ENSAE Paris - IP Paris
Academic year 2023-2024
First exam session (2A S1)

Final exam of the course "Macroeconomics 1"

Duration : 2 hours. No document allowed, no calculator allowed.
The grading scale, which may be modified, is indicated only for general-guidance purposes.

1 Lecture questions (6 points)

Answer very briefly the following questions (one or two sentences are enough for each question, and no equation is necessary).

Question 1 Can the announcement of a future event impact the economy immediately in the Cass-Koopmans-Ramsey model ? And in the Solow-Swan model ? Why ?

Question 2 For what purpose does one consider a "benevolent, omniscient and omnipotent planner" ?

Question 3 What is the socially optimal economic policy in the Cass-Koopmans-Ramsey model, and why ?

Question 4 In the DICE model at the steady state, does the discount rate (r) depend positively or negatively on the uncertainty around the future growth rate of the economy (g), and why ?

Question 5 What is a Pigouvian tax ? Give one example seen in the course or in the tutorials.

Question 6 What is the Ricardian equivalence ? Give one reason why it may not be empirically satisfied.

2 Exercise 1 : Shocks in the growth model with learning by doing (10 points)

We consider the growth model with learning by doing studied in the course and the tutorials, in the specific case of a constant elasticity of intertemporal substitution, equal to $1/\theta$, where $\theta > 0$ and $\theta \neq 1$.

As a reminder, in this model, time is continuous, indexed by t. The production function F, homogeneous of degree one, strictly increasing and strictly concave in each of its arguments, is the same for all firms : for each firm i, $Y_{i,t} = F(K_{i,t}, A_t N_{i,t})$, where $Y_{i,t}$ is its output, $K_{i,t}$ its capital stock, and $N_{i,t}$ its labor demand. The corresponding aggregate variables are $Y_t \equiv \sum_{i=1}^{I} Y_{i,t}$, $K_t \equiv \sum_{i=1}^{I} K_{i,t}$, and $N_t \equiv \sum_{i=1}^{I} N_{i,t}$. The productivity variable A_t is assumed to be equal to

$$A_t = \frac{K_t}{L_t}, \tag{1}$$

where L_t is the population size (assumed to be exogenous, increasing at rate n, and supplying one unit of labor per person). Each household can hold two types of assets : loans to other households, and capital ownership titles. We use exactly the same notations as in the lectures and the tutorials : w_t denotes the real wage, r_t the real interest rate, z_t the real usage cost of capital, ρ the rate of time preference, δ the capital-depreciation rate, B_t the aggregate total amount of assets, and C_t aggregate consumption at date t. We use smaller-case letters to denote per-capita variables, for instance $b_t \equiv B_t/L_t$. It is reminded that the representative household's optimization problem is the following : for $(r_t, w_t)_{t\geq 0}$ and b_0 given,

$$\max_{(c_t)_{t\geq 0}, (b_t)_{t>0}} \left[L_0 \int_0^{+\infty} e^{-(\rho-n)t} \left(\frac{c_t^{1-\theta} - 1}{1-\theta} \right) dt \right]$$

subject to

$$\forall t \geq 0,\ c_t \geq 0, \tag{2}$$

$$\forall t \geq 0,\ \dot{b}_t = (r_t - n)b_t + w_t - c_t, \tag{3}$$

$$\lim_{t\to+\infty} \left[b_t e^{-\int_0^t (r_\tau - n)d\tau} \right] \geq 0. \tag{4}$$

2.1 Differential equation in $\dot{k}_t$

Question 7 Interpret the equation (1). Admitting $Y_t = F(K_t, A_t N_t)$ and using (1) and the labor-market-clearing condition, express Y_t as a function of K_t.

Question 8 Write the goods-market-clearing condition at date t, involving $\dot{K}_t$, Y_t, C_t, K_t, and δ. Deduce the differential equation

$$\dot{k}_t = f(1)k_t - c_t - (n+\delta)k_t, \tag{5}$$

where $f(x) \equiv F(x, 1)$.

2.2 Differential equation in $\dot{c}_t$

Question 9 Interpret the constraints (3) and (4). Write the Hamiltonian of the representative household's problem, and then the first-order condition on the control variable and the costate equation.

Question 10 Deduce that $\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho}{\theta}$. Interpret the effects of r_t and ρ on $\frac{\dot{c}_t}{c_t}$. Briefly explain, without using any equation, why we have $r_t = z_t - \delta$ and $z_t = f'(1)$ in equilibrium. Deduce the differential equation

$$\frac{\dot{c}_t}{c_t} = \frac{f'(1) - (\delta + \rho)}{\theta}. \tag{6}$$

2.3 Equilibrium for a constant δ

As a reminder, using the differential equations (5) and (6), the initial condition (k_0 exogenous) and the terminal condition (coming from the transversality condition), we obtain the following trajectories for k_t and c_t in the competitive equilibrium, when the model's parameters (in particular δ) are constant over time :

$$k_t = k_0 e^{g(\delta)t} \text{ and } c_t = \varphi(\delta) k_0 e^{g(\delta)t},$$

where $g(\delta) \equiv \frac{f'(1)-(\delta+\rho)}{\theta} > 0$ and $\varphi(\delta) \equiv f(1) - (n+\delta) - g(\delta) > 0$.

2.4 Effect of a permanent decrease in δ

Question 11 We assume that initially δ takes the value δ_1 and agents expect (incorrectly) δ to remain at this value thereafter. At $t = T > 0$, δ falls unexpectedly from δ_1 to δ_2, where $\delta_2 < \delta_1$, and then remains at δ_2 thereafter. From date T onwards, agents expect (correctly) that $\delta = \delta_2$ thereafter. (We assume that $\varphi(\delta_j) > 0$ and $g(\delta_j) > 0$ for $j \in \{1, 2\}$.) Draw the form that the trajectories of $ln(k_t)$ and $ln(c_t)$ take over time $t \geq 0$, depending on whether $\theta < 1$, $\theta = 1$, or $\theta > 1$. Interpret.

Question 12 We consider the same scenario as in the previous question, except that the permanent decrease in δ at date T is now expected by the agents from date 0. What equilibrium conditions (two differential equations, one initial condition, and one terminal condition) should we use to analytically obtain the trajectories of $ln(k_t)$ and $ln(c_t)$ as a function of t for $0 \leq t \leq T$? Use these conditions to get

$$c_0 = \frac{\varphi(\delta_1) k_0}{\left[1 - e^{-\varphi(\delta_1)T}\right] + \frac{\varphi(\delta_1)}{\varphi(\delta_2)} \left[e^{-\varphi(\delta_1)T}\right]} \quad \text{and} \quad c_T = c_0 e^{g(\delta_1)T}.$$

Question 13 Is the value taken by c_0 in Question 12 higher than, equal to, or lower than the value taken by c_0 in Question 11 ? Same question for the value taken by c_t just before T, and for the value taken by c_t just after T. Draw, on the same graph as in Question 11, the form that the trajectory of $ln(c_t)$ takes over time $t \geq 0$ in Question 12, depending on whether $\theta < 1$, $\theta = 1$, or $\theta > 1$. Interpret. What does c_0 become as $T \to +\infty$, and why ?

3 Exercise 2 : The "green Solow-Swan model" and the "environmental Kuznets curve" (4 points)

This exercise is derived from an article by Brock and Taylor, published in 2010.[1] The goal of this article is to propose an explanation to the "environmental Kuznets curve". This curve is a hypothesis according to which, as an economy grows, the total quantity of pollutants emitted in this economy first increases, and then decreases. This hypothesis seems to be verified in the data for certain pollutants and certain countries (but not for all pollutants and all countries).

We consider the Solow-Swan model studied in the course and the tutorials, with a Cobb-Douglas production function. As a reminder, in this model, time is continuous, indexed

1. Brock, W.A., and Taylor, M.S., 2010, "The Green Solow Model", Journal of Economic Growth, 15, 127-153.

by t. The aggregate production Y_t is $Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}$ with $0 < \alpha < 1$, where K_t is the aggregate stock of capital, A_t labor productivity (exogenous, increasing at rate g), and L_t the population size (exogenous, increasing at rate n, and supplying one unit of labor per person) at date t. We denote by s the saving rate (exogenous), δ the capital-depreciation rate, and $\kappa_t \equiv K_t/(A_t L_t)$ the aggregate capital stock per unit of effective labor at date t.

We assume that producing goods emits pollutants. More specifically, in the absence of any effort to reduce pollution, producing Y_t units of good at date t would emit $E_t = \Omega_t Y_t$ units of pollutants, where Ω_t is exogenous and decreases at rate $g_\Omega > 0$: $\Omega_t = \Omega_0 e^{-g_\Omega t}$, where $\Omega_0 > 0$. The fact that Ω_t decreases over time can be interpreted as the consequence of an exogenous technological progress in the "greening" of the production process (requiring no particular effort).

We also assume that there exists, in addition, a technology to reduce pollution : with X_t units of good, one reduces pollution by $\Omega_t R(Y_t, X_t)$ units of pollutants, where the function R is strictly increasing and strictly concave in each of its arguments, and homogeneous of degree one. Lastly, we assume that an exogenous and constant fraction of production is devoted to reducing pollution : $X_t/Y_t = \chi$, where $0 < \chi < 1$. The reduction in pollution is therefore $\Omega_t R(Y_t, X_t) = \Omega_t Y_t R(1, X_t/Y_t) = \Omega_t Y_t R(1, \chi) = \eta \Omega_t Y_t$, where $\eta \equiv R(1, \chi) > 0$. Taking into account this reduction, the total quantity of pollutants emitted is therefore $E_t = (1 - \eta)\Omega_t Y_t$.

The saving rate s applies to the goods that are not devoted to reducing pollution $(Y_t - X_t)$: the quantity $s(Y_t - X_t)$ of these goods is saved, and the quantity $(1-s)(Y_t - X_t)$ of these goods is consumed.

Question 14 Show that we have the following differential equation :

$$\dot{\kappa}_t = s(1 - \chi)\kappa_t^{\alpha} - (n + g + \delta)\,\kappa_t.$$

Define the steady state. Show that κ_t is constant at the steady state, and determine its value κ^*. Briefly interpret the way in which κ^* depends on the parameters s, χ, n, g, and δ. Show that for any κ_0, κ_t converges over time to κ^*.

Question 15 Show that

$$\frac{\dot{E}_t}{E_t} = (n + g - g_\Omega) + \alpha s(1 - \chi)\left[\left(\frac{1}{\kappa_t}\right)^{1-\alpha} - \left(\frac{1}{\kappa^*}\right)^{1-\alpha}\right].$$

What are the necessary and sufficient conditions on κ_0 and the model's parameters to obtain an environmental Kuznets curve, i.e. to ensure that from date 0 onwards the growth rate of the economy is always positive and the growth rate of pollutant emissions $\dot{E}_t/E_t$ is first positive, and then negative ? Interpret.