (D.4), and then log-linearizing the resulting equation, leads to

$$\hat{M}_t - \hat{P}_t = (1 - \alpha_c)\hat{m}_t + \alpha_c \hat{y}_t, \tag{D.7}$$

where $\alpha_c \equiv f(h)/[f(h) + \mathcal{M}(h)] \in (0, 1)$ denotes the steady-state share of household cash in the monetary base. Finally, we derive the Phillips curve by following the same steps as in Appendix B.1. More specifically, the log-linearized first-order condition is now

$$\hat{P}_t^* = (1 - \beta\theta)\,\mathbb{E}_t \left\{ \sum_{k=0}^{+\infty} (\beta\theta)^k \left(\alpha_\phi i_{t+k}^\ell + \hat{w}_{t+k} + \hat{P}_{t+k} - \hat{mp}_{t+k|t} \right) \right\},$$

which corresponds to (B.1) without the i_{t+k} term; and this log-linearized first-order condition leads to the Phillips curve

$$\pi_t = \beta \mathbb{E}_t \{\pi_{t+1}\} + \kappa (\hat{y}_t - \delta_m \hat{m}_t + \delta_i i_t) \tag{D.8}$$

with $\delta_i \equiv \alpha_\phi \Psi/\kappa > 0$, where the term $\delta_i i_t$ captures the opportunity cost for firms of holding their cash from one period to the next.

D.3 Derivation of a Sufficient Determinacy Condition

Using the IS equation (21), the reserves-demand equation (23), the money-market-clearing condition (D.7), the Phillips curve (D.8), and the identity $\pi_t = \hat{P}_t - \hat{P}_{t-1}$, we get the following dynamic system in $\hat{m}_t$ and $\hat{y}_t$:

$$\mathbb{E}_t\left\{\begin{bmatrix}\hat{m}_{t+1}\\ \hat{m}_t\\ \hat{y}_{t+1}\\ \hat{y}_t\end{bmatrix}\right\} = \tilde{\mathbf{A}}\begin{bmatrix}\hat{m}_t\\ \hat{m}_{t-1}\\ \hat{y}_t\\ \hat{y}_{t-1}\end{bmatrix} + \mathbb{E}_t\left\{\tilde{\mathbf{B}}\begin{bmatrix}i_t^m\\ \hat{\mu}_{t+1}\\ \hat{\mu}_t\\ r_t\end{bmatrix}\right\}, \tag{D.9}$$

where $\hat{\mu}_t = \hat{M}_t - \hat{M}_{t-1}$ denotes the log-deviation of the gross growth rate of nominal reserves $\mu_t \equiv M_t/M_{t-1}$ from its steady-state value 1,

$$\tilde{\mathbf{A}} \equiv \begin{bmatrix}\frac{1+\beta-\delta_m\kappa}{\beta}-\frac{\delta_i\kappa}{\beta\chi_i} & \frac{-1}{\beta} & \frac{\kappa}{\beta}+\frac{\delta_i\chi_y\kappa}{\beta\chi_i} & 0\\ 1 & 0 & 0 & 0\\ \frac{1-\delta_m\kappa}{\beta\sigma}-\frac{1}{\sigma\chi_i}-\frac{\delta_i\kappa}{\beta\chi_i\sigma} & \frac{-1}{\beta\sigma} & 1+\frac{\chi_y}{\sigma\chi_i}+\frac{\kappa}{\beta\sigma}+\frac{\delta_i\chi_y\kappa}{\beta\chi_i\sigma} & 0\\ 0 & 0 & 1 & 0\end{bmatrix} + \alpha_c\begin{bmatrix}\frac{-1}{\beta\sigma} & \frac{1}{\beta\sigma} & \frac{1}{\beta\sigma} & \frac{-1}{\beta\sigma}\\ 0&0&0&0\\ \frac{-1}{\beta\sigma} & \frac{1}{\beta\sigma} & \frac{1}{\beta\sigma} & \frac{-1}{\beta\sigma}\\ 0&0&0&0\end{bmatrix}$$

$$+\frac{\alpha_c}{1-\alpha_c}\begin{bmatrix}\frac{1}{\sigma\chi_i}+\frac{(1-\sigma)\delta_m\kappa}{\beta\sigma}+\frac{(1-\sigma)\delta_i\kappa}{\beta\chi_i\sigma} & 0 & \frac{-(1-\sigma)}{\beta\sigma}-\frac{\chi_y}{\sigma\chi_i}-\frac{(1-\sigma)\kappa}{\beta\sigma}-\frac{(1-\sigma)\delta_i\chi_y\kappa}{\beta\chi_i\sigma} & \frac{1-\sigma}{\beta\sigma}\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\end{bmatrix}$$

$$\text{and}\quad \tilde{\mathbf{B}} \equiv \begin{bmatrix}\frac{\delta_i\kappa}{\beta} & 1 & \frac{-1}{\beta} & 0\\ 0&0&0&0\\ \frac{1}{\sigma}+\frac{\delta_i\kappa}{\beta\sigma} & 0 & \frac{-1}{\beta\sigma} & \frac{-1}{\sigma}\\ 0&0&0&0\end{bmatrix} + \frac{\alpha_c}{1-\alpha_c}\begin{bmatrix}\frac{-1}{\sigma}+\frac{(1-\sigma)\delta_i\kappa}{\beta\sigma} & 1 & \frac{1-\sigma}{\beta\sigma} & \frac{1}{\sigma}\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\end{bmatrix}.$$

Since this system has two predetermined variables ($\hat{m}_{t-1}$ and $\hat{y}_{t-1}$) and two non-predetermined variables ($\hat{m}_{t+1}$ and $\hat{y}_{t+1}$), the necessary and sufficient condition for local-equilibrium determinacy is that the matrix $\tilde{\mathbf{A}}$ have two eigenvalues inside the unit circle and two eigenvalues outside. We note that $\tilde{\mathbf{A}}$ can be obtained from $\mathbf{A}$ by replacing q_P, q_y, κ, and δ_m by respectively $1/(1-\alpha_c)$, $\alpha_c/(1-\alpha_c)$, $\tilde{\kappa} \equiv (1+\delta_i\chi_y/\chi_i)\kappa$, and $\tilde{\delta}_m \equiv (\delta_m\chi_i+\delta_i)/(\chi_i+\delta_i\chi_y)$ in $\mathbf{A}$. Therefore, we deduce from Appendix C.2 that $\tilde{\mathbf{A}}$ has two eigenvalues inside the unit circle and two eigenvalues outside for any $\theta \in (0,1)$ if and only if

$$\tilde{K} \equiv \frac{-1}{\sigma} + \left[1+\left(\frac{\alpha_c}{1-\alpha_c}\right)\left(\frac{\sigma-1}{\sigma}\right)\right]\tilde{\delta}_m < \frac{1-\tilde{\delta}_m\chi_y}{2\sigma\chi_i\,(1-\alpha_c)}, \tag{D.10}$$

where the right-hand side is positive since $1-\tilde{\delta}_m\chi_y = (1-\delta_m\chi_y)\chi_i/(\chi_i+\delta_i\chi_y) > 0$ (as follows from the second inequality in (B.11)). We have

$$\begin{aligned}\tilde{K} &< \frac{-1}{\sigma}+\frac{\tilde{\delta}_m}{1-\alpha_c}\\ &< \frac{-1}{\sigma}+\frac{1}{1-\alpha_c}\left(\delta_m+\frac{\delta_i}{\chi_i}\right)\\ &< \frac{1}{\sigma}\left[-1+\frac{1}{1-\alpha_c}\left(-\alpha_\ell\frac{\Gamma_{\ell m}m}{\Gamma_\ell}-\alpha_m\frac{\Gamma_{mm}m}{\Gamma_m}\right)\right]\\ &\leq \bar{K} \equiv \frac{1}{\sigma}\left[-1+\frac{1}{1-\alpha_c}\left(-\alpha_\ell\frac{\Gamma_{\ell m}m}{\Gamma_\ell}+\alpha_m\frac{\Gamma_{\ell m}\ell}{\Gamma_m}\right)\right],\end{aligned}$$

where the last inequality follows from (B.14). In turn, using first (6) and (7), and then (10) with equality and (D.3), we get sequentially

$$\begin{aligned}
\bar{K} &= \frac{1}{\sigma}\left[-1+\left(\frac{1-\beta I^m}{1-\alpha_c}\right)\left(\frac{I}{I^\ell}\frac{\Gamma_{\ell m} m}{\Gamma_m}+\frac{1}{\beta I^m}\frac{\Gamma_{\ell m}\ell}{\Gamma_m}\right)\right] \\
&= \frac{1}{\sigma}\left\{-1+\left(\frac{1-\beta I^m}{1-\alpha_c}\right)\left[\left[\left(\frac{\varepsilon-1}{\varepsilon}\right)\left(\frac{f'h}{f}\right)-\left(\frac{1-\phi}{\phi}\right)\left(\frac{\ell}{y}\right)\right]^{-1}\left(\frac{m}{\beta y}\right)+\frac{1}{\beta I^m}\right]\frac{\Gamma_{\ell m}\ell}{\Gamma_m}\right\} \\
&\leq \frac{1}{\sigma}\left\{-1+\left(\frac{1-\beta I^m}{1-\alpha_c}\right)\left[\frac{1}{\beta}\left(\frac{\varepsilon}{\varepsilon-1}\right)\left(\frac{f}{f'h}\right)\left(\frac{m}{y}\right)+\frac{1}{\beta I^m}\right]\frac{\Gamma_{\ell m}\ell}{\Gamma_m}\right\}.
\end{aligned}$$

Now consider the following condition, which states that the last expression is negative:

$$\left(\frac{1-\beta I^m}{1-\alpha_c}\right)\left\{\frac{1}{\beta}\left(\frac{\varepsilon}{\varepsilon-1}\right)\left[\frac{f(h)}{hf'(h)}\right]\left(\frac{m}{y}\right)+\frac{1}{\beta I^m}\right\}\left[\frac{\ell\Gamma_{\ell m}(\ell,m)}{\Gamma_m(\ell,m)}\right]<1. \tag{D.11}$$

This condition implies $\bar{K}<0$, which in turn implies $\tilde{K}<0$, which in turn implies (D.10), which in turn implies that $\tilde{\mathbf{A}}$ has two eigenvalues inside the unit circle and two eigenvalues outside, which finally implies determinacy.

D.4 Assessment of the Sufficient Determinacy Condition

To assess whether Condition (D.11) is likely to be met or not, we proceed broadly along the same lines as in Appendix C.3. We consider, for simplicity, a Cobb-Douglas specification for the production function f^b:

$$f^b\left(h_t^b,m_t\right)\equiv A_b\left(h_t^b\right)^{1-\varsigma}(m_t)^\varsigma,$$

where $A_b>0$ and $0<\varsigma<1$. This specification implies that the steady-state elasticity of Γ_m that appears in Condition (D.11) can be rewritten as

$$\frac{\ell\Gamma_{\ell m}(\ell,m)}{\Gamma_m(\ell,m)}=\left(\frac{1}{1-\varsigma}\right)\left[1+\frac{v^{b\prime\prime}\left(h^b\right)h^b}{v^{b\prime}\left(h^b\right)}\right].$$

Our Cobb-Douglas specification for f^b also implies that households' first-order conditions (6) and (7), at the steady state, can be combined to get

$$\varsigma=\left(\frac{m}{\ell}\right)\left(\frac{I-I^m}{I^\ell-I}\right).$$

Therefore, Condition (D.11) can be rewritten as

$$\left(\frac{I-I^m}{1-\alpha_c}\right)\left\{\left(\frac{\varepsilon}{\varepsilon-1}\right)\left[\frac{f(h)}{hf'(h)}\right]\left(\frac{m}{y}\right)+\frac{1}{I^m}\right\}\left[\frac{1+\frac{v^{b\prime\prime}(h^b)h^b}{v^{b\prime}(h^b)}}{1-\left(\frac{m}{\ell}\right)\left(\frac{I-I^m}{I^\ell-I}\right)}\right]<1. \tag{D.12}$$

We set the steady-state variables I^m, I^ℓ, m/ℓ, α_c, and m/y to match some features of the US economy during the 2008-2015 ZLB episode. More specifically, as in Appendix C.3, we set the net interest rates I^m-1 and $I^\ell-1$ to 0.25% and 3.25% per annum respectively, and the ratio

m/ℓ to 0.18. We set the steady-state share of household cash in the monetary base, α_c, to 0.39, which is the average value of the ratio between the currency component of M1 and the monetary base from January 2009 to November 2015. And we set the ratio m/y to 0.40, which is the average value of the ratio between total reserves of depository institutions and quarterly GDP from 2009Q1 to 2015Q4.

We make standard assumptions about the steady-state elasticity of output to labor $hf'(h)/f(h)$ and the elasticity of substitution between goods ε. More specifically, we set the former to 0.66, and the latter to 6 (implying a 20% markup).

Finally, we make the same conservative assumptions as in Appendix C.3 about the values of the steady-state variables $h^b v^{b\prime\prime}(h^b)/v^{b\prime}(h^b)$ and I. More specifically, we set the elasticity $h^b v^{b\prime\prime}(h^b)/v^{b\prime}(h^b)$ to 5, and the net interest rate $I-1$ to 0.75% per annum. These assumptions are conservative because they tend to overestimate these two steady-state variables and, therefore, to make Condition (D.12) harder to satisfy.

The left-hand side of Condition (D.12) depends on the period length through the ratio $(I - I^m)/I^m$ (and only through this ratio, since the ratio $(I - I^m)/y$ does not depend on the period length). We set the period length to one quarter, as in Appendix C.3. Thus, we express all the interest rates in the ratio $(I - I^m)/I^m$ as quarterly rates.

We then get the value 0.02 for the left-hand side of Condition (D.12). This value is one to two orders of magnitude smaller than 1. We thus find that Condition (D.12) is met by a large margin even under our conservative assumptions. We conclude that the introduction of household cash into the monetary base does not affect the ability of our model to deliver local-equilibrium determinacy under an exogenous IOR rate and an exogenous monetary base, except for implausible calibrations.

Appendix E: Extended Model With Liquid Government Bonds

In this appendix, we motivate, present and analyze our extended model with liquid government bonds. We show that this model can account not only for the three key features of inflation during ZLB episodes (like our benchmark model), but also for the negative spread between T-bill and IOR rates observed during these episodes (unlike our benchmark model). More specifically, we show that this model has an equilibrium in which the return on government bonds is below the IOR rate, while demand for bank reserves is not satiated. This equilibrium coincides with the equilibrium of our benchmark model (without liquid bonds), in the sense that all the endogenous variables that are common to both models, except the lump-sum transfer T_t, take the same equilibrium values. We show the existence of this equilibrium under two simple parameter restrictions, and we show how one of these parameter restrictions can be relaxed without affecting our results.

E.1 Satiation vs. Non-Satiation of Demand for Reserves

In Sections 2 and 3, we have shown that our benchmark model can broadly account for three key observations about US inflation during ZLB episodes (no significant deflation, little inflation volatility, and no significant inflation following QE policies). These results rest on the assumption that demand for bank reserves got close to satiation, but did not reach full satiation, i.e. that bank reserves carried a small but positive convenience yield ($I_t^m < I_t$ and $\Gamma_m > 0$). If we allowed for a finite satiation point in the demand for reserves and if demand for reserves were fully satiated ($I_t^m = I_t$ and $\Gamma_m = 0$), then our results would fall apart as follows: (i) in the local analysis of Section 2, our model would be isomorphic to the basic NK model ($\delta_m = 0$ and $i_t = i_t^m$), and would generate indeterminacy under a permanent interest-rate peg; and (ii) without price-level determinacy, the numerical simulation of QE2 in Section 3 would not be possible.

As we noted in the Introduction, our non-satiation assumption stands in contrast to views often expressed about the US economy in recent years. One argument making a case for satiation of demand for reserves is the fact that the second and third rounds of quantitative easing (QE2 and QE3) had no apparent inflationary consequences, as Reis (2016) and Cochrane (2018) point out. On this front, our counter-argument is simply that this fact may also be consistent with demand for reserves being close to satiation, rather than fully satiated, as our numerical simulation of QE2 in Section 3 suggests.

In the present appendix, we address a second argument that goes against our non-satiation view. This argument is the fact that T-bill returns were often below the IOR rate during ZLB episodes. We do not think this fact contradicts our claim that reserves still had a positive marginal convenience yield during this period. The lower T-bill returns, we argue, could reflect strong demand by non-bank entities – using T-bills as collateral or international reserve asset, for instance. We formalize our counter-argument by introducing government bonds providing liquidity services into our benchmark model. We show that our model with liquid bonds has an equilibrium in which the return on government bonds is below the IOR rate. Moreover, this equilibrium of our model with liquid bonds coincides with the equilibrium of our benchmark model (without liquid bonds), in the sense that all the endogenous variables that are common to both models, except the lump-sum transfer T_t, take the same equilibrium values. So, all the results that we have obtained in our benchmark model in Sections 2 and 3 still hold in our model with liquid bonds.

E.2 Liquidity of Government Bonds

Our benchmark model abstracts from government bonds and any role they may play in facilitating transactions. In reality, banks may hold government bonds (or other liquid assets), in addition to reserves, for liquidity management. Some regulatory constraints that give rise to a convenience yield for reserves – like the constraint on "high-quality liquid assets" imposed on US banks – can also be satisfied by holding government bonds. From this (regulatory) vantage point, bonds and reserves are perfect substitutes in satisfying liquidity needs. But government bonds are not as useful as reserves in satisfying the intra-day liquidity needs that arise from banking transactions, because bonds can either be sold for next-day settlement or used in repo transactions arranged to obtain liquidity, while reserves are readily available for any transaction – as Bush et al. (2019) elaborate.

Government bonds also provide a convenience yield to many non-bank entities (e.g. by serving as collateral or international reserve asset) and benefit from regulations (like restrictions on the asset portfolios of US money-market mutual funds). So, the observed returns on government bonds may reflect their convenience yield. If the returns are sufficiently attractive compared to the IOR rate, banks may hold government bonds to satisfy liquidity needs and regulatory constraints. If not, banks may hold mostly reserves for liquidity management.

Our model abstracts from non-bank financial institutions and foreign entities that may hold bonds. To formalize our main point, we will assume that workers get utility from government bonds (instead of modeling, say, a pension fund that holds bonds on workers' behalf). We will show that bankers may use government bonds for liquidity management if the IOR rate is sufficiently low compared to the equilibrium return from holding liquid bonds; but bankers will only use reserves for liquidity management when the IOR rate is sufficiently high. Although we don't explicitly model inside assets like federal-funds loans, we have in mind that our equilibrium with a relatively high IOR rate can also represent observed episodes in which banks don't lend federal funds, and the federal-funds rate is below the IOR rate. Our main point is that financial institutions that don't have direct access to the IOR rate may hold these assets in equilibrium, while banks hold reserves with a positive marginal convenience yield.[3]

E.3 Equilibrium Conditions Related to Households

As in our benchmark model (presented in Section 1), the representative household consists of workers and bankers, and gets utility from consumption (c_t) and disutility from labor (h_t for workers, h_t^b for bankers). We now assume that workers also get utility from holding government bonds (b_t^w), and bankers may use government bonds (b_t^b) as well as reserves (m_t) to produce loans (ℓ_t). As before, we make a substitution for h_t^b in households' primitive utility function and get the following reduced-form utility function:

$$U_t = \mathbb{E}_t \left\{ \sum_{k=0}^{\infty} \beta^k \zeta_{t+k} \left[u\left(c_{t+k}\right) - v\left(h_{t+k}\right) - \Gamma\left(\ell_{t+k}, m_{t+k} + \eta b_{t+k}^b\right) + z\left(b_{t+k}^w\right) \right] \right\},$$

[3]Bech and Klee (2012) present a model of segmentation in the federal-funds market in which limited access to the IOR rate (and other restrictions on trading) push the federal-funds rate below the IOR rate.

where $\eta \in (0,1]$. The function z, defined over $\mathbb{R}_{>0}$, is twice differentiable, strictly increasing ($z' > 0$), and strictly concave ($z'' < 0$); it also satisfies the usual Inada conditions. Values of η below unity may capture the fact that in reality reserves are more useful than government bonds for liquidity management because they provide immediate intra-day liquidity to banks (as discussed above). We allow for $\eta < 1$ to show that T-bill returns can be below the IOR rate even when T-bills provide smaller liquidity services than reserves to banks.

In the interest of realism (to make sure some reserves are always held in equilibrium), we also assume that the central bank imposes reserve requirements on banks. Since our model consolidates bankers and workers into households (thus, abstracting from deposits), we specify the reserve requirement as

$$m_t \geq \psi \ell_t, \tag{E.1}$$

where $\psi > 0$. The household budget constraint, expressed in real terms, is

$$c_t + b_t + b_t^b + b_t^w + \ell_t + m_t \leq \frac{I_{t-1}}{\Pi_t} b_{t-1} + \frac{I_{t-1}^b}{\Pi_t}\left(b_{t-1}^b + b_{t-1}^w\right) + \frac{I_{t-1}^\ell}{\Pi_t}\ell_{t-1} + \frac{I_{t-1}^m}{\Pi_t} m_{t-1} + w_t h_t + \tau_t,$$

where I_t^b denotes the gross nominal interest rate on government bonds (and b_t represents a private bond in zero net supply, as we indicated in Section 1). We let λ_t and λ_t^r denote the Lagrange multipliers on the period-t budget constraint and reserve requirement (respectively). The optimality conditions are

$$\lambda_t = \zeta_t u'(c_t),$$

$$\lambda_t w_t = \zeta_t v'(h_t),$$

$$\lambda_t = \beta I_t \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\Pi_{t+1}} \right\}, \tag{E.2}$$

$$\lambda_t = \zeta_t z'(b_t^w) + \beta I_t^b \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\Pi_{t+1}} \right\}, \tag{E.3}$$

$$\zeta_t \Gamma_\ell \left(\ell_t, m_t + \eta b_t^b\right) + \lambda_t + \psi \lambda_t^r = \beta I_t^\ell \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\Pi_{t+1}} \right\}, \tag{E.4}$$

$$\zeta_t \Gamma_m \left(\ell_t, m_t + \eta b_t^b\right) + \lambda_t = \lambda_t^r + \beta I_t^m \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\Pi_{t+1}} \right\}, \tag{E.5}$$

and

$$(m_t - \psi \ell_t)\lambda_t^r = 0.$$

We must also have

$$\eta \zeta_t \Gamma_m \left(\ell_t, m_t + \eta b_t^b\right) + \lambda_t \geq \beta I_t^b \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\Pi_{t+1}} \right\} \tag{E.6}$$

and $b_t^b \geq 0$, with complementary slackness.

E.4 Other Equilibrium Conditions

The remaining equilibrium conditions involve minor adjustments to our presentation in Subsections 1.2-1.4 (for firms, the government, and market clearing), as we describe below. The equilibrium conditions associated with firms don't change. For the government, we replace the consolidated budget constraint (14) by

$$M_t + B_t = I_{t-1}^m M_{t-1} + I_{t-1}^b B_{t-1} + T_t, \tag{E.7}$$

where B_t denotes the nominal stock of one-period public debt (held outside the central bank). A Ricardian fiscal policy adjusts the lump-sum transfer T_t to stabilize the real public debt around a steady-state target $b^* > 0$. The market-clearing conditions of Subsection 1.4 still apply – in particular the condition (15), given that private bonds are in zero net supply. In addition, we now have the market-clearing condition for government bonds:

$$b_t^w + b_t^b = \frac{B_t}{P_t}. \tag{E.8}$$

E.5 Characterization of our Equilibrium of Interest

The household optimality conditions above admit a solution with $1 = I_t^m < I_t^b$ that may represent the period before interest payment on reserves in the US (i.e. before 2008). If η is large enough, such a solution may have binding reserve requirements and $b_t^b > 0$. In this case, banks use government bonds – and if we extended our model, they could use inside assets like commercial paper – to manage the liquidity of their portfolios.

We are, however, interested in a candidate equilibrium in which banks do not use government bonds ($b_t^b = 0$), the reserve requirement is not binding ($\lambda_t^r = 0$), and the IOR rate is above the government-bond yield ($I_t^m > I_t^b$). This candidate equilibrium may capture – admittedly, in a stark way – some features of US bank portfolios and T-bill returns in the aftermath of the financial crisis.

In Appendix E.6, we prove the existence of such an equilibrium under two parameter restrictions. The first restriction is that the minimal reserves-to-loans ratio imposed by the central bank, ψ, should be lower than the steady-state value taken by the reserves-to-loans ratio in our benchmark model. This restriction is necessary for the reserve requirement (E.1) to be slack ($\lambda_t^r = 0$). The second restriction is that the steady-state marginal liquidity service of government bonds, $z'(b^*)$, should be large enough for the interest rate on government bonds to be lower than the IOR rate ($I_t^b < I_t^m$).

We now show that this equilibrium of interest (with $b_t^b = \lambda_t^r = 0$ and $I_t^b < I_t^m$) coincides with the equilibrium of our benchmark model, in the sense that all the endogenous variables that are common to both models, except the lump-sum transfer T_t, take the same equilibrium values.

Intuitively, in this equilibrium, banks hold only reserves ($b_t^b = 0$) because they pay more interest than government bonds ($I_t^b < I_t^m$) and are at least as liquid as government bonds ($\eta \leq 1$); and since banks do not hold government bonds ($b_t^b = 0$) and face a non-binding reserve requirement ($\lambda_t^r = 0$), they behave in exactly the same way as in our benchmark model.

To show that our equilibrium of interest (in the model with liquid bonds) coincides with the equilibrium of our benchmark model (without liquid bonds), we first use (E.2) to rewrite households' optimality conditions (E.3), (E.4), (E.5), and (E.6) in the following simpler forms:

$$\frac{I_t^b}{I_t} = 1 - \frac{\zeta_t z'\left(b_t^w\right)}{\lambda_t}, \tag{E.9}$$

$$\frac{I_t^\ell}{I_t} = 1 + \frac{\zeta_t \Gamma_\ell\left(\ell_t, m_t + \eta b_t^b\right)}{\lambda_t} + \psi \frac{\lambda_t^r}{\lambda_t}, \tag{E.10}$$

$$\frac{I_t^m}{I_t} = 1 + \frac{\zeta_t \Gamma_m\left(\ell_t, m_t + \eta b_t^b\right)}{\lambda_t} - \frac{\lambda_t^r}{\lambda_t}, \tag{E.11}$$

and

$$\frac{I_t^b}{I_t} \leq (1 - \eta) + \eta \frac{I_t^m}{I_t} + \eta \frac{\lambda_t^r}{\lambda_t}. \tag{E.12}$$

In our equilibrium of interest, because $b_t^b = \lambda_t^r = 0$, the equilibrium conditions (E.10) and (E.11) collapse to

$$\frac{I_t^\ell}{I_t} = 1 + \frac{\zeta_t \Gamma_\ell\left(\ell_t, m_t\right)}{\lambda_t} \quad \text{and} \quad \frac{I_t^m}{I_t} = 1 + \frac{\zeta_t \Gamma_m\left(\ell_t, m_t\right)}{\lambda_t}.$$

These conditions are identical to the equilibrium conditions (6) and (7) of our benchmark model. Therefore, our equilibrium of interest satisfies all the equilibrium conditions of our benchmark model (listed in Section 1), except the government budget constraint (14), which is replaced by (E.7). As a consequence, all the endogenous variables that are present in both models take the same values in our equilibrium of interest as in the equilibrium of our benchmark model, except the lump-sum transfer T_t appearing in the government budget constraint. So, we conclude that the introduction of liquid government bonds into our benchmark model enables us to account for the negative spread between Treasury-bill and IOR rates observed during ZLB episodes, without affecting in any way the ability of the model to account for the three key features of inflation during ZLB episodes (i.e., without affecting any of the results obtained in Sections 2 and 3).

E.6 Existence of our Equilibrium of Interest

We prove the existence of our equilibrium of interest under two parameter restrictions. The first restriction is

$$\psi < \bar{\psi}, \tag{E.13}$$

where $\bar{\psi}$ denotes the steady-state value of the reserves-to-loans ratio m_t/ℓ_t in our benchmark model (without liquid bonds). As we will see, this restriction will ensure that the reserve requirement (E.1) is not binding in our model with liquid bonds. The second restriction is

$$\max\left[1, \frac{1}{\beta} - \frac{z'(b^*)}{\beta\bar{\lambda}}\right] < I^m < \frac{1}{\beta}, \tag{E.14}$$

where $\bar{\lambda}$ denotes the upper bound of the steady-state values taken by the marginal utility of consumption λ_t as I^m varies from 1 to $1/\beta$ in our benchmark model (this upper bound being reached for $I^m = 1$). As we will see, that restriction will ensure that the interest rate on government bonds I_t^b is lower than the IOR rate I_t^m in our model with liquid bonds. In fact, that restriction will turn out to be sufficient but not necessary for $I_t^b < I_t^m$; for simplicity, we relegate to Appendix E.7 the statement of the (more complex) parameter restriction that is necessary and sufficient for $I_t^b < I_t^m$.

We proceed in two steps: we show first the existence of our *steady-state* equilibrium of interest, and then the existence of our *dynamic* equilibrium of interest. In the first step, to show that our model with liquid bonds, under the parameter restrictions (E.13) and (E.14), has a steady-state equilibrium with $I^b < I^m$ and $b^b = \lambda^r = 0$, we start from a candidate steady-state equilibrium with $b^b = \lambda^r = 0$. In this candidate equilibrium, as follows from the analysis above, all the endogenous variables that also appear in our benchmark model, except the lump-sum transfer T_t, take the same steady-state values as in that model. Using these values and $b^b = 0$, we then get residually the steady-state values of the other endogenous variables: (i) b^w and B from the market-clearing condition (E.8) and the steady-state target $B/P = b^*$; (ii) I^b from the first-order condition (E.9); and (iii) T from the consolidated budget constraint of the government (E.7).

At this stage, all equality conditions for steady-state equilibrium are satisfied, and the steady-state value of all endogenous variables is pinned down. What remains to be shown is that: (i) the inequality conditions for steady-state equilibrium, i.e. the steady-state versions of (E.1) and (E.12), are satisfied as strict inequalities, implying that the candidate steady-state equilibrium is indeed a steady-state equilibrium; and (ii) this equilibrium has the property that $I^b < I^m$. We first establish this last inequality by using in turn the first-order condition (E.9) with $I = 1/\beta$ and $b^w = b^*$, the inequality $\lambda \leq \bar{\lambda}$, and the parameter restriction (E.14), to get

$$I^b = \frac{1}{\beta} - \frac{z'(b^*)}{\beta\lambda} \leq \frac{1}{\beta} - \frac{z'(b^*)}{\beta\bar{\lambda}} < I^m.$$

In turn, the property $I^b < I^m$, together with $I^m < I$ and $\lambda^r = 0$, implies that the steady-state version of (E.12) is satisfied as a strict inequality:

$$\frac{I^b}{I} < (1-\eta) + \eta\frac{I^m}{I} + \eta\frac{\lambda^r}{\lambda}.$$

Finally, the parameter restriction (E.13) straightforwardly implies that the steady-state version of (E.1) is satisfied as a strict inequality. We conclude that our model with liquid bonds does indeed have a steady-state equilibrium with $I^b < I^m$ and $b^b = \lambda^r = 0$ that coincides with the steady-state equilibrium of our benchmark model. In this equilibrium, banks hold only reserves

($b^b = 0$) because they pay more interest than government bonds ($I^b < I^m$) and are at least as liquid as government bonds ($\eta \leq 1$).

In the second step, we proceed similarly to show the existence of a dynamic equilibrium with $I_t^b < I_t^m$ and $b_t^b = \lambda_t^r = 0$. More specifically, we start from a candidate equilibrium with $b_t^b = \lambda_t^r = 0$. In this candidate equilibrium, as follows from the analysis above, all the endogenous variables that also appear in our benchmark model, except the lump-sum transfer T_t, take the same equilibrium values as in that model. Using these values and $b_t^b = 0$, we then get residually the equilibrium values of the other endogenous variables (expressed as log-deviations from their steady-state values, and denoted by letters with hats): (i) $\hat{b}_t^w$ and $\hat{B}_t$ from the log-linearized version of the market-clearing condition (E.8) and the fiscal-policy rule; (ii) $\hat{I}_t^b$ from the log-linearized version of the first-order condition (E.9); and (iii) $\hat{T}_t$ from the log-linearized version of the government's consolidated budget constraint (E.7).

At this stage, all equality conditions for equilibrium are satisfied, and the equilibrium value of all endogenous variables is pinned down. What remains to be shown is that: (i) the inequality conditions for equilibrium, i.e. (E.1) and (E.12), are satisfied as strict inequalities, implying that the candidate equilibrium is indeed an equilibrium; and (ii) this equilibrium has the property that $I_t^b < I_t^m$. Now, we have just shown that these three strict inequalities are satisfied at the steady state; therefore, by continuity, they are also satisfied in the neighborhood of this steady state, under the standard assumption of small enough shocks. As a consequence, our model with liquid bonds does indeed have a dynamic equilibrium with $I_t^b < I_t^m$ and $b_t^b = \lambda_t^r = 0$, and this equilibrium coincides with the dynamic equilibrium of our benchmark model.

E.7 Relaxation of a Parameter Restriction

To prove the existence of our equilibrium of interest in Appendix E.6, we have used the parameter restriction (E.14), which involves the reduced-form parameter $\bar{\lambda}$. This restriction, however, can be harmlessly relaxed to some extent, because our proof only rests on the weaker condition

$$\max\left[1, \frac{1}{\beta} - \frac{z'(b^*)}{\beta\lambda}\right] < I^m < \frac{1}{\beta}, \tag{E.15}$$

where the steady-state value λ depends on several parameters of the model − in particular β and I^m, but not b^*. To re-state (E.15) as a condition involving only parameters, we write λ as

$$\lambda = \Lambda(\beta I^m),$$

where the function Λ is defined by

$$\Lambda(x) \equiv u'\left\{f\left[\mathcal{F}^{-1}(x-1)\right]\right\}$$

for $x \in (-\infty, 1]$, where in turn the function $\mathcal{F}$ is defined in Subsection 2.1. Given the properties of $\mathcal{F}$, the function Λ is strictly decreasing ($\Lambda' < 0$), with $\lim_{x\to-\infty}\Lambda(x) = +\infty$. Therefore, there exists a unique $x^* \in (-\infty, 1)$ such that

$$1 - \frac{z'(b^*)}{\Lambda(x^*)} = x^*.$$

We can then re-state (E.15) as

$$\max\left(1, \frac{x^*}{\beta}\right) < I^m < \frac{1}{\beta},$$

where the reduced-form parameter x^* depends on several parameters of the model – in particular b^*, but not β nor I^m.

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