Supplement to "The Implementation of Stabilization Policy"

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August 6, 2020

This supplementary appendix contains the (standard) algebraic derivations of the two optimal feasible paths considered in Section 4, which are used in the numerical simulations (to produce Figures 1 and 2). It also provides the expression of some reduced-form parameters introduced in Section 5, as functions of the structural parameters, so as to make the results ready-to-use for any numerical application.

S.1 Optimal Feasible Path in the Basic NK Model

In this subsection, I derive the expression (30) for the timeless-perspective optimal feasible path $\widetilde{P}$. Given that the timeless-perspective optimal feasible path is defined as the limit of the date-t_0 Ramsey-optimal feasible path as $t_0 \to -\infty$, I start by deriving the date-t_0 Ramsey-optimal feasible path. To do so, I follow the undetermined-coefficients method. More specifically, I write the inflation rate and output as $\pi_{t_0+k} = \sum_{j=0}^{k} a_j^{\pi} \varepsilon_{t_0+k-j}^{\eta} + \sum_{j=0}^{k} b_j^{\pi} \varepsilon_{t_0+k-j}^{u}$ and $y_{t_0+k} = \sum_{j=0}^{k} a_j^{y} \varepsilon_{t_0+k-j}^{\eta} + \sum_{j=0}^{k} b_j^{y} \varepsilon_{t_0+k-j}^{u}$ for $k \geq 0$.[1] I look for the values of the coefficients a_j^{π}, b_j^{π}, a_j^{y}, and b_j^{y} for $j \geq 0$ that minimize

$$L_{t_0} = \mathbb{E}_{t_0} \left\{ \sum_{k=0}^{+\infty} \beta^k \left[\left(\sum_{j=0}^{k} a_j^{\pi} \varepsilon_{t_0+k-j}^{\eta} + \sum_{j=0}^{k} b_j^{\pi} \varepsilon_{t_0+k-j}^{u} \right)^2 + \lambda \left(\sum_{j=0}^{k} a_j^{y} \varepsilon_{t_0+k-j}^{\eta} + \sum_{j=0}^{k} b_j^{y} \varepsilon_{t_0+k-j}^{u} \right)^2 \right] \right\} \tag{S.1}$$

[1] To lighten the exposition, I do not consider a deterministic term c_k^{π} (respectively c_k^{y}) in the expression of π_{t_0+k} (respectively y_{t_0+k}), because this term is clearly zero on the timeless-perspective optimal feasible path ($\lim_{t_0 \to -\infty} c_k^{\pi} = \lim_{t_0 \to -\infty} c_k^{y} = 0$). Thus, the "date-$t_0$ Ramsey-optimal feasible path" that I consider is, in fact, the date-t_0 Ramsey-optimal feasible path up to a deterministic term.

subject to the constraints

$$a_0^y - a_1^y - \sigma^{-1} a_1^\pi - 1 = 0, \tag{S.2}$$

$$b_0^y - b_1^y - \sigma^{-1} b_1^\pi = 0, \tag{S.3}$$

$$a_j^\pi - \beta a_{j+1}^\pi - \kappa a_j^y = 0 \quad \text{for} \quad j \geq 0, \tag{S.4}$$

$$b_0^\pi - \beta b_1^\pi - \kappa b_0^y - 1 = 0, \tag{S.5}$$

$$b_j^\pi - \beta b_{j+1}^\pi - \kappa b_j^y - (\rho_u + \theta_u)\rho_u^{j-1} = 0 \quad \text{for} \quad j \geq 1. \tag{S.6}$$

The constraints (S.2)-(S.3) come from the IS equation (26) and the observation set $\widetilde{O}_t \equiv \{\varepsilon^{\eta,t-1}, \varepsilon^{u,t-1}\}$, which implies that i_t cannot depend on $(\varepsilon_t^\eta, \varepsilon_t^u)$. The constraints (S.4)-(S.6) come from the Phillips curve (27). I denote respectively by γ_a, γ_b, $(\delta_j^a)_{j\geq 0}$, δ_0^b, and $(\delta_j^b)_{j\geq 1}$ the Lagrange multipliers associated with these constraints.

I start with the determination of the coefficients a_j^π and a_j^y for $j \geq 0$. The first-order conditions of the Lagrangian minimization with respect to these coefficients are

$$2V_\eta(1-\beta)^{-1} a_0^\pi - \delta_0^a = 0,$$

$$2V_\eta(1-\beta)^{-1}\beta a_1^\pi + \sigma^{-1}\gamma_a - \delta_1^a + \beta\delta_0^a = 0,$$

$$2V_\eta(1-\beta)^{-1}\beta^j a_j^\pi - \delta_j^a + \beta\delta_{j-1}^a = 0 \quad \text{for} \quad j \geq 2,$$

$$2V_\eta(1-\beta)^{-1}\lambda a_0^y - \gamma_a + \kappa\delta_0^a = 0,$$

$$2V_\eta(1-\beta)^{-1}\beta\lambda a_1^y + \gamma_a + \kappa\delta_1^a = 0,$$

$$2V_\eta(1-\beta)^{-1}\beta^j\lambda a_j^y + \kappa\delta_j^a = 0 \quad \text{for} \quad j \geq 2,$$

where V_η denotes the variance of ε_t^η. By getting rid of the Lagrange multipliers, I can rewrite these first-order conditions as

$$\beta\lambda a_1^y + (1+\kappa\sigma^{-1})\lambda a_0^y + \beta\kappa a_1^\pi + (1+\beta+\kappa\sigma^{-1})\kappa a_0^\pi = 0, \tag{S.7}$$

$$\beta\kappa a_2^\pi + \beta\lambda a_2^y + \beta\kappa a_1^\pi + \kappa\lambda\sigma^{-1} a_0^y + (\beta+\kappa\sigma^{-1})\kappa a_0^\pi = 0, \tag{S.8}$$

$$\kappa a_j^\pi + \lambda a_j^y - \lambda a_{j-1}^y = 0 \quad \text{for} \quad j \geq 3. \tag{S.9}$$

The equations (S.4) and (S.9) imply the recurrence equation $\beta\lambda a_{j+2}^\pi - (\beta\lambda + \kappa^2 + \lambda) a_{j+1}^\pi + \lambda a_j^\pi = 0$ for $j \geq 2$. The roots of the corresponding characteristic polynomial are μ (defined in the main text) and $\mu' \equiv (2\beta\lambda)^{-1}[\lambda + \beta\lambda + \kappa^2 + \sqrt{(\lambda+\beta\lambda+\kappa^2)^2 - 4\beta\lambda^2}]$. Since $0 < \mu < 1$ and $\beta\mu'^2 \geq 1$, as can be readily checked, the solution of the recurrence equation that minimizes L_{t_0} is of the form $a_j^\pi = a_2^\pi \mu^{j-2}$ for $j \geq 2$. The equation (S.4) then implies that $a_j^y = (1-\beta\mu)\kappa^{-1}a_2^\pi \mu^{j-2}$ for $j \geq 2$. Coefficients a_0^π, a_1^π, a_2^π, a_0^y, and a_1^y are then determined by the linear system made of (S.2), (S.4) for $j \in \{0,1\}$, (S.7), (S.8), and $a_2^y = (1-\beta\mu)\kappa^{-1}a_2^\pi$. I thus eventually obtain $a_0^\pi = a_0$, $a_1^\pi = a_1$, $a_j^\pi = a_2\mu^{j-2}$ for $j \geq 2$, $a_0^y = \kappa^{-1}(a_0 - \beta a_1)$, $a_1^y = \kappa^{-1}(a_1 - \beta a_2)$, and $a_j^y = (1-\beta\mu)\kappa^{-1}a_2\mu^{j-2}$ for $j \geq 2$, where $[\,a_0 \quad a_1 \quad a_2\,]^T \equiv \mathbf{M}^{-1}[\,0 \quad 0 \quad \kappa\,]^T$ and

$$\mathbf{M} \equiv \begin{bmatrix} (\beta\kappa^2 + \kappa\lambda\sigma^{-1} + \kappa^2 + \kappa^3\sigma^{-1} + \lambda) & \beta\kappa\,(\kappa - \lambda\sigma^{-1}) & -\beta^2\lambda \\ (\beta\kappa + \kappa^2\sigma^{-1} + \lambda\sigma^{-1})\,\kappa & \beta\kappa\,(\kappa - \lambda\sigma^{-1}) & \beta\,(-\beta\lambda\mu + \kappa^2 + \lambda) \\ 1 & -\,(1 + \beta + \kappa\sigma^{-1}) & \beta \end{bmatrix}.$$

I now turn to the determination of the coefficients b_j^π and b_j^y for $j \geq 0$. The first-order conditions of the Lagrangian minimization with respect to these coefficients are the same as those with respect to the coefficients a_j^π and a_j^y for $j \geq 0$, except that a_j^π, a_j^y, γ_a, δ_j^a, and V_η should be respectively replaced by b_j^π, b_j^y, γ_b, δ_j^b, and V_u, where V_u denotes the variance of ε_t^u. Therefore, by getting rid of the Lagrange multipliers, I can rewrite these first-order conditions as

$$\beta\lambda b_1^y + (1 + \kappa\sigma^{-1})\lambda b_0^y + \beta\kappa b_1^\pi + (1 + \beta + \kappa\sigma^{-1})\kappa b_0^\pi = 0, \tag{S.10}$$

$$\beta\kappa b_2^\pi + \beta\lambda b_2^y + \beta\kappa b_1^\pi + \kappa\lambda\sigma^{-1} b_0^y + (\beta + \kappa\sigma^{-1})\kappa b_0^\pi = 0, \tag{S.11}$$

$$\kappa b_j^\pi + \lambda b_j^y - \lambda b_{j-1}^y = 0 \;\text{ for }\; j \geq 3, \tag{S.12}$$

which correspond to (S.7), (S.8), and (S.9) in which a_j^π and a_j^y have been respectively replaced by b_j^π and b_j^y for all $j \geq 0$. The equations (S.6) and (S.12) imply the recurrence equation $\beta\lambda b_{j+2}^\pi - (\beta\lambda + \kappa^2 + \lambda)b_{j+1}^\pi + \lambda b_j^\pi = \lambda(1-\rho_u)(\rho_u + \theta_u)\rho_u^{j-1}$ for $j \geq 2$, which is identical to the recurrence equation obtained above for $(a_j^\pi)_{j\geq 2}$ except for the term on the right-hand side. Therefore, the roots of the corresponding characteristic polynomial are μ, μ', and ρ_u. Given that $0 < \mu < 1$ and $\beta\mu'^2 \geq 1$, and given that I focus on the generic case $\rho_u \neq \mu$, the solution of the recurrence equation that minimizes L_{t_0} is of the form $b_j^\pi = (b_2^\pi - b)\mu^{j-2} + b\rho_u^{j-2}$ for $j \geq 2$ with $b \in \mathbb{R}$. The recurrence equation for $j = 2$ implies that $b = [\beta\lambda\rho_u^2 - (\beta\lambda + \kappa^2 + \lambda)\rho_u + \lambda]^{-1}\lambda(1-\rho_u)(\rho_u + \theta_u)\rho_u$. The equation (S.6) then implies that $b_j^y = (1-\beta\mu)\kappa^{-1}(b_2^\pi - b)\mu^{j-2} + [(1-\beta\rho_u)b - (\rho_u + \theta_u)\rho_u]\kappa^{-1}\rho_u^{j-2}$ for $j \geq 2$. The coefficients b_0^π, b_1^π, b_2^π, b_0^y, and b_1^y are then determined by the linear system made of (S.3), (S.5), (S.6) for $j = 1$, (S.10), (S.11), and $b_2^y = \kappa^{-1}[(1-\beta\mu)b_2^\pi + \beta(\mu - \rho_u)b - (\rho_u + \theta_u)\rho_u]$. I thus eventually obtain $b_0^\pi = b_0$, $b_1^\pi = b_1$, $b_j^\pi = (b_2 - b)\mu^{j-2} + b\rho_u^{j-2}$

for $j \geq 2$, $b_0^y = \kappa^{-1}(b_0 - \beta b_1 - 1)$, $b_1^y = \kappa^{-1}[b_1 - \beta b_2 - (\rho_u + \theta_u)]$, and $b_j^y = (1 - \beta\mu)\kappa^{-1}(b_2 - b)\mu^{j-2} + [(1-\beta\rho_u)b - (\rho_u + \theta_u)\rho_u]\kappa^{-1}\rho_u^{j-2}$ for $j \geq 2$, where $[\,b_0 \quad b_1 \quad b_2\,]^T \equiv \mathbf{M}^{-1}[\,\lambda\,[1 + \beta(\rho_u + \theta_u) + \kappa\sigma^{-1}] \quad \lambda\,[\beta(\rho_u + \theta_u)\rho_u - \beta^2(\mu - \rho_u)b + \kappa\sigma^{-1}] \quad 1-(\rho_u+\theta_u)\,]^T$.

The coefficients a_j^π, b_j^π, a_j^y, and b_j^y for $j \geq 0$ that I have obtained give me the inflation rate and output on the date-t_0 Ramsey-optimal feasible path as functions of shocks having occurred since date t_0. By making t_0 tend towards $-\infty$, I straightforwardly get these two variables on the timeless-perspective optimal feasible path as functions of all current and past shocks:

$$\mathbf{Z}_t = \mathbf{T}_0^Z \boldsymbol{\varepsilon}_t + \mathbf{T}_1^Z \boldsymbol{\varepsilon}_{t-1} + \sum_{j=2}^{+\infty} \left(\rho_u^{j-2}\mathbf{T}_u^Z + \mu^{j-2}\mathbf{T}_\mu^Z\right) \boldsymbol{\varepsilon}_{t-j}, \tag{S.13}$$

where $\mathbf{T}_0^Z \equiv \begin{bmatrix} a_0 & b_0 \\ \frac{a_0-\beta a_1}{\kappa} & \frac{b_0-\beta b_1-1}{\kappa} \end{bmatrix}$, $\mathbf{T}_1^Z \equiv \begin{bmatrix} a_1 & b_1 \\ \frac{a_1-\beta a_2}{\kappa} & \frac{b_1-\beta b_2-(\rho_u+\theta_u)}{\kappa} \end{bmatrix}$,
$\mathbf{T}_u^Z \equiv \begin{bmatrix} 0 & b \\ 0 & \frac{(1-\beta\rho_u)b-(\rho_u+\theta_u)\rho_u}{\kappa} \end{bmatrix}$, and $\mathbf{T}_\mu^Z \equiv \begin{bmatrix} a_2 & b_2 - b \\ \frac{(1-\beta\mu)a_2}{\kappa} & \frac{(1-\beta\mu)(b_2-b)}{\kappa} \end{bmatrix}$.

Multiplying the left- and right-hand sides of (S.13) by $(1-\rho_u L)(1-\mu L)$ leads to the first two lines of (30) with

$$\begin{aligned} \mathbf{T}_Z(X) \;\equiv\; & \mathbf{T}_0^Z + \left[-(\rho_u+\mu)\,\mathbf{T}_0^Z + \mathbf{T}_1^Z\right] X + \left[\rho_u\mu\mathbf{T}_0^Z - (\rho_u+\mu)\mathbf{T}_1^Z + \mathbf{T}_u^Z + \mathbf{T}_\mu^Z\right] X^2 \\ & + \left[\rho_u\mu\mathbf{T}_1^Z - \mu\mathbf{T}_u^Z - \rho_u\mathbf{T}_\mu^Z\right] X^3. \end{aligned}$$

Moreover, $\text{rank}[\mathbf{T}_Z(0)] = 2$, since $\text{rank}(\mathbf{T}_0^Z) = 2$.

Then, using the IS equation (26) and (S.13), I residually obtain the interest rate on the timeless-perspective optimal feasible path as a function of all past shocks:

$$i_t = \mathbf{T}_1^i \boldsymbol{\varepsilon}_{t-1} + \sum_{j=2}^{+\infty} \left(\rho_\eta^{j-2}\mathbf{T}_\eta^i + \rho_u^{j-2}\mathbf{T}_u^i + \mu^{j-2}\mathbf{T}_\mu^i\right) \boldsymbol{\varepsilon}_{t-j}, \tag{S.14}$$

where $\mathbf{T}_1^i \equiv \begin{bmatrix} \frac{\left(1+\beta-\beta\mu+\kappa\sigma^{-1}\right)a_2-a_1+\kappa(\rho_\eta+\theta_\eta)}{\kappa\sigma^{-1}} & \frac{\left(1+\beta-\beta\mu+\kappa\sigma^{-1}\right)b_2-b_1+\beta(\mu-\rho_u)b+(\rho_u+\theta_u)(1-\rho_u)}{\kappa\sigma^{-1}} \end{bmatrix}$,
$\mathbf{T}_\eta^i \equiv \begin{bmatrix} (\rho_\eta+\theta_\eta)\rho_\eta\sigma & 0 \end{bmatrix}$, $\mathbf{T}_u^i \equiv \begin{bmatrix} 0 & b\rho_u - \frac{[(1-\beta\rho_u)b-(\rho_u+\theta_u)\rho_u](1-\rho_u)\sigma}{\kappa} \end{bmatrix}$,
and $\mathbf{T}_\mu^i \equiv \begin{bmatrix} \frac{-(\kappa\sigma-\lambda)a_2\mu}{\lambda} & \frac{-(\kappa\sigma-\lambda)(b_2-b)\mu}{\lambda} \end{bmatrix}$.

Multiplying the left- and right-hand sides of (S.14) by $(1-\rho_\eta L)(1-\rho_u L)(1-\mu L)$ leads to the last line of (30) with

$$\begin{aligned} \mathbf{T}_i(X) \;\equiv\; & \mathbf{T}_1^i + \left[-(\rho_\eta+\rho_u+\mu)\mathbf{T}_1^i + \mathbf{T}_\eta^i + \mathbf{T}_u^i + \mathbf{T}_\mu^i\right] X \\ & + \left[(\rho_\eta\rho_u + \rho_\eta\mu + \rho_u\mu)\mathbf{T}_1^i - (\rho_u+\mu)\mathbf{T}_\eta^i - (\rho_\eta+\mu)\mathbf{T}_u^i - (\rho_\eta+\rho_u)\mathbf{T}_\mu^i\right] X^2 \\ & + \left[-\rho_\eta\rho_u\mu\mathbf{T}_1^i + \rho_u\mu\mathbf{T}_\eta^i + \rho_\eta\mu\mathbf{T}_u^i + \rho_\eta\rho_u\mathbf{T}_\mu^i\right] X^3. \end{aligned}$$

Finally, it can be checked that $[1 \quad 0]\mathbf{T}_Z(\rho_u^{-1}) \neq \mathbf{0}$, $[1 \quad 0]\mathbf{T}_Z(\mu^{-1}) \neq \mathbf{0}$, $[0 \quad 1]\mathbf{T}_Z(\rho_u^{-1}) \neq \mathbf{0}$, $[0 \quad 1]\mathbf{T}_Z(\mu^{-1}) \neq \mathbf{0}$, $\mathbf{T}_i(\rho_\eta^{-1}) \neq \mathbf{0}$, $\mathbf{T}_i(\rho_u^{-1}) \neq \mathbf{0}$, and $\mathbf{T}_i(\mu^{-1}) \neq \mathbf{0}$, except possibly in zero-measure cases. Therefore, the ARMA(p, q) representation (30) of the path $\widetilde{P}$ is generically of *minimal* orders p and q.

S.2 Optimal Feasible Path in Svensson and Woodford's Model

Svensson and Woodford (2005) compute the timeless-perspective optimal feasible path when $\mathcal{CB}$'s observation set is $\{\varepsilon^{\eta,t-1}, \varepsilon^{u,t-1}\}$. They provide the following expressions for $\mathbb{E}_t\{\pi_{t+1}\}$, $\mathbb{E}_t\{y_{t+1}\}$, and i_t on this path, as functions of η_{t-1} and u^t:[2]

$$\mathbb{E}_t\{\pi_{t+1}\} = \frac{\rho_u\mu}{1-\beta\rho_u\mu}u_t - \frac{(1-\mu)\rho_u\mu}{1-\beta\rho_u\mu}\sum_{j=1}^{+\infty}\mu^{j-1}u_{t-j}, \tag{S.15}$$

$$\mathbb{E}_t\{y_{t+1}\} = \frac{-\kappa\rho_u\mu}{\lambda(1-\beta\rho_u\mu)}\sum_{j=0}^{+\infty}\mu^j u_{t-j}, \tag{S.16}$$

$$i_t = \sigma\rho_\eta\eta_{t-1} + \frac{(\lambda-\kappa\sigma)\rho_u^2\mu}{\lambda(1-\beta\rho_u\mu)}u_{t-1} - \frac{(\lambda-\kappa\sigma)(1-\mu)\rho_u\mu}{\lambda(1-\beta\rho_u\mu)}\sum_{j=1}^{+\infty}\mu^{j-1}u_{t-j}. \tag{S.17}$$

Using these expressions, the IS equation (38), the Phillips curve (39), and the definition of μ, I easily get π_t and y_t on this path as functions of ε_t^η and u^t:

$$\pi_t = u_t + \left(\frac{\rho_u\mu}{1-\beta\rho_u\mu} - \rho_u\right)u_{t-1} - \frac{(1-\mu)\rho_u\mu}{1-\beta\rho_u\mu}\sum_{j=2}^{+\infty}\mu^{j-2}u_{t-j}, \tag{S.18}$$

$$y_t = \varepsilon_t^\eta - \frac{\kappa\rho_u\mu}{\lambda(1-\beta\rho_u\mu)}\sum_{j=1}^{+\infty}\mu^{j-1}u_{t-j}. \tag{S.19}$$

Multiplying the left- and right-hand sides of (S.18) and (S.19) by $(1-\rho_u L)(1-\mu L)$ leads to the first two lines of (40), with

$$\mathbf{T}_Z^{SW}(X) \equiv \begin{bmatrix} 0 & 1 + \left(\frac{\rho_u\mu}{1-\beta\rho_u\mu} - \rho_u - \mu\right)X - \frac{\beta\rho_u^2\mu^2}{1-\beta\rho_u\mu}X^2 \\ (1-\rho_u X)(1-\mu X) & \frac{-\kappa\rho_u\mu}{\lambda(1-\beta\rho_u\mu)}X \end{bmatrix}.$$

Multiplying the left- and right-hand sides of (S.17) by $(1-\rho_\eta L)(1-\rho_u L)(1-\mu L)$ leads to the last line of (40), with

$$\mathbf{T}_i^{SW}(X) \equiv \begin{bmatrix} \sigma\rho_\eta(1-\rho_u X)(1-\mu X) & \frac{-(\lambda-\kappa\sigma)\rho_u\mu}{\lambda(1-\beta\rho_u\mu)}(1-\rho_\eta X)(1-\rho_u-\mu+\rho_u\mu X) \end{bmatrix}.$$

It is easy to check that $[1 \quad 0]\mathbf{T}_Z^{SW}(\rho_u^{-1}) \neq \mathbf{0}$, $[1 \quad 0]\mathbf{T}_Z^{SW}(\mu^{-1}) \neq \mathbf{0}$, $[0 \quad 1]\mathbf{T}_Z^{SW}(\rho_u^{-1}) \neq \mathbf{0}$, $[0 \quad 1]\mathbf{T}_Z^{SW}(\mu^{-1}) \neq \mathbf{0}$, $\mathbf{T}_i^{SW}(\rho_\eta^{-1}) \neq \mathbf{0}$, $\mathbf{T}_i^{SW}(\rho_u^{-1}) \neq \mathbf{0}$, and $\mathbf{T}_i^{SW}(\mu^{-1}) \neq \mathbf{0}$, except possibly in zero-measure cases. Therefore, the ARMA(p, q) representation (40) is

[2] There are two differences between Equations (S.15), (S.16), (S.17) in this supplementary appendix, and Equations (26), (27), (32) in Svensson and Woodford (2005). First, as mentioned in the main text, I have set the mean of η_t to zero for simplicity. Second, I have corrected a typo in their Equation (27); more specifically, I have removed the negative sign just after the equality sign in this equation.

generically of *minimal* orders p and q. Finally, it is also easy to check that the polynomials $(1-\rho_\eta X)\det[\mathbf{T}_Z^{SW}(X)]$ and $\mathbf{T}_i^{SW}(X)adj[\mathbf{T}_Z^{SW}(X)]$ are divisible by $D(X) \equiv (1-\rho_u X)(1-\mu X)$, but not by any scalar polynomial of higher degree. Therefore, $D(X)$ is the greatest common divisor (defined up to a non-zero real-number multiplicative factor) of these two polynomials.

S.3 Some Reduced-Form Parameters

$$\mathbf{A} \equiv \frac{1}{\alpha + \left(1+\chi-\frac{\alpha}{1-\tau}\right)\frac{s_c}{\sigma}}\left[\mathbf{A}_1 \quad \mathbf{A}_2\right],$$

where

$$\mathbf{A}_1 \equiv \begin{bmatrix} \frac{\alpha s_x}{\delta} & \frac{-(1-\delta)\alpha s_x}{\delta} + (1-\alpha)(1+\chi)\frac{s_c}{\sigma} \\ \frac{\alpha}{\delta} + \left(1+\chi-\frac{\alpha}{1-\tau}\right)\frac{s_c}{\delta\sigma} & \frac{-(1-\delta)}{\delta}\left[\alpha + \left(1+\chi-\frac{\alpha}{1-\tau}\right)\frac{s_c}{\sigma}\right] \\ \frac{s_x}{\delta} & \frac{-(1-\delta)s_x}{\delta} - (1-\alpha)\left[1-\frac{s_c}{(1-\tau)\sigma}\right] \\ -\left(1+\chi-\frac{\alpha}{1-\tau}\right)\frac{s_x}{\delta} & \left(1+\chi-\frac{\alpha}{1-\tau}\right)\frac{(1-\delta)s_x}{\delta} + (1-\alpha)(1+\chi) \\ \left(1+\frac{\omega\tau}{1-\tau}\right)\frac{\alpha s_x}{\delta} & \frac{(1-\alpha)(1+\chi)\omega\tau s_c}{(1-\tau)\sigma} - \left(1+\frac{\omega\tau}{1-\tau}\right)\frac{(1-\delta)\alpha s_x}{\delta} - \alpha\left[1+\left(\chi-\frac{\tau}{1-\tau}\right)\frac{s_c}{\sigma}\right] \\ -\left(1-\frac{\alpha}{1-\tau}\right)\frac{s_x}{\delta} & \left(1-\frac{\alpha}{1-\tau}\right)\frac{(1-\delta)s_x}{\delta} + (1-\alpha)\left[1+\frac{\chi s_c}{(1-\tau)\sigma}\right] \\ \frac{-\alpha s_x}{\delta} & \frac{(1-\delta)\alpha s_x}{\delta} - (1-\alpha)(1+\chi)\frac{s_c}{\sigma} \end{bmatrix},$$

$$\mathbf{A}_2 \equiv \begin{bmatrix} (1+\chi)\frac{s_c}{\sigma} & \left(1-\frac{s_c}{\varphi\sigma}\right)\alpha s_g \\ 0 & 0 \\ -\left[1-\frac{s_c}{(1-\tau)\sigma}\right] & \left(1-\frac{s_c}{\varphi\sigma}\right)s_g \\ 1+\chi & -\left(1+\chi+\frac{\alpha}{\varphi}-\frac{\alpha}{1-\tau}\right)s_g \\ (1+\chi)\left(1+\frac{\omega\tau}{1-\tau}\right)\frac{s_c}{\sigma} & \left(1-\frac{s_c}{\varphi\sigma}\right)\left(1+\frac{\omega\tau}{1-\tau}\right)\alpha s_g - \left[\alpha+\left(1+\chi-\frac{\alpha}{1-\tau}\right)\frac{s_c}{\sigma}\right]\frac{s_g\omega}{\varphi} \\ \left[1+\frac{\chi s_c}{(1-\tau)\sigma}\right] & -\left(1+\frac{\alpha\sigma+\chi s_c}{\varphi\sigma}-\frac{\alpha}{1-\tau}\right)s_g \\ -(1+\chi)\frac{s_c}{\sigma} & \left[\alpha+\left(1+\chi-\frac{\alpha}{1-\tau}\right)\frac{s_c}{\sigma}\right](1-\tau)\frac{s_g}{\varphi\tau} - \left(1-\frac{s_c}{\varphi\sigma}\right)\alpha s_g \end{bmatrix},$$

with $s_c \equiv 1 - s_g - s_x$ and $\varphi \equiv (1-\tau)\left[\alpha + (1-\alpha)\omega\right]$;

$$\mathbf{B} \equiv \frac{s_c}{(1+\chi)s_c + \alpha(\sigma - s_c)}\left[\mathbf{B}_1 \quad \mathbf{B}_2\right],$$

where

$$\mathbf{B}_1 \equiv \begin{bmatrix} \frac{\alpha\sigma s_x}{\delta s_c} & (1-\alpha)(1+\chi) - \frac{(1-\delta)\alpha\sigma s_x}{\delta s_c} \\ \frac{1+\chi}{\delta} + \frac{\alpha(\sigma-s_c)}{\delta s_c} & \frac{-(1-\delta)}{\delta}\left[1+\chi+\frac{\alpha(\sigma-s_c)}{s_c}\right] \\ \frac{\sigma s_x}{\delta s_c} & \frac{-(1-\alpha)(\sigma-s_c)}{s_c} - \frac{(1-\delta)\sigma s_x}{\delta s_c} \\ \frac{(\alpha\sigma-s_c)\sigma s_x}{\delta s_c^2} & \frac{(1-\alpha)(1+\chi)\sigma}{s_c} - \frac{(1-\delta)(\alpha\sigma-s_c)\sigma s_x}{\delta s_c^2} \\ \frac{\alpha\sigma s_x}{\delta s_c} & -\alpha\left(\chi+\frac{\sigma}{s_c}\right) - \frac{\alpha(1-\delta)\sigma s_x}{\delta s_c} \\ \left(\chi+\frac{\alpha\sigma-s_c}{s_c}\right)\frac{\sigma s_x}{\delta s_c} & (1-\alpha)\left(\chi+\frac{\sigma}{s_c}\right) - \frac{(1-\delta)\sigma s_x}{\delta s_c}\left(\chi+\frac{\alpha\sigma-s_c}{s_c}\right) \end{bmatrix},$$

$$\mathbf{B}_2 \equiv \left[\begin{array}{ccc} \frac{-\alpha\tau}{1-\tau} & 1+\chi & \frac{\alpha\sigma s_g}{s_c} \\ 0 & 0 & 0 \\ \frac{-\tau}{1-\tau} & \frac{-(\sigma-s_c)}{s_c} & \frac{\sigma s_g}{s_c} \\ \frac{-\alpha\sigma\tau}{(1-\tau)s_c} & \frac{(1+\chi)\sigma}{s_c} & \frac{(\alpha\sigma-s_c)\sigma s_g}{s_c^2} \\ \frac{-\alpha\tau}{1-\tau} - \left[1+\chi+\frac{\alpha(\sigma-s_c)}{s_c}\right]\frac{\omega\tau}{1-\tau} & 1+\chi & \frac{\alpha\sigma s_g}{s_c} \\ -\left(\chi+\frac{\alpha\sigma}{s_c}\right)\frac{\tau}{1-\tau} & \chi+\frac{\sigma}{s_c} & \left(\chi+\frac{\alpha\sigma-s_c}{s_c}\right)\frac{\sigma s_g}{s_c} \end{array} \right];$$

and

$$\begin{aligned}
P_k(X) &\equiv \left[B_{41} - \frac{1-\beta(1-\delta)}{\sigma}B_{51}\right] + \left[B_{42} - B_{41} - \frac{1-\beta(1-\delta)}{\sigma}B_{52}\right]X - B_{42}X^2, \\
P_\tau(X) &\equiv \left[B_{43} - \frac{1-\beta(1-\delta)}{\sigma}B_{53}\right] - B_{43}X, \\
Q_b(X) &\equiv 1 - \beta^{-1}X, \\
Q_k(X) &\equiv [\alpha + (1-\alpha)\omega]\,\tau\,(B_{11} + B_{12}X), \\
P_a &\equiv -B_{44}, \\
P_g &\equiv -B_{45}, \\
Q_\tau &\equiv [\alpha + (1-\alpha)\omega]\,\tau\,(B_{13} + 1), \\
Q_a &\equiv [\alpha + (1-\alpha)\omega]\,\tau B_{14}, \\
Q_g &\equiv [\alpha + (1-\alpha)\omega]\,\tau B_{15} - s_g.
\end{aligned}$$