# Stabilization Policy, Lags, and Determinacy

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**Abstract**: Macroeconomic stabilization policy is notoriously subject to inside lags (which delay the reaction of policy to the state of the economy) and outside lags (which delay the effects of policy on the economy). In a broad class of dynamic rational-expectations models, I show that under a weak condition, neither inside lags nor outside lags of any length hinder the ability of the policymaker to ensure local-equilibrium determinacy, no matter how many variables the policymaker observes. To establish this result, I invert the problem usually tackled in the literature: I start from a targeted characteristic polynomial, and I derive a corresponding policy-instrument rule. For any lags, this method offers degrees of freedom that can be exploited, beyond determinacy, for implementation purposes (controlling the response of the economy to non-news and news shocks) and robustness purposes (designing non-superinertial rules that may prove more robust under model uncertainty).

**Keywords**: stabilization policy, inside lags, outside lags, local-equilibrium determinacy, implementation, non-superinertial rules.

**JEL codes**: E32, E52.

## 1 Introduction

One of the main problems faced by macroeconomic stabilization policy is the existence of lags. Economists distinguish between two kinds of lags. Recognition, decision, and implementation lags, called "inside lags," delay the reaction of policy to the state of the economy. Transmission lags, called "outside lags," delay the effects of policy on the economy. Inside and outside lags are not equally problematic for all policies: for instance, monetary policy is thought to have a shorter inside lag and a longer outside lag than fiscal policy (Mankiw, 2019, Chapter 16). But all policies are subject to lags, to some extent or another.

The existence of lags has been known for a long time. Friedman (1961), for instance, famously emphasized long and variable outside lags for monetary policy. Much has been written about the implications of lags for stabilization policy in dynamic models without rational expectations (e.g.,

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Fisher and Cooper, 1973); much less so, however, in dynamic models with rational expectations. Now, the latter models raise the distinctive issue of local-equilibrium determinacy (i.e. existence and uniqueness of a stationary solution to the locally log-linearized model). Indeterminacy opens the door to sunspot-driven macroeconomic fluctuations, which are typically detrimental to welfare. So, it may be feared that inside and outside lags, by putting the policymaker behind the curve, could prevent her from ensuring determinacy.[1] Do they? The main contribution of the paper is to provide a general, negative answer to this question.

The literature (reviewed below) addresses this question only in some sparse examples, and provides little insight into whether and how the results could be extended beyond these examples. The approach is typically the following: (i) consider a specific dynamic rational-expectations model, with outside lags of a specific length (of length zero if the focus is on inside lags); (ii) consider a specific parametric family of policy-instrument rules, consistent with inside lags of a specific length (of length zero if the focus is on outside lags); and (iii) derive the characteristic polynomial of the resulting system, and find under what inequality conditions on the model's parameters and the rule's coefficients the roots of this polynomial satisfy Blanchard and Kahn's (1980) root-counting condition for determinacy. These inequality conditions can be obtained analytically for very simple models and rules; for more complex models and rules, however, the results are necessarily numerical, and hold only for some specific calibrations.

In this paper, I invert the problem: for a given model, I choose a characteristic polynomial, and I derive a corresponding policy-instrument rule, i.e. a rule such that the system consisting of the structural equations and this rule has this characteristic polynomial. I choose a characteristic polynomial that has as many roots outside the unit circle of the complex plane as there are non-predetermined variables in the system, so as to meet Blanchard and Kahn's (1980) root-counting condition. I can do that for a broad class of dynamic rational-expectations models, for any value of their structural parameters, and for inside and outside lags of any length. I thus get the general result that inside and outside lags do not hinder the ability of the policymaker to ensure determinacy.

Start with inside lags. These lags prevent the policymaker from reacting to current or recent endogenous variables set by the private sector. Many rules considered in the literature, for instance Taylor's (1993) popular interest-rate rule – and its extensions – in monetary-policy models, are not consistent with such lags because they make the policy instrument react to current endogenous variables. My results show that this contemporaneous reaction is not needed to ensure determinacy. In essence, the reason why inside lags do not reduce the policymaker's ability to ensure determinacy is that, in models that raise non-trivial determinacy issues, the private sector's current decisions depend directly on its expected future decisions and, therefore, indirectly on the expected future policy instrument. Now, by making the current policy instrument react to the private sector's past decisions, a rule also makes the expected future policy

[1]For convenience, throughout the paper, I refer to the policymaker with the female pronoun "she."

instrument react to the private sector's current decisions. The latter reaction can be viewed as the feedback mechanism that enables the policymaker to ensure determinacy.

Now turn to outside lags. These lags make current endogenous variables depend only on past variables and past expectations, so that they are predetermined and cannot be affected (neither directly nor indirectly) by current or recent policy shocks. In their simplest form (which I call "non-distributed outside lags"), outside lags make the private sector decide on its actions a single fixed number of periods $\ell'$ in advance. In this case, outside lags are neutral for determinacy. The reason is simple. Compared to the situation without outside lags, on the one hand, the policymaker is lagging behind because she reacts to variables that were actually set $\ell'$ periods earlier; on the other hand, these variables were set on the basis of expectations whose horizon is $\ell'$ periods longer, and there is perfect foresight up to horizon $\ell'$. So, the feedback loop between endogenous variables and the policy instrument is essentially the same as without outside lags.

In the more sophisticated form of outside lags (which I call "distributed outside lags"), an endogenous variable results from several decisions made by the private sector at different dates in the past. In this case, the structural equations involve expectations formed at different past dates. To rewrite the dynamic system in Blanchard and Kahn's (1980) form, which involves only current expectations, one needs to treat these expectations as new variables. These variables are *latent* in the sense that they play a key role in the dynamics of the system but are not observed by the policymaker (if, as I assume throughout the paper, the policymaker does not observe private agents' expectations). So, the question becomes whether the policymaker can ensure determinacy when she observes only a subset of variables. This question is, of course, of broader interest, as one can think of other kinds of unobserved variables (e.g., Lagrange multipliers of private agents' optimization problems).

I show that the answer to this question is positive under a certain condition on the subset of variables observed by the policymaker. I argue that this condition is likely to be met in practice in monetary-policy models, as long as the subset of variables is not empty. Indeed, I show that this condition is necessarily met if the model has at least one stationary solution when the policy instrument is set exogenously (and stationarily). Now, it is well known that monetary-policy models typically have multiple stationary solutions when the interest rate is set exogenously – a property that Giannoni and Woodford (2002) and Woodford (2003, Chapter 8) call the "Sargent-Wallace property," after Sargent and Wallace (1975). So, these models necessarily have the weaker form of Sargent-Wallace property that I am considering. In these models, therefore, observing a single variable is enough for the policymaker to ensure determinacy, and distributed outside lags of any length are not an obstacle (even in conjunction with inside lags of any length).

To satisfy Blanchard and Kahn's (1980) root-counting condition, I only need the characteristic polynomial of the dynamic system to have a certain number of roots outside the unit circle (as many as there are non-predetermined variables in the system). However, the method that I use − starting from a characteristic polynomial and finding a corresponding rule − enables me to choose not only the *number* of roots outside the unit circle, but also the *values* of these roots, as well as the number of roots *inside* the unit circle and their values. I show that these degrees of freedom can be exploited for implementation purposes. More specifically, suppose that the policymaker observes all exogenous shocks with the same inside lags as endogenous variables.[2] Then, choosing the number of roots inside the unit circle and their values enables her to implement, as the unique local equilibrium, any given stationary VARMA path consistent with the structural equations and the inside lags − even if she has finite memory, i.e. even if the policy-instrument rule has to involve a finite number of terms. This result also obtains if some shocks are unobserved but can be inferred from the observed variables and shocks using only the structural equations (as, e.g., a technology shock can be inferred from the observed input and output levels using only the production function). The reason is that using the structural equations to replace, in a policy-instrument rule, these unobserved shocks by functions of observed variables and shocks is neutral for determinacy. Finally, choosing the values of the roots outside the unit circle enables the policymaker to partially control the response of the economy to news shocks.

Even for a given characteristic polynomial, there remain some degrees of freedom, which opens the door to the design of rules with additional properties. To illustrate these degrees of freedom, I design policy-instrument rules that not only are consistent with inside lags, ensure determinacy, and fully or partially control the response of the economy to non-news and news shocks, but also are not *superinertial* in the sense of Woodford (1999). As I document below, optimal interest-rate rules are often found to be superinertial in monetary-policy models with rational expectations (also called "forward-looking models"); however, superinertial interest-rate rules are often found to lead to non-existence of a local equilibrium in monetary-policy models without rational expectations (also called "backward-looking models"). These findings suggest that superinertial rules should be avoided when there is a non-zero probability that the true model is backward-looking, even if this probability is arbitrarily small. They provide, therefore, a motivation for designing non-superinertial rules. To be clear, the design of rules whose properties are robust across alternative models is mostly beyond the scope of this paper. I make only a small step in this direction by identifying some degrees of freedom that can be exploited for robustness purposes.

I design these non-superinertial rules under a certain condition on the set of variables observed by the policymaker. This condition is stronger than the condition under which I design rules ensuring determinacy (which is itself implied by the weak Sargent-Wallace property discussed above). To get a sense of how strong or weak these two conditions are in practice, I consider four stabilization-policy models: the basic New Keynesian (NK) model, presented in detail in

[2] Examples of observable exogenous shocks include exogenous policy measures and foreign macroeconomic developments (considered as exogenous from the point of view of a small open economy).

Woodford (2003) and Galí (2015), and the models of Rotemberg and Woodford (1997, 1999), Smets and Wouters (2007), and Schmitt-Grohé and Uribe (1997). These models differ from each other in several dimensions: small vs. medium scale, with vs. without distributed outside lags, monetary vs. fiscal policy. I show that, in these models, for all structural-parameter values (except possibly a zero-measure subset) and in the presence of inside lags of any length, the policymaker can ensure determinacy for any non-empty set of observed variables, and she can ensure determinacy with a non-superinertial rule for almost any non-empty and non-singleton set of observed variables.

A few remarks may serve to put my contribution in the context of the literature. McCallum (1999) argues that interest-rate rules need to take inside lags into account to be operational for monetary policy. Bullard and Mitra (2002) and Carlstrom and Fuerst (2002), among others, analytically derive determinacy conditions in a simple model under a simple interest-rate rule with one-period inside lags. Rotemberg and Woodford (1999) also study determinacy under a simple interest-rate rule with one-period inside lags, but numerically, in a more complex model. Benhabib (2004) does the same, but considers much longer inside lags (up to 30 or 60 periods in the discrete-time version of his model). He finds (numerically) that inside lags tend to increase the likelihood of indeterminacy, for a given model and a given parametric family of rules. The reason is, presumably, that they increase the number of roots of the characteristic polynomial without increasing the number of non-predetermined variables. Turning to fiscal policy, Schmitt-Grohé and Uribe (1997) numerically study determinacy under a tax-rate rule with inside lags that stabilizes the expected future stock of public debt (in the discrete-time version of their model).

Giannoni and Woodford (2003), Woodford (2003, Chapter 5), and Svensson and Woodford (2005) consider the basic New Keynesian model with non-distributed outside lags. Examples of more complex, quantitative monetary-policy models with outside lags include Christiano et al. (2005), Giannoni and Woodford (2005), Boivin and Giannoni (2006) for non-distributed lags (of length one or two), and Rotemberg and Woodford (1997, 1999) for distributed lags (of lengths one and two). Woodford (2003, Chapter 5) and Svensson and Woodford (2005) analytically derive determinacy conditions under a simple interest-rate rule in the basic New Keynesian model with non-distributed outside lags. Rotemberg and Woodford (1999) numerically study the determinacy properties of some simple interest-rate rules in their model with distributed outside lags.

As I explained above, the key difference with all these papers is that I do not start from a parametric family of rules and derive determinacy conditions, either analytically or numerically. Instead, I start from a characteristic polynomial and I derive analytically a corresponding rule. This method enables me to establish much more general determinacy results with inside and outside lags than were previously established in these sparse examples.

Giannoni and Woodford (2017) also establish general determinacy results. More specifically, building on their earlier work (reported in Woodford, 2003, Chapter 8), they design analytically, in a general dynamic rational-expectations framework, "target criteria" that are consistent with the optimal path (for a given objective function) and ensure determinacy. Giannoni and Woodford (2003, 2005) can be viewed as early applications of their general theory to specific models. These target criteria, however, are not required to be formulated as a policy-instrument rule consistent with a given information set of the policymaker – in particular an information set excluding current or recent variables, because of inside lags, or excluding some variables altogether, because of distributed outside lags. In fact, in many applications, these target criteria will not even be formulated as a policy-instrument rule (as they will not involve the policy instrument). They are proposed as, in Svensson and Woodford's (2005) terminology, a "higher-level policy specification."

In this sense, my paper is complementary to Giannoni and Woodford (2017): to evaluate the implications of inside and outside lags on the ability of the policymaker to ensure determinacy, I need to focus on a lower level of policy specification, namely the operational level of policy-instrument rules. Like Bassetto (2002, 2004, 2005), I need to confront the constraints faced out of equilibrium by the policymaker. These constraints are informational in my paper (impossibility of setting the policy instrument as a function of variables that are unobserved due to inside or outside lags), while they are of a different nature in Bassetto's papers – e.g. physical (impossibility of spending resources that do not exist).

In Loisel (2021), I study the implementability and implementation of some specific paths of interest in the basic New Keynesian model and in the Real-Business-Cycle model, as well as in a class of univariate models (i.e. models with only one variable set by the private sector). I do not allow for inside or outside lags – except in one model, but only to show that a path may not be implementable. In the present paper, by contrast, the central issue is the implications of inside and outside lags for the ability of the policymaker to ensure determinacy. Another difference is that the implementation analysis there is fundamentally univariate; here, the determinacy and implementation analyses are multivariate, and require the use of different mathematical tools (namely Bézout's identity and the Euclidean division, rather than Sylvester matrices, as I elaborate in the text). Still another difference is that I show here how to exploit some degrees of freedom in the design of rules ensuring determinacy to partially control the response of the economy to news shocks and to avoid superinertial rules.

Examples of papers that numerically find superinertial optimal interest-rate rules in forward-looking models include Rotemberg and Woodford (1999), Woodford (1999), and Levin, Wieland and Williams (1999, 2003). In these papers, superinertial rules may be optimal in the sense of implementing the welfare-optimal path as the unique local equilibrium, or in the sense of minimizing, within a given parametric family of rules, a given loss function subject to the determinacy constraint. Giannoni and Woodford (2002) and Woodford (2003, Chapter 8) show, in a broad class of forward-looking models, that their optimal "targeting rules" (which are related to the "target criteria" of Giannoni and Woodford, 2017) are superinertial policy-instrument rules if the loss function includes a separate quadratic term in the policy instrument and if the model has the Sargent-Wallace property discussed above. In their applications to monetary-policy models, Giannoni and Woodford (2003, 2005) and Woodford (2003, Chapter 8) obtain interest-rate rules that are superinertial for all structural-parameter values. Rudebusch and Svensson (1999), Taylor (1999a, 1999b), and Levin and Williams (2003) find that superinertial interest-rate rules often lead to non-existence of a local equilibrium in backward-looking models. McCallum (1988, 1999) is among the first to have called for the design of monetary-policy rules that are robust across alternative models.

The rest of the paper is organized as follows. Section 2 studies the implications of inside lags and non-distributed outside lags for determinacy in univariate models. Section 3 generalizes the analysis to multivariate models, and broadens it to distributed outside lags. The next two sections extend the analysis in two different directions: path implementation with finite memory in Section 4, and non-superinertial rules in Section 5. Section 6 applies these general results to four specific models. I then conclude and provide a technical appendix.

# 2 Lags and Determinacy in Univariate Models

I start with a class of univariate models, i.e. models with only one variable set by the private sector, and therefore only one structural equation. I show that inside lags and non-distributed outside lags do not prevent the policymaker from ensuring determinacy in this class of models. I also illustrate, with a simple example, that distributed outside lags require a multivariate analysis with latent variables, which I postpone to the next section. I do not state the results in formal propositions, since the analysis is a special case of the more general analysis conducted in the next section (in which results will be stated in formal propositions).

## 2.1 Setup with Inside Lags

The agents are a private sector ($\mathcal{PS}$) and a policymaker ($\mathcal{PM}$). At each date $t \in \mathbb{Z}$, $\mathcal{PS}$ sets an endogenous variable $z_t$ according to the following (locally log-linearized) structural equation:

$$\mathbb{E}_t \left\{ L^{-\delta} \left[ A(L) z_t + L^{-\gamma} B(L) i_t \right] \right\} = 0, \tag{1}$$

where $i_t$ denotes the policy instrument set by $\mathcal{PM}$ at date $t$, $L$ is the lag operator, and $\mathbb{E}_t\{.\}$ is the rational-expectations operator conditionally on all endogenous variables until date $t$. I abstract from exogenous disturbances for now, because they are irrelevant for determinacy issues; I will consider them in Section 4 when I address implementation issues.

The structural equation (1) is parametrized by $(A(X), B(X)) \in \mathbb{R}[X]^2$, $\gamma \in \mathbb{Z}$, and $\delta \in \mathbb{N} \setminus \{0\}$.[3]

[3]Throughout the paper, $\mathbb{R}[X]$ denotes the set of polynomials in $X$ with real-number coefficients.

I rule out the uninteresting case $\delta = 0$, in which the structural equation involves no expected future variable set by $\mathcal{PS}$. Without any loss in generality, I assume that $A(0) \neq 0$ (so that $\delta$ is the highest horizon of the expected future variables set by $\mathcal{PS}$ in the structural equation). For the policy instrument to affect the endogenous variable set by $\mathcal{PS}$, I need $B(X) \neq 0$. Without any loss in generality (since $\gamma \in \mathbb{Z}$), I assume that $B(0) \neq 0$.

Finally, I assume that $A(X)$ and $B(X)$ have at most $\delta$ common non-zero roots in the unit disk of the complex plane, $\mathbb{D} \equiv \{c \in \mathbb{C} : |c| \leq 1\}$. When $\gamma \geq 0$, this assumption is not restrictive for the following reason: if $A(X)$ and $B(X)$ had more than $\delta$ common roots in $\mathbb{D} \setminus \{0\}$, then the structural equation (1) would not have any stationary solution in $(z_t, i_t)$ (in the presence of exogenous disturbances). Indeed, this equation can be rewritten as

$$\mathbb{E}_t \left\{ L^{-\delta} D(L) \, \tilde{z}_t \right\} = 0, \tag{2}$$

where $D(X) \in \mathbb{R}[X]$ denotes the greatest common divisor of $A(X)$ and $B(X)$, defined up to a multiplicative non-zero real-number scalar, and $\tilde{z}_t \equiv [A(L)/D(L)]z_t + [L^{-\gamma}B(L)/D(L)]i_t$ is a well-defined variable set at date $t$ (when $\gamma \geq 0$). If $D(X)$ had more than $\delta$ roots in $\mathbb{D} \setminus \{0\}$ and if we added an exogenous disturbance to (2), then (2) would have no stationary solution in $\tilde{z}_t$, as follows from Blanchard and Kahn's (1980) analysis, and hence no stationary solution in $(z_t, i_t)$ either.

I assume that $\mathcal{PM}$'s information set, when she sets the policy instrument $i_t$, is $I_t = \{z^{t-\ell}, i^{t-1}\}$, where $\ell \in \mathbb{N}$ and, for any variable $v$ and any date $t$, $v^t \equiv \{v_{t-k} | k \in \mathbb{N}\}$ denotes the history of variable $v$ until date $t$. The parameter $\ell$ captures inside policy lags (i.e. recognition lags, decision lags, and implementation lags). It compels the current policy instrument $i_t$ to be set as a function of variables set at date $t - \ell$ or earlier.[4]

## 2.2 Determinacy with Inside Lags

I now design a policy-instrument rule consistent with the information set $I_t$ and ensuring determinacy. I start from the class of rules of type

$$F(L) i_t = L^h G(L) z_t \tag{3}$$

with $(F(X), G(X)) \in \mathbb{R}[X]^2$, $F(0) \neq 0$, $G(X) \neq 0$, $h \in \mathbb{N}$, and $h \geq \max(\ell, \gamma + \delta + 1)$. Since $F(0) \neq 0$ and $h \geq \ell$, (3) expresses $i_t$ as a function of only elements of $I_t$, i.e. it is a policy-instrument rule consistent with $I_t$.

The system consisting of the structural equation (1) and the rule (3) can easily be written in Blanchard and Kahn's (1980) form with exactly $\delta$ non-predetermined variables (since $A(0) \neq 0$ and $h \geq \gamma + 1$). For this system to satisfy Blanchard and Kahn's (1980) root-counting condition,

[4] An alternative way to introduce inside lags would involve the parameter $\gamma$. It is more convenient, however, to capture them separately with the parameter $\ell$.

we need its characteristic polynomial to have exactly $\delta$ roots outside the unit circle (as many as there are non-predetermined variables in the system). The characteristic polynomial of this system is the same, up to a multiplicative factor of type $X^p$ with $p \in \mathbb{N}$, as the characteristic polynomial $\mathcal{C}(X)$ of the corresponding perfect-foresight deterministic system, which consists of the equations $A(L)z_t + L^{-\gamma}B(L)i_t = 0$ and $F(L)i_t = L^h G(L)z_t$. The *reciprocal* polynomial of $\mathcal{C}(X)$ is, straightforwardly, $\mathcal{R}(X) \equiv A(X)F(X) + X^{h-\gamma}B(X)G(X)$. For Blanchard and Kahn's (1980) root-counting condition to be satisfied, therefore, we need $\mathcal{R}(X)$ to have exactly $\delta$ roots in $\mathbb{D} \setminus \{0\}$. Equivalently, we need

$$\tilde{\mathcal{R}}(X) \equiv \tilde{A}(X)F(X) + X^{h-\gamma}\tilde{B}(X)G(X)$$

to have exactly $\delta - d$ roots in $\mathbb{D} \setminus \{0\}$, where $\tilde{A}(X) \equiv A(X)/D(X)$, $\tilde{B}(X) \equiv B(X)/D(X)$, and $d$ denotes the number of roots of $D(X)$ in $\mathbb{D} \setminus \{0\}$.

Let $a$ and $b$ denote the degrees of $\tilde{A}(X)$ and $\tilde{B}(X)$ respectively. Since $h \geq \gamma + \delta + 1$ implies $a + b + h - \gamma - 1 \geq \delta - d$, I can consider an arbitrary polynomial $\Phi(X) \in \mathbb{R}[X]$ that: (i) is of degree $a+b+h-\gamma-1$, (ii) has exactly $\delta - d$ roots in $\mathbb{D}\setminus\{0\}$ and all its other roots in $\mathbb{C}\setminus\mathbb{D}$, and (iii) is not a multiple of $\tilde{A}(X)$ (except, of course, if $\tilde{A}(X)$ is of degree zero). In the following, I design $(F^*(X), G^*(X)) \in \mathbb{R}[X]^2$ such that $F^*(0) \neq 0$, $G^*(X) \neq 0$, and

$$\tilde{\mathcal{R}}^*(X) \equiv \tilde{A}(X)F^*(X) + X^{h-\gamma}\tilde{B}(X)G^*(X) = \Phi(X).$$

Therefore, $\tilde{\mathcal{R}}^*(X)$ will have exactly $\delta - d$ roots in $\mathbb{D}\setminus\{0\}$. As a consequence, the system consisting of the structural equation (1) and the rule (3) with $F(X) = F^*(X)$ and $G(X) = G^*(X)$ will satisfy Blanchard and Kahn's (1980) root-counting condition.

I start with the case in which $a = 0$. In this case, the design of $F^*(X)$ and $G^*(X)$ is trivial: since $\tilde{A}(X) = \tilde{A}(0) \neq 0$, one can choose, e.g., $F^*(X) = [\Phi(X) - X^{h-\gamma}\tilde{B}(X)]/\tilde{A}(0)$ and $G^*(X) = 1$. This choice is, straightforwardly, such that $F^*(0) \neq 0$, $G^*(X) \neq 0$, and $\tilde{\mathcal{R}}^*(X) = \Phi(X)$.

Now turn to the alternative case in which $a \geq 1$. In this case, I use the Sylvester matrix of $\tilde{A}(X)$ and $X^{h-\gamma}\tilde{B}(X)$ to design some $F^*(X)$ of degree lower than $b+h-\gamma$ and some $G^*(X)$ of degree lower than $a$. For any polynomial $P(X)$ and any $k \in \mathbb{N}$, let $P_k$ denote the coefficient of $X^k$ in $P(X)$ (with $P_k = 0$ if $k$ is higher than the degree of $P(X)$). The equation $\tilde{\mathcal{R}}^*(X) = \Phi(X)$ can then be rewritten as

$$\mathbf{S} \begin{bmatrix} G^*_{a-1} \\ \vdots \\ G^*_0 \\ F^*_{b+h-\gamma-1} \\ \vdots \\ F^*_0 \end{bmatrix} = \begin{bmatrix} \Phi_{a+b+h-\gamma-1} \\ \vdots \\ \vdots \\ \vdots \\ \Phi_0 \end{bmatrix}, \quad \text{where} \quad \mathbf{S} \equiv \Bigg[ \underbrace{\begin{matrix} \tilde{B}_b & & \\ \vdots & \ddots & \\ \tilde{B}_0 & \ddots & \tilde{B}_b \\ & \ddots & \vdots \\ & & \tilde{B}_0 \\ & & \end{matrix}}_{(a+b+h-\gamma)\times a} \Bigg| \underbrace{\begin{matrix} \tilde{A}_a & & \\ \vdots & \ddots & \\ \tilde{A}_0 & \ddots & \ddots \\ & \ddots & \ddots & \tilde{A}_a \\ & & \ddots & \vdots \\ & & & \tilde{A}_0 \end{matrix}}_{(a+b+h-\gamma)\times(b+h-\gamma)} \Bigg]$$

is the transpose of the Sylvester matrix of $\tilde{A}(X)$ and $X^{h-\gamma}\tilde{B}(X)$.[5] A Sylvester matrix of two polynomials (with real-number coefficients) is invertible if and only if these polynomials have no common (real or complex) roots. Now, $\tilde{A}(X)$ and $X^{h-\gamma}\tilde{B}(X)$ have no common roots, since $\tilde{A}(X)$ and $\tilde{B}(X)$ are coprime and $\tilde{A}(0) \neq 0$. Therefore, their Sylvester matrix is invertible, and so is its transpose $\mathbf{S}$, so that the coefficients of $F^*(X)$ and $G^*(X)$ can be obtained as[6]

$$\left[\, G^*_{a-1} \quad \ldots \quad G^*_0 \quad F^*_{b+h-\gamma-1} \quad \ldots \quad F^*_0 \,\right]^T = \mathbf{S}^{-1} \left[\, \Phi_{a+b+h-\gamma-1} \quad \ldots \quad \Phi_0 \,\right]^T .$$

By construction, these polynomials $F^*(X)$ and $G^*(X)$ are such that $\tilde{\mathcal{R}}^*(X) = \Phi(X)$. They are also such that $F^*(0) \neq 0$ and $G^*(X) \neq 0$. The first inequality follows from $\tilde{\mathcal{R}}^*(X) = \Phi(X)$, which implies $\tilde{\mathcal{R}}^*(0) = \tilde{A}(0)F^*(0) = \Phi(0)$ (since $h \geq \gamma + 1$), and hence $F^*(0) = \Phi(0)/\tilde{A}(0) \neq 0$ (since $\Phi(0) \neq 0$). The second inequality follows from $\tilde{\mathcal{R}}^*(X) = \Phi(X)$ and the fact that $\Phi(X)$ is not a multiple of $\tilde{A}(X)$, which together imply $G^*(X) \neq 0$.

So, the system consisting of the structural equation (1) and the rule (3) with $F(X) = F^*(X)$ and $G(X) = G^*(X)$ satisfies Blanchard and Kahn's (1980) *root-counting* condition. In addition, this system also satisfies Blanchard and Kahn's (1980) *no-decoupling* condition, except possibly for a zero-measure set of polynomials $\Phi(X)$.[7] More specifically, the rule in this system has two properties that preclude two variants of decoupling. First, $G^*(X) \neq 0$ ensures that the dynamics of $i_t$ are not decoupled from the dynamics of $z_t$ in the rule (so that the rule is, in this sense, a *feedback* rule). Second, $F^*(X)$ and $G^*(X)$ have no common roots, in particular no common roots in $\mathbb{D} \setminus \{0\}$, except possibly for a zero-measure set of polynomials $\Phi(X)$. If they had a common root in $\mathbb{D} \setminus \{0\}$, then the rule would not have any stationary solution in $(z_t, i_t)$ (in the presence of exogenous policy disturbances).

Since the system satisfies both the root-counting and the no-decoupling conditions of Blanchard and Kahn (1980), the rule (3) with $F(X) = F^*(X)$ and $G(X) = G^*(X)$ ensures local-equilibrium determinacy. Thus, although they put the policymaker behind the curve, inside lags do not prevent her from ensuring determinacy, provided that her choice is not arbitrarily restricted to a specific family of policy-instrument rules.

In essence, a policy-instrument rule can still ensure determinacy in the presence of such lags via the private sector's expectation that it will be followed in the future. To see how, consider a structural equation that makes $z_t$ depend on both $i_t$ and $\mathbb{E}_t\{z_{t+\delta}\}$. By recurrence, it also makes $z_t$ depend on $\mathbb{E}_t\{z_{t+n\delta}\}$ for $n \in \mathbb{N} \setminus \{0\}$ such that $n\delta \geq h$ and, therefore, on $\mathbb{E}_t\{i_{t+n\delta}\}$. Now, by making $i_t$ react to $z_{t-n\delta}$, a rule also makes $\mathbb{E}_t\{i_{t+n\delta}\}$ react to $z_t$. The latter reaction can be viewed as the feedback mechanism that ensures determinacy.

[5] Throughout the paper, letters in bold denote vectors and matrices that have potentially more than one element. To lighten the exposition, I have displayed only the elements of $\mathbf{S}$ that may be non-zero.

[6] Throughout the paper, the superscript $T$ denotes the transpose operator.

[7] The "no-decoupling condition" requires that the system should not be "decoupled" in the sense of Sims (2007). It is formulated as a matrix-rank condition in Blanchard and Kahn (1980, p. 1308), and is often called the "rank condition" in the literature. Sims' (2007) bare-bones example of a system meeting the root-counting condition but not the no-decoupling condition is $x_t = 1.1x_{t-1} + \varepsilon_t$ and $\mathbb{E}_t\{y_{t+1}\} = 0.9y_t + \nu_t$.

## 2.3 Setup and Determinacy with Non-Distributed Outside Lags

In the presence of non-distributed outside lags of length $\ell' \in \mathbb{N}$, the structural equation becomes

$$\mathbb{E}_{t-\ell'}\left\{L^{-\delta}\left[A(L)\,z_t + L^{-\gamma}B(L)\,i_t\right]\right\} + A_\delta\left(z_t - \mathbb{E}_{t-\ell'}\left\{z_t\right\}\right) = 0, \tag{4}$$

where $A_\delta \neq 0$. This equation expresses the current endogenous variable $z_t$ as a function of expectations formed at date $t-\ell'$. In this sense, $z_t$ is predetermined at date $t-\ell'$, and cannot be affected by current or recent policy shocks. When $\ell' = 0$, the equation (4) is the same as (1).

I consider the class of rules of type (3), and hence I allow for inside lags of length $\ell$. To get rid of the past-expectation terms in (4), I introduce the variables

$$\tilde{z}_t \equiv \mathbb{E}_t\{z_{t+\ell'}\}, \tag{5}$$

$$\tilde{i}_t \equiv \mathbb{E}_t\{i_{t+\ell'}\}, \tag{6}$$

and I rewrite the system (3)-(6) as the following system:

$$\mathbb{E}_t\left\{L^{-\delta}\left[A(L)\,\tilde{z}_t + L^{-\gamma}B(L)\,\tilde{i}_t\right]\right\} = 0, \tag{7}$$

$$F(L)\tilde{i}_t = L^h G(L)\tilde{z}_t, \tag{8}$$

$$z_t = \tilde{z}_{t-\ell'}, \tag{9}$$

$$i_t = \tilde{i}_{t-\ell'}. \tag{10}$$

The two systems are equivalent to each other. To see why, note that (4) and (5) imply (9); (3), (6), and (9) imply (10); (3), (9), and (10) imply (8); and (4), (9), and (10) imply

$$\mathbb{E}_{t-\ell'}\left\{L^{-\delta}\left[A(L)\,\tilde{z}_{t-\ell'} + L^{-\gamma}B(L)\,\tilde{i}_{t-\ell'}\right]\right\} = 0,$$

and hence (7). Conversely, (9) implies (5); (10) implies (6); (8), (9), and (10) imply (3); and (5), (6), (7), and (9) imply (4).

The system (7)-(10) is block-recursive: the sub-system (7)-(8) determines $(\tilde{z}_t, \tilde{i}_t)$, uniquely or not; and, for any value of $(\tilde{z}_t, \tilde{i}_t)$, the sub-system (9)-(10) uniquely determines $(z_t, i_t)$.

So, there is a unique solution in $(z_t, i_t)$ to the system (3)-(4) if and only if there is a unique solution in $(\tilde{z}_t, \tilde{i}_t)$ to the system (7)-(8). In turn, there is a unique solution in $(\tilde{z}_t, \tilde{i}_t)$ to the system (7)-(8) if and only if there is a unique solution in $(z_t, i_t)$ to the system consisting of (1) and (3), because the two systems are identical to each other. So, we are back to the situation without outside lags (and with inside lags). Thus, non-distributed outside lags do not affect the ability of the policymaker to ensure determinacy, for any given inside-lag length.

The reason for this neutrality is the following. At date $t$, $\mathcal{PM}$ eventually observes $z_{t-\ell}$ (due to inside lags), which was actually decided at date $t-\ell-\ell'$ (due to non-distributed outside lags). But $z_{t-\ell}$ was decided at date $t-\ell-\ell'$ on the basis of expectations whose horizon is $\ell'$-periods longer than without outside lags; and all variables until date $t-\ell$ were already known with perfect foresight at date $t-\ell-\ell'$. So, the situation is essentially the same as without outside lags ($\ell' = 0$), and the policymaker can therefore ensure determinacy.

### 2.4 Example of Setup with Distributed Outside Lags

I now turn to a very simple example of setup with distributed outside lags. The goal is to illustrate how determinacy with this kind of lags can be analyzed by introducing new variables and increasing the dimension of the system. I start from a univariate model without outside lags, in which $\mathcal{PS}$ sets $z_t$ according to the structural equation $z_t = \alpha \mathbb{E}_t\{z_{t+1}\} + i_t$ with $\alpha \in \mathbb{R} \setminus \{0\}$. Then, I assume that half of the private agents make their decision two periods in advance, and the other half one period in advance, and that as a result the structural equation becomes

$$z_t = \frac{1}{2}\mathbb{E}_{t-2}\left\{\alpha z_{t+1} + i_t\right\} + \frac{1}{2}\mathbb{E}_{t-1}\left\{\alpha z_{t+1} + i_t\right\}. \tag{11}$$

The endogenous variable $z_t$ is predetermined at date $t-1$, but not at date $t-2$. To write the system consisting of the rule (3) and the structural equation (11) in Blanchard and Kahn's (1980) form, I need to get rid of past-expectation terms. I can use the same transformation as in the previous subsection to get rid of the term in $\mathbb{E}_{t-2}\{.\}$. More specifically, I can define

$$\tilde{z}_t \equiv \mathbb{E}_t\{z_{t+1}\}, \tag{12}$$

$$\tilde{i}_t \equiv \mathbb{E}_t\{i_{t+1}\}, \tag{13}$$

and rewrite the system consisting of (3) and (11)-(13) as the following, equivalent system:

$$\tilde{z}_t = \frac{1}{2}\mathbb{E}_{t-1}\left\{\alpha \tilde{z}_{t+1} + \tilde{i}_t\right\} + \frac{1}{2}\mathbb{E}_t\left\{\alpha \tilde{z}_{t+1} + \tilde{i}_t\right\}, \tag{14}$$

$$F(L)\tilde{i}_t = L^h G(L)\tilde{z}_t, \tag{15}$$

$$z_t = \tilde{z}_{t-1}, \tag{16}$$

$$i_t = \tilde{i}_{t-1}. \tag{17}$$

But this transformation still leaves me with a term in $\mathbb{E}_{t-1}\{.\}$. To get rid of this term, I need to introduce a new variable:

$$v_t \equiv \mathbb{E}_t\left\{\alpha \tilde{z}_{t+2} + \tilde{i}_{t+1}\right\}, \tag{18}$$

so that I can rewrite (14) as

$$\tilde{z}_t = \frac{1}{2}v_{t-1} + \frac{1}{2}\mathbb{E}_t\left\{\alpha \tilde{z}_{t+1} + \tilde{i}_t\right\}. \tag{19}$$

For convenience, I rewrite the system (18)-(19) as the following, equivalent system:

$$v_t = \mathbb{E}_t\left\{\tilde{z}_{t+1}\right\}, \tag{20}$$

$$\tilde{z}_t = \frac{1}{2}\left(v_{t-1} + \alpha v_t + \tilde{i}_t\right). \tag{21}$$

The system consisting of (15)-(17) and (20)-(21) is block-recursive: the sub-system consisting of (15) and (20)-(21) determines $(\tilde{z}_t, v_t, \tilde{i}_t)$, uniquely or not; and, for any value of $(\tilde{z}_t, v_t, \tilde{i}_t)$, the sub-system (16)-(17) uniquely determines $(z_t, i_t)$.

So, there is a unique solution in $(z_t, i_t)$ to the system consisting of (3) and (11) if and only if there is a unique solution in $(\tilde{z}_t, v_t, \tilde{i}_t)$ to the system consisting of (15) and (20)-(21). Unlike the former system, the latter system does not involve any past-expectation term, and can thus be written in Blanchard and Kahn's (1980) form. There are two main changes, however, compared to the previous subsection. First, the model is now multivariate in the sense that $\mathcal{PS}$ sets two variables ($\tilde{z}_t$ and $v_t$), not just one ($\tilde{z}_t$). Second, $\mathcal{PM}$ no longer observes all the endogenous variables set by $\mathcal{PS}$: her information set when she sets $\tilde{i}_t$ is $I_t = \{\tilde{z}^{t-\ell}, \tilde{i}^{t-1}\}$, not $I_t = \{\tilde{z}^{t-\ell}, v^{t-\ell}, \tilde{i}^{t-1}\}$.

To study the implications of distributed outside lags for $\mathcal{PM}$'s ability to ensure determinacy, therefore, I need to consider multivariate models with *latent* endogenous variables, i.e. variables set by $\mathcal{PS}$ and never observed by $\mathcal{PM}$. I do that in the next section.[8]

# 3 Lags and Determinacy in Multivariate Models

In this section, I generalize the determinacy results of the previous section to multivariate models, and I extend them to distributed outside lags. More specifically, I show that inside lags, non-distributed outside lags, and distributed outside lags do not hinder the ability of the policymaker to ensure determinacy, under a certain condition on the set of variables observed by the policymaker. I also show that this condition is necessarily met, for any non-empty set of observed variables, if the model has at least one stationary solution for any exogenous stationary process of the policy instrument (a property that I call "weak Sargent-Wallace property").

## 3.1 Setup with Inside and Outside Lags

At each date $t \in \mathbb{Z}$, the policymaker ($\mathcal{PM}$) still sets the policy instrument $i_t$, while the private sector ($\mathcal{PS}$) now sets an $n$-dimension vector of endogenous variables $\mathbf{Z}_t$ according to the following (locally log-linearized) structural equations:

$$\mathbb{E}_t \left\{ \mathbf{\Delta}\left(L^{-1}\right) \left[ \mathbf{A}\left(L\right) \mathbf{Z}_t + L^{-\gamma} \mathbf{B}\left(L\right) i_t \right] \right\} = \mathbf{0}, \tag{22}$$

where $n \in \mathbb{N} \setminus \{0\}$, $\gamma \in \mathbb{Z}$, $\mathbf{A}(X) \in \mathbb{R}^{n\times n}[X]$, $\mathbf{B}(X) \in \mathbb{R}^{n\times 1}[X]$, and where $\mathbf{\Delta}(X) \in \mathbb{R}^{n\times n}[X]$ is a diagonal matrix whose $j^{th}$ diagonal element is $X^{\delta_j}$ with $\delta_j \in \mathbb{N}$.[9] This system of structural equations explicitly features an arbitrary finite number of lags and expected leads of the endogenous variables. As is well known, it could be rewritten in an equivalent reduced form with no lags and only one expected lead, by gathering past, current, and expected future variables into the same vector. Generic systems of structural equations are typically written in such a compact reduced form in the literature (e.g., in Giannoni and Woodford, 2002, and Woodford, 2003, Chapter 8). I depart from the literature in this respect because I need to distinguish

[8] I focus on *finitely* distributed outside lags, which involve a finite number of past-expectation terms. Infinitely distributed outside lags, such as those considered in Mankiw and Reis (2002), raise specific difficulties whose study is beyond the scope of this paper.

[9] Throughout the paper, $\mathbf{0}$ denotes a vector or a matrix whose elements are all equal to zero and whose dimensions depend on the specific context in which it is used. For any $(p, q) \in (\mathbb{N} \setminus \{0\})^2$, $\mathbb{R}^{p\times q}[X]$ denotes the set of polynomials in $X$ whose coefficients are $p \times q$ matrices with real-number elements.

endogenous variables according to the date at which they are set, in order to be able to specify $\mathcal{PM}$'s information set explicitly.

Let $\Psi_j(X) \in \mathbb{R}[X]$, for each $j \in \{1, ..., n+1\}$, denote the determinant of the $n \times n$ matrix obtained by removing the $j^{th}$ column of the $n \times (n+1)$ matrix $[\,\mathbf{A}(X) \quad \mathbf{B}(X)\,]$. Let $\hat{\mathbf{A}}_0 \in \mathbb{R}^{n \times n}$ denote the coefficient of $X^0$ in the "Laurent polynomial" $\hat{\mathbf{A}}(X) \equiv \mathbf{\Delta}(X^{-1})\mathbf{A}(X)$. I make the following four non-restrictive assumptions on $\mathbf{A}(X)$ and $\mathbf{B}(X)$:

**Assumption 1**: $\det[\mathbf{A}(0)] \neq 0$.

**Assumption 2**: $\mathbf{B}(0) \neq \mathbf{0}$.

**Assumption 3**: $\forall j \in \{1, ..., n\}$, $\Psi_j(X) \neq 0$.

**Assumption 4**: $\det[\hat{\mathbf{A}}_0] \neq 0$.

These assumptions are made without any loss in generality for the following reasons. First, any system of *independent* structural equations of type (22) that does not satisfy Assumption 1 can be equivalently rewritten as a system of type (22) that satisfies this assumption. Second, any system of type (22) needs to satisfy $\mathbf{B}(X) \neq \mathbf{0}$ for $\mathcal{PM}$'s policy instrument to have an effect on the endogenous variables set by $\mathcal{PS}$, and any system of type (22) satisfying $\mathbf{B}(X) \neq \mathbf{0}$ but not Assumption 2 can be equivalently rewritten as a system of type (22) satisfying Assumption 2 (simply by changing the value of $\gamma$). Third, if Assumption 3 were not satisfied, i.e. if there existed $j \in \{1, ..., n\}$ such that $\Psi_j(X) = 0$, then there would exist a linear combination of the structural equations that would involve only elements of $\{\mathbb{E}_t\{z_{j,t+k}\} | k \in \mathbb{Z}\}$, where $z_{j,t}$ denotes the $j^{th}$ element of $\mathbf{Z}_t$, so that the variable $z_{j,t}$ should then be considered as exogenous, not endogenous. And fourth, if Assumption 4 were not satisfied, then there would exist a linear combination of the structural equations that would not involve any element of $\mathbf{Z}_t$, so that the structural equations would not describe how $\mathcal{PS}$ sets $\mathbf{Z}_t$ at date $t$.

Assumptions 1 and 2 are the generalization, to a multivariate context, of the assumptions $A(0) \neq 0$ and $B(0) \neq 0$ made in the previous section. Assumption 3 is specific to the multivariate context: indeed, in the case $n = 1$ (i.e., the univariate case), Assumption 3 boils down to $\mathbf{B}(X) \neq \mathbf{0}$ and is necessarily satisfied under Assumption 2. Assumption 4 is also specific to the multivariate context in the following sense. I make this assumption to give a chance to each element of $\mathbf{Z}_t$ to be pinned down even if $\mathcal{PM}$ never observes this element. If I did not make Assumption 4, then there could exist $j \in \{1, ..., n\}$ such that the structural equations involve some elements of $\{\mathbb{E}_t\{z_{j,t+k}\} | k \in \mathbb{N} \setminus \{0\}\}$, but no element of $\{z_{j,t-k} | k \in \mathbb{N}\}$. In this case, $z_{j,t}$ could not be pinned down unless $\mathcal{PM}$ sets $i_t$ as a function of some $z_{j,t-k}$ with $k \in \mathbb{N}$, which requires that she observe it. I did not need to make Assumption 4 in the univariate analysis of the previous section (i.e. I did not need to assume $A_\delta \neq 0$ in Subsections 2.1-2.2) simply because, in order to satisfy Blanchard and Kahn's (1980) no-decoupling condition, I had (and still have) to assume that $\mathcal{PM}$ observes at least one variable set by $\mathcal{PS}$, and in univariate models there is only one such variable.

In addition to Assumptions 1-4, I make the following assumption on $\mathbf{A}(X)$, which is also specific to the multivariate context:

**Assumption 5**: $\mathbf{A}(X)$ *is not block-triangular, and the system (22) cannot be rewritten in an equivalent form of type (22) satisfying Assumptions 1-4 with a block-triangular* $\mathbf{A}(X)$.

Assumption 5, which rules out block-recursive systems of structural equations, may seem restrictive at first sight. Indeed, dynamic general-equilibrium models typically have a subset of *static* structural equations, i.e. structural equations involving only current endogenous variables (like, often, the production function and the goods-market-clearing condition). These static equations can be used to replace some variables by functions of other contemporaneous variables in the dynamic equations. This replacement leads to a block-recursive system, with a first block composed of the dynamic equations and involving only some variables, and a second block composed of the static equations and residually determining the other variables. The determinacy issue, however, arises only in the first block. So, one can focus on the first block of structural equations and the corresponding subset of endogenous variables. Assumption 5, thus, does not exclude dynamic models with some static structural equations, since they can be reduced to dynamic models with no static structural equation. I will illustrate this point with some specific models in Section 6.

I make Assumption 5 for simplicity, in order to overcome difficulties raised by Blanchard and Kahn's (1980) *no-decoupling* condition. To illustrate these difficulties in the simplest possible way, consider the system consisting of the following two structural equations: $z_{1,t} = \alpha_1 \mathbb{E}_t\{z_{1,t+1}\} + i_t$ and $z_{2,t} = \alpha_2 \mathbb{E}_t\{z_{2,t+1}\} + i_t$, where $(\alpha_1, \alpha_2) \in (\mathbb{R} \setminus \{0\})^2$. This system does not satisfy Assumption 5 because its $2 \times 2$ matrix $\mathbf{A}(X)$ is diagonal. To ensure that the system consisting of these structural equations and the policy-instrument rule meets Blanchard and Kahn's (1980) no-decoupling condition, I would need to assume either that the rule makes $i_t$ react to at least some $z_{1,t-k}$ and some $z_{2,t-k'}$ with $(k, k') \in \mathbb{N}^2$; or that the rule makes $i_t$ react to at least some $z_{1,t-k}$ with $k \in \mathbb{N}$ and $|\alpha_2| < 1$; or that the rule makes $i_t$ react to at least some $z_{2,t-k'}$ with $k' \in \mathbb{N}$ and $|\alpha_1| < 1$; or that $|\alpha_1| < 1$ and $|\alpha_2| < 1$. More generally, for any block-diagonal matrix $\mathbf{A}(X)$, I would need to assume either that the rule involves at least one variable from each block, or that each block satisfies Blanchard and Kahn's (1980) conditions. Similar difficulties arise when the matrix $\mathbf{A}(X)$ is block-recursive without being block-diagonal.

As this discussion suggests, Assumption 5 could be relaxed without affecting the results, but at the cost of greater complexity. My view is that it strikes a good balance between generality and simplicity. In fact, I suspect that most of the locally log-linearized dynamic rational-expectations models commonly used for monetary-policy analysis can be cast in a form of type (22) satisfying Assumptions 1 to 5. I cannot, of course, verify this claim for a large number of models; but I verify it in Section 6 for two well known monetary-policy models in particular: the small-scale basic New Keynesian model, and the medium-scale model of Smets and Wouters (2007). Moreover, as I elaborate in Section 6, fiscal-policy models do not naturally satisfy Assumption 5, but a simple trick can be used to rewrite them in a form that does. I illustrate this trick with the model of Schmitt-Grohé and Uribe (1997) in Section 6.

Let $I_t$ denote the information set of $\mathcal{PM}$ when she sets $i_t$. I consider the class of alternative information sets of type

$$I_t = \{\mathbf{Z}^{J,t-\ell}, i^{t-1}\} \tag{23}$$

with $\emptyset \subsetneq J \subseteq \{1, ..., n\}$ and $\ell \in \mathbb{N}$, where $\mathbf{Z}_t^J$ denotes the vector whose elements are the $j^{th}$ elements of $\mathbf{Z}_t$ for $j \in J$. As previously, the parameter $\ell$ captures inside lags, which may be of any length. The novelty is that I now allow for the non-observation of some endogenous variables (when $J \subsetneq \{1, ..., n\}$). Examples of unobserved variables may include Lagrange multipliers of $\mathcal{PS}$'s optimization problems, or more generally variables that do not have natural empirical counterparts. More to the point, however, the presence of unobserved variables may result from distributed outside lags, as I have illustrated in Subsection 2.4. In Section 6, I show that the model of Rotemberg and Woodford (1997, 1999), which has distributed outside lags, can be rewritten in a form of type (22) satisfying Assumptions 1 to 5 with one unobserved variable.

Unlike distributed outside lags, non-distributed outside lags do not generate unobserved latent variables. When these lags are of the same length in all structural equations, they are neutral for determinacy, for the same reason as in Subsection 2.3. When their lengths differ across the structural equations, the model can typically be rewritten in a form of type (22) satisfying Assumptions 1 to 5, with an information set that *includes* a set of type (23), where $\ell$ then denotes the sum of the inside-lags length and the maximum outside-lags length. Since the policymaker can ensure determinacy with the latter (smaller) information set, as I show in the next subsection, she can also ensure determinacy with the former (larger) information set.

### 3.2 Determinacy

I now design a policy-instrument rule consistent with the information set $I_t$ and ensuring determinacy. I start from the class of rules of type

$$F(L)\, i_t = L^h \mathbf{G}(L)\, \mathbf{Z}_t \tag{24}$$

with $F(X) \in \mathbb{R}[X]$, $F(0) \neq 0$, $\mathbf{G}(X) \equiv [G_1(X) \quad \cdots \quad G_n(X)] \in \mathbb{R}^{1\times n}[X]$, $\mathbf{G}(X) \neq \mathbf{0}$, $G_j(X) = 0$ for all $j \in \{1, ..., n\} \setminus J$, $h \in \mathbb{N}$, and $h \geq \max(\ell, \gamma + 1)$.[10] Since $F(0) \neq 0$ and $h \geq \ell$, (24) expresses $i_t$ as a function of only elements of $I_t$, i.e. it is a policy-instrument rule consistent with $I_t$.

I first establish a useful preliminary result:

[10]Unlike in the previous section, I do not need to impose a restriction of type $h \geq \gamma + \delta + 1$, because I will not use Sylvester matrices to design the rule.

**Lemma 1**: *The system consisting of the structural equations (22) and any rule of type (24) can be written in Blanchard and Kahn's (1980) form with* $\delta \equiv \sum_{j=1}^{n} \delta_j$ *non-predetermined variables, and the reciprocal polynomial of its characteristic polynomial is*

$$\sum_{j=1}^{n} (-1)^{n-j} X^{h-\gamma} \Psi_j (X) G_j (X) + \Psi_{n+1} (X) F (X) . \tag{25}$$

**Proof**: See Appendix A.1. ■ The first part of the lemma, about the number of non-predetermined variables, is essentially a consequence of Assumption 1 and the restriction $h \geq \gamma + 1$. The second part of the lemma, about the characteristic polynomial, is a consequence of Laplace's expansion.

Lemma 1 implies that the system consisting of the structural equations (22) and a rule of type (24) meets Blanchard and Kahn's (1980) root-counting condition if and only if the polynomial (25) has exactly $\delta$ roots in $\mathbb{D} \setminus \{0\}$. So, I need to design some polynomials $F^*(X)$ and $\mathbf{G}^*(X)$ such that the polynomial (25) with $F(X) = F^*(X)$ and $\mathbf{G}(X) = \mathbf{G}^*(X)$ has exactly $\delta$ roots in $\mathbb{D} \setminus \{0\}$.

In Section 2, I used a Sylvester matrix to design $F^*(X)$ and $G^*(X)$. Here, I cannot use a Sylvester matrix to design $F^*(X)$ and $\mathbf{G}^*(X)$, because $\mathbf{G}(X)$ has generically more than one non-zero element, and Sylvester matrices are associated with only two scalar polynomials. Instead, I use Bézout's identity (together with the Euclidean division), which can be applied to an arbitrary finite number of scalar polynomials.[11]

In Section 2, I designed $F^*(X)$ and $G^*(X)$ under the assumption that $D(X)$ had at most $\delta$ roots in $\mathbb{D} \setminus \{0\}$. Here, I design $F^*(X)$ and $\mathbf{G}^*(X)$ under a similar condition. For any non-empty and non-singleton set $S \subseteq \{1, ..., n+1\}$, let $D_S(X) \equiv \gcd[\Psi_j(X)]_{j \in S} \in \mathbb{R}[X]$ denote the greatest common divisor, defined up to a multiplicative non-zero real-number scalar, of all the polynomials $\Psi_j(X)$ for $j \in S$ (none of which is zero, given Assumptions 1 and 3). The condition is that $D_{\bar{J}}(X)$, where $\bar{J} \equiv J \cup \{n+1\}$, should have at most $\delta$ roots in $\mathbb{D} \setminus \{0\}$. I thus get the following proposition:

**Proposition 1 (Determinacy with Inside and Outside Lags)**: *For any non-empty set* $J \subseteq \{1, ..., n\}$ *and any* $\ell \in \mathbb{N}$, *if* $D_{\bar{J}}(X)$ *has at most* $\delta$ *roots in* $\mathbb{D} \setminus \{0\}$, *then there exists a policy-instrument rule consistent with* $I_t = \{\mathbf{Z}^{J,t-\ell}, i^{t-1}\}$ *and ensuring determinacy.*

**Proof**: For any polynomial $P(X)$, let $d(P)$ denote the degree of $P(X)$. I start with the case in which $d(\Psi_{n+1}) > d(D_{\bar{J}})$. Since $\Psi_{n+1}(0) = \det[\mathbf{A}(0)] \neq 0$ (given Assumption 1), the greatest common divisor of the polynomials $[X^{h-\gamma}\Psi_j(X)]_{j \in J}$ and $\Psi_{n+1}(X)$ is $D_{\bar{J}}(X)$. Therefore, Bézout's identity implies that there exist $U_j(X) \in \mathbb{R}[X]$ for $j \in \bar{J}$ such that

$$\sum_{j \in J} X^{h-\gamma} \Psi_j (X) U_j (X) + \Psi_{n+1} (X) U_{n+1} (X) = D_{\bar{J}} (X) . \tag{26}$$

[11] Bézout's identity is sometimes unnamed and presented as a corollary of the Euclidean algorithm (as in, e.g., Prasolov, 2004, Chapter 2, Theorem 2.1.1).

Replacing $X$ by 0 in this equation, and using $h-\gamma \geq 1$, $\Psi_{n+1}(0) \neq 0$, and $D_{\bar{J}}(0) \neq 0$, I get $U_{n+1}(0) \neq 0$. In turn, using $U_{n+1}(X) \neq 0$ and $d(\Psi_{n+1}) > d(D_{\bar{J}})$, I obtain that there exists $j \in J$ such that $U_j(X) \neq 0$.

Let $d \in \mathbb{N}$ denote the number of roots of $D_{\bar{J}}(X)$ in $\mathbb{D} \setminus \{0\}$. Assume that $d \leq \delta$, as stated in the proposition. Let $\Phi(X) \in \mathbb{R}[X]$ be an arbitrary polynomial that: (i) is of higher degree than $\Psi_{n+1}(X)$, and (ii) has exactly $\delta - d$ roots in $\mathbb{D} \setminus \{0\}$ and all its other roots in $\mathbb{C} \setminus \mathbb{D}$. Since $\Psi_{n+1}(X) = \det[\mathbf{A}(X)] \neq 0$ (given Assumption 1), I can consider the Euclidean division of $\Phi(X)$ by $X\Psi_{n+1}(X)$ (in order to satisfy Blanchard and Kahn's (1980) no-decoupling condition, as I explain below). Let $Q(X) \in \mathbb{R}[X]$ and $R(X) \in \mathbb{R}[X]$ denote respectively the quotient and the remainder of this division, i.e. the unique polynomials such that

$$\Phi(X) = X\Psi_{n+1}(X)Q(X) + R(X) \tag{27}$$

and $d(R) \leq d(\Psi_{n+1})$. Replacing $X$ by 0 in (27), and using $\Phi(0) \neq 0$, I get $R(0) \neq 0$. Using $d(\Phi) > d(\Psi_{n+1}) \geq d(R)$, I also get $Q(X) \neq 0$. Multiplying the left- and right-hand sides of (26) by $R(X)$, and using (27), leads to

$$\sum\nolimits_{j=1}^{n} (-1)^{n-j} X^{h-\gamma} \Psi_j(X) G_j^*(X) + \Psi_{n+1}(X) F^*(X) = D_{\bar{J}}(X) \Phi(X), \tag{28}$$

where

$$\begin{aligned} F^*(X) &\equiv R(X) U_{n+1}(X) + XQ(X) D_{\bar{J}}(X), \\ G_j^*(X) &\equiv (-1)^{n-j} R(X) U_j(X) \text{ for } j \in J, \\ G_j^*(X) &\equiv 0 \text{ for } j \in \{1, ..., n\} \setminus J. \end{aligned}$$

The polynomials $F^*(X)$ and $\mathbf{G}^*(X) \equiv [\, G_1^*(X) \quad \cdots \quad G_n^*(X) \,]$ are admissible choices for $F(X)$ and $\mathbf{G}(X)$ in the rule (24), because $F^*(0) \neq 0$ and $\mathbf{G}^*(X) \neq \mathbf{0}$. The first inequality follows from $R(0) \neq 0$ and $U_{n+1}(0) \neq 0$ (both of which I have proved above). The second inequality follows from $R(X) \neq 0$ and from the existence of $j \in J$ such that $U_j(X) \neq 0$ (which I have also proved above).

In the alternative case in which $d(\Psi_{n+1}) = d(D_{\bar{J}})$, the polynomials $\Psi_{n+1}(X)$ and $D_{\bar{J}}(X)$ are equal to each other up to a multiplicative non-zero real-number scalar: $D_{\bar{J}}(X)/\Psi_{n+1}(X) = \psi \in \mathbb{R} \setminus \{0\}$. So, one can choose, e.g., $G_j^*(X) = 1$ for $j \in J$, $G_j^*(X) = 0$ for $j \in \{1, ..., n\} \setminus J$, and $F^*(X) = \psi\Phi(X) - \sum_{j=1}^{n} (-1)^{n-j} \psi X^{h-\gamma} [\Psi_j(X)/D_{\bar{J}}(X)] G_j^*(X)$. This choice is admissible because $\Psi_j(X)/D_{\bar{J}}(X) \in \mathbb{R}[X]$ for $j \in \{1, ..., n\}$, $F^*(0) \neq 0$, and $\mathbf{G}^*(X) \neq \mathbf{0}$. Moreover, it straightforwardly implies (28).

The left-hand side of (28) is the polynomial (25) with $F(X) = F^*(X)$ and $\mathbf{G}(X) = \mathbf{G}^*(X)$. The right-hand side of (28) has exactly $\delta$ roots in $\mathbb{D} \setminus \{0\}$, since $D_{\bar{J}}(X)$ and $\Phi(X)$ have respectively $d$ and $\delta - d$ such roots. Therefore, Lemma 1 implies that the system consisting of the structural equations (22) and the rule (24) with $F(X) = F^*(X)$ and $\mathbf{G}(X) = \mathbf{G}^*(X)$ meets Blanchard and Kahn's (1980) *root-counting* condition.

In addition, this system also satisfies Blanchard and Kahn's (1980) *no-decoupling* condition, except possibly for a zero-measure set of polynomials $\Phi(X)$. More specifically, the system has two properties that preclude two variants of decoupling. First, $\mathbf{G}^*(X) \neq \mathbf{0}$, Assumption 2, and Assumption 5 ensure that there is no element of the vector $[\mathbf{Z}_t^T \quad i_t]^T$ whose dynamics are decoupled from the dynamics of the other elements. Second, because $Q(X) \neq 0$ (as I have proved above), $F^*(X)$ and $\mathbf{G}^*(X)$ have no common roots, in particular no common roots in $\mathbb{D} \setminus \{0\}$, except possibly for a zero-measure set of polynomials $\Phi(X)$.

Since it satisfies Blanchard and Kahn's (1980) root-counting and no-decoupling conditions, the system consisting of the structural equations (22) and the rule (24) with $F(X) = F^*(X)$ and $\mathbf{G}(X) = \mathbf{G}^*(X)$ has a unique stationary solution. Proposition 1 follows. ■

The rule designed in the proof of Proposition 1 does actually more than ensuring determinacy: it also makes the resulting system (consisting of the structural equations and this rule) have a given characteristic polynomial. More specifically, the reciprocal polynomial of this characteristic polynomial is $D_{\bar{J}}(X)\Phi(X)$. The number of roots of $\Phi(X)$ in $\mathbb{D}\setminus\{0\}$ is fixed (set to $\delta - d$), but the values of these roots are arbitrary, and the number and the values of roots of $\Phi(X)$ in $\mathbb{C} \setminus \mathbb{D}$ are also arbitrary (except that the number of such roots should be higher than $d(\Psi_{n+1}) - (\delta - d)$). I will exploit these degrees of freedom in Section 4.

Proposition 1 implies that under a certain condition on $D_{\bar{J}}(X)$, neither inside lags (captured by the parameter $\ell$), nor non-distributed outside lags (also captured by $\ell$ when their lengths differ across the structural equations, as discussed above), nor distributed outside lags (captured by the set $J$) hinder the ability of the policymaker to ensure determinacy, no matter the length of these lags. The condition on $D_{\bar{J}}(X)$, which involves the set $J$ but not the parameter $\ell$, is that $D_{\bar{J}}(X)$ should have at most $\delta$ roots in $\mathbb{D} \setminus \{0\}$. How restrictive is this condition? The next subsection provides an answer to this question.

### 3.3 A Weak Sargent-Wallace Property

I now show that the condition stated in Proposition 1 is met by any model that has at least one stationary solution for any exogenous stationary process of the policy instrument:

**Proposition 2 (A "Weak Sargent-Wallace Property" as a Sufficient Determinacy Condition)**: *If the system of structural equations (22) has at least one stationary solution in* $(\mathbf{Z}_t)_{t\in\mathbb{Z}}$ *for any exogenous stationary process for* $(i_t)_{t\in\mathbb{Z}}$*, then* $D_{\bar{J}}(X)$ *has at most* $\delta$ *roots in* $\mathbb{D} \setminus \{0\}$ *for any non-empty set* $J \subseteq \{1, ..., n\}$.

**Proof**: Suppose that the system of structural equations (22) has at least one stationary solution in $(\mathbf{Z}_t)_{t\in\mathbb{Z}}$ for any exogenous stationary process for $(i_t)_{t\in\mathbb{Z}}$. Consider the policy-instrument rule $i_t = 0$. It is straightforward to adjust the proof of Lemma 1 and show that this lemma still holds for such a degenerate rule. So, the system consisting of (22) and this rule can be written in Blanchard and Kahn's (1980) form with $\delta$ non-predetermined variables, and the reciprocal polynomial of its characteristic polynomial is the polynomial (25) with $F(X) = 1$ and $\mathbf{G}(X) = \mathbf{0}$, that is to say the polynomial $\Psi_{n+1}(X)$. Since this system has, by assumption, at least one stationary solution in $(\mathbf{Z}_t)_{t\in\mathbb{Z}}$, Blanchard and Kahn's (1980) root-counting condition implies that $\Psi_{n+1}(X)$ has at most $\delta$ roots in $\mathbb{D} \setminus \{0\}$. Therefore, $D_{\bar{J}}(X)$ has also at most $\delta$ roots in $\mathbb{D} \setminus \{0\}$ for any non-empty set $J \subset \{1, ..., n\}$. Proposition 2 follows. ■

The condition stated in Proposition 2, which implies the one stated in Proposition 1, seems little restrictive in the context of monetary-policy models. Indeed, it is well known that monetary-policy models typically have what Giannoni and Woodford (2002) and Woodford (2003, Chapter 8) call the "Sargent-Wallace property," after Sargent and Wallace (1975), that is to say that rules setting the interest rate exogenously typically lead to local-equilibrium multiplicity in these models. These models, therefore, typically have the weaker form of Sargent-Wallace property stated in Proposition 2. As a consequence, they typically meet the condition stated in Proposition 1.

In Section 6, I consider four stabilization-policy models: three monetary-policy models, and one fiscal-policy model. I find that each of these models satisfies the condition stated in Proposition 1 for any non-empty set $J$ and for all parameter values (except possibly a zero-measure subset of values). Both this finding and Proposition 2 suggest, strongly in my view, that the condition stated in Proposition 1 is typically met by existing stabilization-policy models.

## 4 Beyond Determinacy: Implementation

So far, I have shown that neither inside lags nor outside lags prevent the policymaker from ensuring determinacy. I have done so by starting from a given characteristic polynomial (or equivalently its reciprocal polynomial $D_{\bar{J}}(X)\Phi(X)$), and deriving a corresponding policy-instrument rule. To establish determinacy, I have only used the *number* of characteristic-polynomial roots outside the unit circle. I have not played with the *values* of these roots, nor with the number and the values of the roots *inside* the unit circle.

In this section, I show how to exploit these degrees of freedom for implementation purposes (i.e. for the policymaker to implement a given shock-contingent path for the endogenous variables as the unique local equilibrium). To do so, I start from the class of deterministic multivariate models considered in the previous section, which allows for inside and outside lags of any length, and I introduce stochastic exogenous disturbances into this class of models. More specifically, I replace (22) by

$$\mathbb{E}_t \left\{ \mathbf{\Delta} \left( L^{-1} \right) \left[ \mathbf{A} \left( L \right) \mathbf{Z}_t + L^{-\gamma} \mathbf{B} \left( L \right) i_t \right] \right\} = \mathbf{C}(L) \boldsymbol{\xi}_t, \tag{29}$$

where $\boldsymbol{\xi}_t$ denotes a $n_\xi$-dimension vector of exogenous disturbances, with $n_\xi \in \mathbb{N} \setminus \{0\}$, and $\mathbf{C}(X) \in \mathbb{R}^{n \times n_\xi}[X]$. The other notations are unchanged, and I maintain Assumptions 1-5. I assume that $\boldsymbol{\xi}_t$ follows a stationary (but possibly non-invertible) VARMA process driven by $\boldsymbol{\varepsilon}_t$, a $n_\varepsilon$-dimension vector of orthogonal i.i.d. exogenous shocks of mean zero, where $n_\varepsilon \in \mathbb{N} \setminus \{0\}$. I consider three cases in turn: observed shocks, unobserved but inferable shocks, and unobserved news shocks.

### 4.1 Implementation with Observed Shocks

I start with the case in which: (i) the shocks are fully unexpected by $\mathcal{PS}$, i.e. $\mathbb{E}_t\{\boldsymbol{\varepsilon}_{t+k}\} = \mathbf{0}$ for any $k \in \mathbb{N} \setminus \{0\}$ (non-news shocks), and (ii) $\mathcal{PM}$'s information set includes all exogenous shocks (with inside lags). So, the class of alternative information sets that I consider for $\mathcal{PM}$, instead of (23), is now

$$I_t = \{\mathbf{Z}^{J,t-\ell}, i^{t-1}, \boldsymbol{\varepsilon}^{t-\ell}\} \tag{30}$$

with $\emptyset \subsetneq J \subseteq \{1, ..., n\}$ and $\ell \in \mathbb{N}$. One may think of exogenous policy measures, or foreign macroeconomic developments (considered as exogenous from the point of view of a small open economy), as examples of observed exogenous shocks. The case in which all shocks belong to $I_t$ is admittedly restrictive, but the results obtained will also serve, in the next subsection, as a useful starting point to investigate an alternative case in which $I_t$ does not include all shocks (and may even include none).

I consider all paths for the endogenous variables that: (i) satisfy the structural equations (29); (ii) make $i_t$ independent of $\boldsymbol{\varepsilon}_t$, ..., $\boldsymbol{\varepsilon}_{t-\ell+1}$ (if $\ell \geq 1$); and (iii) can be written as a stationary VARMA process driven by the vector of exogenous shocks $\boldsymbol{\varepsilon}_t$, i.e. written in a form of type

$$\mathbf{S}(L) \begin{bmatrix} \mathbf{Z}_t \\ i_t \end{bmatrix} = \mathbf{T}(L)\, \boldsymbol{\varepsilon}_t \tag{31}$$

with $\mathbf{S}(X) \in \mathbb{R}^{(n+1)\times(n+1)}[X]$ and $\mathbf{T}(X) \in \mathbb{R}^{(n+1)\times n_\varepsilon}[X]$, where $\det[\mathbf{S}(X)]$ has no root in $\mathbb{D}$. Note, for later use, that one can use Cramer's rule to rewrite (31) as

$$\det[\mathbf{S}(L)] \begin{bmatrix} \mathbf{Z}_t \\ i_t \end{bmatrix} = \begin{bmatrix} \mathbf{T}_Z(L) \\ \mathbf{T}_i(L)\, L^\ell \end{bmatrix} \boldsymbol{\varepsilon}_t \tag{32}$$

with $\mathbf{T}_Z(X) \in \mathbb{R}^{n \times n_\varepsilon}[X]$ and $\mathbf{T}_i(X) \in \mathbb{R}^{1\times n_\varepsilon}[X]$. The presence of $L^\ell$ in factor of $\mathbf{T}_i(L)$ in (32) comes from the fact that $i_t$ does not depend on $\boldsymbol{\varepsilon}_{t-k}$ for $k \in \{0, ..., \ell-1\}$ on this path (when $\ell \geq 1$).

At its most basic level, the implementation question that I ask is the following: for any path of type (31), does there exist a policy-instrument rule consistent with the information set (30) and implementing this path as the unique local equilibrium? The answer to this question is clearly positive, if $D_{\bar{J}}(X)$ has at most $\delta$ roots in $\mathbb{D} \setminus \{0\}$. In this case, indeed, one can start from a rule of type (24) consistent with the information set (23) and ensuring local equilibrium determinacy, whose existence is established by Proposition 1; and add to this rule (denoted by $R$) an exogenous term of type $\mathbf{H}(L)\boldsymbol{\varepsilon}_{t-\ell}$ such that the resulting rule (denoted by $R'$) is satisfied on the targeted path. The rule $R'$ has the following three properties. First, it is consistent with the information set (30). Second, since $R$ ensures determinacy and $R'$ is derived from $R$ simply by adding an exogenous term, $R'$ ensures determinacy too. Third, the targeted path is, by construction, a stationary solution of the system consisting of the structural equations (29) and $R'$. Since this system has a unique stationary solution, this unique stationary solution must be the targeted path.

However, the exogenous term that would need to be added to an *arbitrary* rule $R$ ensuring determinacy, for the resulting rule $R'$ to be satisfied on the targeted path, would typically be written as the sum of an *infinite* number of terms (involving all past exogenous shocks $\boldsymbol{\varepsilon}_{t-\ell-k}$ for $k \in \mathbb{N}$). In other words, $\mathbf{H}(X)$ would typically be an infinite power series, not a polynomial. For the sake of practical relevance, I impose the constraint that the rule should express the policy instrument as a function of a finite (but unbounded) number of arguments – a property that I call "finite memory:"

**Definition 1 (Policy-Instrument Rules with Finite Memory)**: *A policy-instrument rule of type $F(L)i_t = \mathbf{G}(L)\mathbf{Z}_t + \mathbf{H}(L)\boldsymbol{\varepsilon}_t$ is said to have finite memory when $F(X)$, $\mathbf{G}(X)$, and $\mathbf{H}(X)$ are polynomials, not infinite power series.*

So, the more challenging implementation question that I ask is: for any path of type (31), does there exist a policy-instrument rule consistent with the information set (30), with finite memory, and implementing this path as the unique local equilibrium? The following proposition provides a positive answer to this question, if $D_{\bar{J}}(X)$ has at most $\delta$ roots in $\mathbb{D} \setminus \{0\}$:

**Proposition 3 (Implementation with Observed Shocks)**: *For any non-empty set $J \subseteq \{1, ..., n\}$ and any $\ell \in \mathbb{N}$, if $D_{\bar{J}}(X)$ has at most $\delta$ roots in $\mathbb{D} \setminus \{0\}$, then, for any path of type (31), there exists a policy-instrument rule consistent with $I_t = \{\mathbf{Z}^{J,t-\ell}, i^{t-1}, \boldsymbol{\varepsilon}^{t-\ell}\}$, with finite memory, and implementing this path as the unique local equilibrium.*

**Proof**: Proposition 1 establishes the existence of a policy-instrument rule consistent with the information set (23) and ensuring determinacy. The proof of Proposition 1, more specifically, designs such a rule of type (24) with $F(X) = F^*(X)$ and $\mathbf{G}(X) = \mathbf{G}^*(X)$. Now consider a given path of type (31), characterized by some $\mathbf{S}(X)$ and $\mathbf{T}(X)$, and consider the rule

$$\det[\mathbf{S}(L)]F^*(L)i_t = \det[\mathbf{S}(L)]L^h\mathbf{G}^*(L)\mathbf{Z}_t + L^\ell\mathbf{H}^*(L)\boldsymbol{\varepsilon}_t, \tag{33}$$

where

$$\mathbf{H}^*(X) \equiv F^*(X)\mathbf{T}_i(X) - X^{h-\ell}\mathbf{G}^*(X)\mathbf{T}_Z(X).$$

The rule (33) is consistent with the information set (30), and has finite memory (since $\mathbf{H}^*(X)$ is a polynomial, i.e. $\mathbf{H}(X) \in \mathbb{R}^{1\times n_\varepsilon}[X]$). Moreover, because $\det[\mathbf{S}(X)]$ has no root in $\mathbb{D}$, the rule

(33) is equivalent to the rule

$$F^*(L)i_t = L^h \mathbf{G}^*(L)\mathbf{Z}_t + \{\det[\mathbf{S}(L)]\}^{-1} L^\ell \mathbf{H}^*(L)\boldsymbol{\varepsilon}_t$$

and therefore ensures determinacy, like the rule $F^*(L)i_t = L^h \mathbf{G}^*(L)\mathbf{Z}_t$, by construction of $F^*(X)$ and $\mathbf{G}^*(X)$. Finally, the rule (33) is satisfied on the targeted path (31) rewritten as (32), by construction of $\mathbf{H}^*(X)$. The targeted path is one stationary solution of the system consisting of the structural equations (29) and the rule (33), and this system has a unique stationary solution. Proposition 3 follows. ■

To construct a rule with finite memory, I need that the characteristic polynomial of the resulting system (consisting of the structural equations and the rule) be a multiple of the targeted path's characteristic polynomial. In other words, I need to control the values of the characteristic-polynomial roots inside the unit circle. To do so, in the proof of Proposition 2, I multiply Proposition 1's rule by $\det[\mathbf{S}(L)]$, before adding an exogenous term. As a result, the reciprocal polynomial of the system's characteristic polynomial is multiplied by $\det[\mathbf{S}(X)]$, which is the reciprocal polynomial of the targeted path's characteristic polynomial.[12]

## 4.2 Implementation with Inferable Shocks

I now turn to the case in which: (i) the shocks are still fully unexpected by $\mathcal{PS}$, i.e. $\mathbb{E}_t\{\boldsymbol{\varepsilon}_{t+k}\} = \mathbf{0}$ for any $k \in \mathbb{N} \setminus \{0\}$ (non-news shocks), and (ii) $\mathcal{PM}$'s information set $I_t$ does not include all exogenous shocks, but all excluded shocks can be inferred from $I_t$ using *only* the structural equations (29) (as, e.g., a technology shock can be inferred from the observed input and output levels using only the production function). More specifically, the class of alternative information sets that I consider for $\mathcal{PM}$, instead of (30), is now

$$I_t = \{\mathbf{Z}^{J,t-\ell}, i^{t-1}, \boldsymbol{\varepsilon}^{K,t-\ell}\} \tag{34}$$

with $\emptyset \subsetneq J \subseteq \{1, ..., n\}$, $K \subsetneq \{1, ..., n_\varepsilon\}$, and $\ell \in \mathbb{N}$, where $\boldsymbol{\varepsilon}_t^K$ denote the vector whose elements are the $k^{th}$ elements of $\boldsymbol{\varepsilon}_t$ for $k \in K$. And I assume that (29) implies a relationship of type

$$\mathbf{E}(L)\boldsymbol{\varepsilon}_t = \mathbf{D}(L) \begin{bmatrix} \mathbf{Z}_t^J \\ i_{t-1} \\ \boldsymbol{\varepsilon}_t^K \end{bmatrix} \tag{35}$$

with $\mathbf{E}(X) \in \mathbb{R}^{n_\varepsilon \times n_\varepsilon}[X]$ and $\mathbf{D}(X) \in \mathbb{R}^{n_\varepsilon \times (|J|+|K|+1)}[X]$, where $\det[\mathbf{E}(X)]$ has no root in $\mathbb{D}$ (and where $|.|$, when applied to a set, denotes the cardinality operator). Note, for later use, that one can use Cramer's rule to rewrite (35) as

$$\det[\mathbf{E}(L)]\boldsymbol{\varepsilon}_t = \tilde{\mathbf{D}}(L) \begin{bmatrix} \mathbf{Z}_t^J \\ i_{t-1} \\ \boldsymbol{\varepsilon}_t^K \end{bmatrix}, \tag{36}$$

[12] An alternative way to proceed would be to choose $\Phi(X)$, in the design of Proposition 1's rule, as a multiple of $\det[\mathbf{S}(X)]$. The proof that the resulting rule has finite memory would, however, be more complex.

where $\tilde{\mathbf{D}}(X) \in \mathbb{R}^{n_\varepsilon \times (|J|+|K|+1)}[X]$.

Under this assumption of unobserved but inferable shocks, I obtain a similar implementation result as under the previous assumption of observed shocks. The reason is that using the structural equations to replace, in a policy-instrument rule, the unobserved shocks by functions of observed variables and shocks is *neutral* for determinacy. I state this result as follows:

**Proposition 4 (Implementation with Inferable Shocks)**: *For any non-empty set $J \subseteq \{1, ..., n\}$, any set $K \subsetneq \{1, ..., n_\varepsilon\}$, and any $\ell \in \mathbb{N}$, if $D_{\bar{J}}(X)$ has at most $\delta$ roots in $\mathbb{D} \setminus \{0\}$ and if (29) implies a relationship of type (35), then, for any path of type (31), there exists a policy-instrument rule consistent with $I_t = \{\mathbf{Z}^{J,t-\ell}, i^{t-1}, \boldsymbol{\varepsilon}^{K,t-\ell}\}$, with finite memory, and implementing this path as the unique local equilibrium.*

**Proof**: Consider a given path of type (31), characterized by some $\mathbf{S}(X)$ and $\mathbf{T}(X)$. The rule (33) has finite memory and implements this path as the unique local equilibrium; however, it is not consistent with the information set (34), as it involves unobserved shocks. Now, multiplying the left- and right-hand sides of the rule (33) by $\det[\mathbf{E}(L)]$, and using (36), leads to

$$\det[\mathbf{E}(L)] \det[\mathbf{S}(L)] F^*(L) i_t = \det[\mathbf{E}(L)] \det[\mathbf{S}(L)] L^h \mathbf{G}^*(L) \mathbf{Z}_t + L^\ell \mathbf{H}^*(L) \tilde{\mathbf{D}}(L) \begin{bmatrix} \mathbf{Z}_t^J \\ i_{t-1} \\ \boldsymbol{\varepsilon}_t^K \end{bmatrix}. \quad (37)$$

Unlike (33), the rule (37) is consistent with the information set (34). Like (33), it has finite memory and is consistent with the targeted path. Moreover, the system consisting of (29) and (37) is equivalent to the system consisting of (29) and (33), because: (i) the only equation used to transform the rule (33) into the rule (37) is (36), (ii) (36) is implied by the structural equations (29), and (iii) (36) is invertible in the sense that $\{\det[\mathbf{E}(L)]\}^{-1}$ exists (as $\det[\mathbf{E}(X)]$ has no root in $\mathbb{D}$). Since the targeted path is the unique stationary solution of the system consisting of (29) and (33), it is therefore also the unique stationary solution of the system consisting of (29) and (37). Proposition 4 follows. ∎

The assumption that the unobserved shocks can be inferred from the information set $I_t$ using *only* the structural equations (29), *not* the targeted path (31), plays a key role in Proposition 4. Indeed, using the targeted path to replace the unobserved shocks in the rule (33) by functions of observed variables and shocks would not necessarily be neutral for determinacy.

### 4.3 Partial Implementation with Unobserved News Shocks

The last case that I consider is one in which the shocks may be at least partially expected by $\mathcal{PS}$, i.e. one in which $\mathbb{E}_t\{\boldsymbol{\varepsilon}_{t+k}\}$ may be non-zero for $k \in \mathbb{N} \setminus \{0\}$. These non-zero expectations can be viewed as the result of news shocks providing to $\mathcal{PS}$ some information about the realization of future shocks. In this case, the general form of a (locally log-linearized) path for the endogenous variables is of type

$$\begin{bmatrix} \mathbf{Z}_t \\ i_t \end{bmatrix} = \sum_{u \in \mathbb{N}} \sum_{v \in \mathbb{N}} \mathbf{M}_{u,v} \mathbb{E}_{t-u} \left\{ \boldsymbol{\varepsilon}_{t-u+v} \right\}, \tag{38}$$

where $\mathbf{M}_{u,v} \in \mathbb{R}^{(n+1) \times n_{\varepsilon}}$ for $(u, v) \in \mathbb{N}^2$ (with the usual convention that $\mathbb{E}_{t-u}\{\boldsymbol{\varepsilon}_{t-u}\} = \boldsymbol{\varepsilon}_{t-u}$ for $u \in \mathbb{N}$).

I assume that $\mathcal{PM}$ does not observe the news shocks $\mathbb{E}_{t-u}\{\boldsymbol{\varepsilon}_{t-u+v}\}$ for $u \in \mathbb{N}$ and $v \in \mathbb{N} \setminus \{0\}$, just like she does not observe $\mathcal{PS}$'s expectations of future endogenous variables. For simplicity, I also assume that she does not observe the non-news shocks $\boldsymbol{\varepsilon}_{t-u}$ for $u \in \mathbb{N}$. If I assumed that she observes these non-news shocks (possibly with inside lags), then I would get, about the matrices $\mathbf{M}_{u,0}$ for $u \in \mathbb{N}$, the same implementation result as in Subsection 4.1, independently of the result that I am about to get about the matrices $\mathbf{M}_{u,v}$ for $u \in \mathbb{N}$ and $v \in \mathbb{N} \setminus \{0\}$. To focus on the latter result, thus, I consider a class of alternative information sets for $\mathcal{PM}$ that do not include any shock: $I_t = \{\mathbf{Z}^{J,t-\ell}, i^{t-1}\}$, with $\emptyset \subsetneq J \subseteq \{1, ..., n\}$ and $\ell \in \mathbb{N}$.

Given her information set, the policymaker cannot fully control the response of the economy to news shocks, captured by the matrices $\mathbf{M}_{u,v}$ for $u \in \mathbb{N}$ and $v \in \mathbb{N} \setminus \{0\}$. A full control of these matrices would require either that she observes all news shocks, or that she observes a finite number of news shocks and a finite number of $\mathcal{PS}$'s expectations of future endogenous variables (so that the policy-instrument rule with "finite forward memory" could be iterated forward). What I show, however, is that she can partially control the response of the economy to news shocks. More specifically, she can control the rates at which news shocks are discounted in the following sense:

**Definition 2 (News-Shocks Discounting)**: *A path of type (38) is said to discount news shocks at rates* $(\beta_1, \ldots, \beta_q) \in \mathbb{D}^q$, *where* $q \in \mathbb{N} \setminus \{0\}$, *if there exist* $\mathbf{N}_{u,p} \in \mathbb{R}^{(n+1) \times n_{\varepsilon}}$ *for* $u \in \mathbb{N}$ *and* $p \in \{1, ..., q\}$ *such that* $\forall u \in \mathbb{N}$, $\exists \underline{v} \in \mathbb{N}$, $\forall v \geq \underline{v}$, $\mathbf{M}_{u,v} = \sum_{p=1}^{q} \mathbf{N}_{u,p} \beta_p^v$.

Loosely speaking, the discount rates of news shocks measure the speed at which the current impact of news about more and more distant future events vanishes. As clear from Blanchard and Kahn's (1980) analysis, these discount rates are equal to the inverses of the roots of the system's characteristic polynomial $\mathcal{C}(X)$ that lie outside the unit circle, if these roots are simple (i.e. of multiplicity one). Equivalently, the discount rates are equal to the roots of the reciprocal polynomial of $\mathcal{C}(X)$ that belong to $\mathbb{D} \setminus \{0\}$, again if these roots are simple. By choosing not only the number of these roots (as I did in the previous section), but also their values (which I left free in the previous section), I can therefore control the discount rates of news shocks and get the following partial-implementation result:

**Proposition 5 (Partial Implementation with News Shocks**: *For any non-empty set* $J \subseteq \{1, ..., n\}$ *and any* $\ell \in \mathbb{N}$, *if* $D_{\bar{J}}(X)$ *has at most* $\delta$ *roots in* $\mathbb{D} \setminus \{0\}$, *all of them simple, then, for* *any* $\Phi(X) \in \mathbb{R}[X]$ *such that* $D_{\bar{J}}(X)\Phi(X)$ *has exactly* $\delta$ *roots in* $\mathbb{D} \setminus \{0\}$*, all of them simple and denoted by* $(r_p)_{1 \leq p \leq \delta}$*, there exists a policy-instrument rule consistent with* $I_t = \{\mathbf{Z}^{J,t-\ell}, i^{t-1}\}$*, with finite memory, ensuring determinacy, and such that the unique local equilibrium is a path of type (38) that discounts news shocks at rates* $(r_p)_{1 \leq p \leq \delta}$*.*

**Proof**: Proposition 5 straightforwardly follows from the proof of Proposition 1 and Blanchard and Kahn (1980, p. 1308). ∎

Thus, even in the presence of inside lags of any length (i.e. for any $\ell \in \mathbb{N}$), the policymaker can still control some key features of the response of the economy to news shocks. This partial-implementation result can be interpreted in the same way as the determinacy result of Section 2 (for univariate models) and Proposition 1 (for multivariate models): no matter how far behind the curve inside lags put her, and provided that her choice is not arbitrarily restricted to a specific parametric family of policy-instrument rules, the policymaker can still control some key features of the forward-looking behavior of the economy, via the private sector's expectation that she will follow her policy-instrument rule in the future.

# 5 Beyond Determinacy: Non-Superinertial Rules

So far, I have shown that inside and outside lags do not prevent the policymaker from ensuring determinacy (Proposition 1), nor, beyond determinacy, from controlling the response of the economy to non-news shocks with a finite memory (Propositions 3-4) as well as some key features of the response of the economy to news shocks (Proposition 5). The common source of these results is that inside and outside lags do not prevent the policymaker from controlling the *characteristic polynomial* of the dynamic system. In particular, they do not prevent her from choosing the number of roots of this polynomial outside the unit circle (Proposition 1), the number and the values of its roots inside the unit circle (Propositions 3-4), and the values of its roots outside the unit circle (Proposition 5).

The number and the values of the characteristic-polynomial roots, however, are not the only degrees of freedom available to the policymaker. For a given characteristic polynomial, there remain some degrees of freedom. Because her choice is not arbitrarily restricted to a specific parametric family of rules, the policymaker can substitute a reaction of the policy instrument to one variable with a reaction to another variable, or react to variables in the more distant past, without affecting the characteristic polynomial. Reacting to variables in the more distant past need not increase the degree of the characteristic polynomial because the reactions to different past variables may have offsetting effects on the characteristic polynomial.

In the present section, I exploit these additional degrees of freedom by designing, in the class of multivariate models considered in Section 3, policy-instrument rules that not only are consistent with the information set (and thus consistent with lags), not only lead to the targeted characteristic polynomial (and in particular ensure determinacy), but also are not *superinertial*.[13] I define superinertial policy-instrument rules as follows:

**Definition 3 (Superinertial Policy-Instrument Rules)**: *A policy-instrument rule of type (24) is said to be superinertial when $F(X)$ has at least one root in $\mathbb{D}$.*

As I explained and documented in the Introduction, simulation results in the literature provide a motivation for adopting a non-superinertial rule when there is a non-zero probability that the true model is backward-looking, even if this probability is arbitrarily small. Although the design of rules whose properties are robust across alternative models is mostly beyond the scope of this paper, I make here a small step in this direction by identifying and illustrating some degrees of freedom that can be exploited for robustness purposes.

To extend Proposition 1 to determinacy with non-superinertial rules in the presence of inside and outside lags, I first establish the following lemma:

**Lemma 2**: $\forall Q(X) \in \mathbb{R}[X]$ *such that* $Q(0) \neq 0$, $\exists P(X) \in \mathbb{R}[X]$ *such that all the roots of* $Q(X) + X^{d(Q)+1}P(X)$ *belong to* $\mathbb{C} \setminus \mathbb{D}$.

**Proof**: See Appendix A.2. ■ In essence, Appendix A.2 first uses Newton's identities to rewrite the constraint on the coefficients of the polynomial $Q(X) + X^{d(Q)+1}P(X)$ as a constraint on the sums of the $k^{th}$ powers of the roots of this polynomial. It then designs a polynomial that satisfies the latter constraint and has all its roots in $\mathbb{C} \setminus \mathbb{D}$.

I then proceed along the lines of the proof of Proposition 1, but I apply Bézout's identity to $\Psi_j(X)$ for $j \in J$, instead of $j \in \bar{J}$, and I use Lemma 2 and a new condition (namely, that $D_J(X)$ should have no roots in $\mathbb{D} \setminus \{0\}$) to ensure that the policy-instrument rule designed is not superinertial. I thus obtain the following proposition:

**Proposition 6 (Determinacy with a Non-Superinertial Rule)**: *For any non-empty and non-singleton set* $J \subseteq \{1, ..., n\}$ *and any* $\ell \in \mathbb{N}$, *if* $D_J(X)$ *has no root in* $\mathbb{D} \setminus \{0\}$, *then there exists a policy-instrument rule consistent with* $I_t = \{\mathbf{Z}^{J,t-\ell}, i^{t-1}\}$, *ensuring determinacy, and non-superinertial.*

**Proof**: Bézout's identity implies that there exist $U_j(X) \in \mathbb{R}[X]$ for $j \in J$ such that

$$\sum\nolimits_{j \in J} \Psi_j(X) U_j(X) = D_J(X). \tag{39}$$

[13] The term "superinertial" was coined by Woodford (1999).

Since $D_J(X) \neq 0$, there exists $j \in J$ such that $U_j(X) \neq 0$. For later use, let $d_0 \in \mathbb{N}$ denote the multiplicity of the root 0 in $D_J(X)$, and let $\tilde{D}_J(X) \equiv X^{-d_0} D_J(X) \in \mathbb{R}[X]$ (so that $\tilde{D}_J(0) \neq 0$).

Let $\Phi(X) \in \mathbb{R}[X]$ be an arbitrary polynomial that: (i) is of degree $\delta$, (ii) has all its roots in $\mathbb{D} \setminus \{0\}$, and (iii) is not a multiple of $\Psi_{n+1}(X)$. Let $\tilde{\Phi}(X) \equiv X^{d(\Phi)}\Phi(X^{-1})$ and $\tilde{\Psi}_{n+1}(X) \equiv X^{d(\Psi_{n+1})}\Psi_{n+1}(X^{-1})$ denote the reciprocal polynomials of $\Phi(X)$ and $\Psi_{n+1}(X)$. Since $\Phi(0) \neq 0$ and $\Psi_{n+1}(0) = \det[\mathbf{A}(0)] \neq 0$ (given Assumption 1), we have $d(\tilde{\Phi}) = d(\Phi) = \delta$ and $d(\tilde{\Psi}_{n+1}) = d(\Psi_{n+1})$. Let $(h, m) \in \mathbb{N}^2$ be such that

$$h \geq \max(\ell, \gamma + 1), \tag{40}$$

$$m \geq \max\left[2d(\Psi_{n+1}) + (h - \gamma - 1) + d_0 - \delta, d(\Psi_{n+1}) - 1\right]. \tag{41}$$

Since $\tilde{\Psi}_{n+1}(X) \neq 0$ (as $\Psi_{n+1}(X) \neq 0$), I can consider the Euclidean division of $X^m\tilde{\Phi}(X)$ by $\tilde{\Psi}_{n+1}(X)$. Let $\tilde{Q}(X) \in \mathbb{R}[X]$ and $\tilde{R}(X) \in \mathbb{R}[X]$ denote respectively the quotient and the remainder of this division, i.e. the unique polynomials such that

$$X^m\tilde{\Phi}(X) = \tilde{\Psi}_{n+1}(X)\tilde{Q}(X) + \tilde{R}(X) \tag{42}$$

and $d(\tilde{R}) < d(\tilde{\Psi}_{n+1})$. Using (42) and $m + d(\tilde{\Phi}) = m + \delta \geq d(\Psi_{n+1}) = d(\tilde{\Psi}_{n+1})$, where the inequality follows from (40)-(41), I get $\tilde{Q}(X) \neq 0$ and, therefore, $d(\tilde{Q}) = m + d(\tilde{\Phi}) - d(\tilde{\Psi}_{n+1})$, or equivalently

$$d(\tilde{Q}) = m + \delta - d(\Psi_{n+1}). \tag{43}$$

Evaluating (42) at $X^{-1}$ (instead of $X$), multiplying the left- and right-hand sides of the resulting equation by $X^{m+\delta}$, and using (43), I get

$$\Phi(X) = \Psi_{n+1}(X)Q(X) + X^{m+\delta-d(\tilde{R})}R(X), \tag{44}$$

where $Q(X) \equiv X^{d(\tilde{Q})}\tilde{Q}(X^{-1})$ and $R(X) \equiv X^{d(\tilde{R})}\tilde{R}(X^{-1})$ denote the reciprocal polynomials of $\tilde{Q}(X)$ and $\tilde{R}(X)$.

Since $\tilde{Q}(X) \neq 0$ implies $Q(0) \neq 0$, Lemma 2 implies the existence of $P(X) \in \mathbb{R}[X]$ such that all the roots of $Q(X) + X^{d(Q)+1}P(X)$ lie outside the unit circle. Multiplying the left- and right-hand sides of (39) by $X^{-d_0}[X^{m+\delta-d(\tilde{R})}R(X) - X^{d(Q)+1}P(X)\Psi_{n+1}(X)]$ and using (44), I obtain

$$\sum\nolimits_{j=1}^{n} (-1)^{n-j} X^{h-\gamma}\Psi_j(X) G_j^*(X) + \Psi_{n+1}(X) F^*(X) = \tilde{D}_J(X)\Phi(X), \tag{45}$$

where

$$F^*(X) \equiv \left[Q(X) + X^{d(Q)+1}P(X)\right]\tilde{D}_J(X),$$
$$G_j^*(X) \equiv (-1)^{n-j}X^{-d_0-(h-\gamma)}\left[X^{m+\delta-d(\tilde{R})}R(X) - X^{d(Q)+1}P(X)\Psi_{n+1}(X)\right]U_j(X) \text{ for } j \in J,$$
$$G_j^*(X) \equiv 0 \text{ for } j \in \{1, ..., n\} \setminus J.$$

To show that $F^*(X)$ and $\mathbf{G}^*(X) \equiv [\, G_1^*(X) \quad \cdots \quad G_n^*(X)\,]$ are admissible choices for $F(X)$ and $\mathbf{G}(X)$ in the rule (24), I need to prove that: (i) $F^*(0) \neq 0$, (ii) $\mathbf{G}^*(X) \neq \mathbf{0}$, and (iii) $\mathbf{G}^*(X)$ is a polynomial ($\mathbf{G}^*(X) \in \mathbb{R}^{1\times n}[X]$), i.e. it does not involve any negative power of $X$.

The inequality $F^*(0) \neq 0$ follows from the fact that, by construction, $Q(X)+X^{d(Q)+1}P(X)$ has no root in $\mathbb{D}$ and $\tilde{D}_J(0) \neq 0$.

The inequality $\mathbf{G}^*(X) \neq \mathbf{0}$ follows from the existence of $j \in J$ such that $U_j(X) \neq 0$ (which I have shown above) and from $V_1(X) \equiv X^{m+\delta-d(\tilde{R})}R(X) \neq V_2(X) \equiv X^{d(Q)+1}P(X)\Psi_{n+1}(X)$. To prove that $V_1(X) \neq V_2(X)$, I first note that because $\Phi(X)$ is not a multiple of $\Psi_{n+1}(X)$, (44) implies $R(X) \neq 0$, which in turn implies $d(V_1) = m + \delta - d(\tilde{R}) + d(R)$. I also note that $m \geq d(\Psi_{n+1}) - 1 = d(\tilde{\Psi}_{n+1}) - 1 \geq d(\tilde{R})$, where the first inequality follows from (41). Next, using (42), $\tilde{\Phi}(0) \neq 0$, $\tilde{\Psi}_{n+1}(0) \neq 0$, $\tilde{R}(X) \neq 0$ (due to $R(X) \neq 0$), and $m \geq d(\tilde{R})$, I get that $\min\{k \in \mathbb{N} | \tilde{Q}_k \neq 0\} = \min\{k \in \mathbb{N} | \tilde{R}_k \neq 0\}$, and hence that

$$d(\tilde{R}) - d(R) = d(\tilde{Q}) - d(Q). \tag{46}$$

Using (43) and (46), I then get $d(V_1) = d(Q) + d(\Psi_{n+1})$. If $P(X) \neq 0$, then $d(V_2) = d(Q) + d(\Psi_{n+1}) + d(P) + 1 > d(V_1)$, so $V_2(X) \neq V_1(X)$. Alternatively, if $P(X) = 0$, then $V_2(X) = 0 \neq V_1(X)$.

Finally, $\mathbf{G}^*(X) \in \mathbb{R}^{1\times n}[X]$ follows from $v_1 \equiv m + \delta - d(\tilde{R}) - d_0 - (h - \gamma) \geq 0$ and $v_2 \equiv d(Q) + 1 - d_0 - (h - \gamma) \geq 0$. In turn, $v_1 \geq 0$ follows from (41) and $d(\tilde{R}) \leq d(\tilde{\Psi}_{n+1}) - 1 = d(\Psi_{n+1}) - 1$, while $v_2 \geq 0$ follows from (41), (43), (46), and $d(\tilde{R}) \leq d(\Psi_{n+1}) - 1$.

The left-hand side of (45) is the polynomial (25) with $F(X) = F^*(X)$ and $\mathbf{G}(X) = \mathbf{G}^*(X)$. The right-hand side of (45) has exactly $\delta$ roots in $\mathbb{D} \setminus \{0\}$ under the assumption stated in Proposition 6 (namely, the assumption that $D_J(X)$ has no such roots). Therefore, Lemma 1 implies that the system consisting of the structural equations (22) and the rule (24) with $F(X) = F^*(X)$ and $\mathbf{G}(X) = \mathbf{G}^*(X)$ meets Blanchard and Kahn's (1980) *root-counting* condition. In addition, this system also satisfies Blanchard and Kahn's (1980) *no-decoupling* condition, except possibly for a zero-measure set of polynomials $\Phi(X)$, for the same reasons as in the proof of Proposition 1. Therefore, this system has a unique stationary solution.

Under the assumption stated in Proposition 6 (namely, the assumption that $D_J(X)$ has no root in $\mathbb{D} \setminus \{0\}$), $\tilde{D}_J(X)$ has no root in $\mathbb{D}$. Moreover, by construction, $Q(X) + X^{d(Q)+1}P(X)$ has no root in $\mathbb{D}$ either. Therefore, $F^*(X)$ has no root in $\mathbb{D}$, and the rule (24) with $F(X) = F^*(X)$ and $\mathbf{G}(X) = \mathbf{G}^*(X)$ is not superinertial. Proposition 6 follows. ■

Proposition 6's condition that $D_J(X)$ should have no roots in $\mathbb{D}\setminus\{0\}$ is stronger than Proposition 1's condition that $D_{\bar{J}}(X)$ should have at most $\delta$ roots in $\mathbb{D} \setminus \{0\}$. The "weak Sargent-Wallace property" of Proposition 2, in particular, is not enough for Proposition 6's condition to be met. In the next section, I will investigate whether Proposition 6's condition is met in four different stabilization-policy models. I will find that, depending on the model, it is met for all or almost all non-empty and non-singleton sets $J$.

## 6 Application to Four Models

In this section, I consider four stabilization-policy models in turn. For each model, I derive three results that hold for all parameter values (except possibly a zero-measure subset). First, I show that the model (without exogenous disturbances) can be written in a form of type (22) satisfying Assumptions 1-5. Second, I show that the model satisfies the condition stated in Proposition 1 for any non-empty set $J$. Third, I determine the set of sets $J$ for which the model satisfies the condition stated in Proposition 6.

The four models are the basic New Keynesian (NK) model, presented in detail in Woodford (2003) and Galí (2015), and the models of Rotemberg and Woodford (1997, 1999), Smets and Wouters (2007), and Schmitt-Grohé and Uribe (1997). They differ from each other in several dimensions: small vs. medium scale, with vs. without distributed outside lags, monetary vs. fiscal policy. The goal of the section is to show that my general framework encompasses various models; to illustrate that determinacy can be ensured under weak conditions (i.e. even by reacting to a single variable with arbitrarily long inside lags); and to gain more insight into the condition under which determinacy can be ensured with a non-superinertial rule. To save space, I summarize here the main results obtained for each model, and I relegate the technical details to Appendices A.3-A.6.

In the basic NK model, the private sector sets output and inflation, and the policymaker is a central bank setting the short-term nominal interest rate. The system of structural equations can straightforwardly be written in a form of type (22) satisfying Assumptions 1-5 with $n = 2$, $\gamma = -1$, and $\delta = 2$. This system satisfies the condition stated in Proposition 1 for any non-empty set $J$. So, the central bank can ensure determinacy by making the interest rate react to either output, or inflation, or both, in the presence of inside lags of any length. The system also satisfies the condition stated in Proposition 6 for the only non-empty and non-singleton set $J$. So, the central bank can ensure determinacy with a non-superinertial interest-rate rule involving both output and inflation, in the presence of inside lags of any length. The rules whose existence is implied by Propositions 1 and 6 not only ensure determinacy, but also can lead to any given characteristic polynomial with $\delta = 2$ roots outside the unit circle. So, the central bank can fully or partially control the response of the economy to non-news and news shocks (Propositions 3-5).

The model of Rotemberg and Woodford (1997, 1999) introduces distributed outside lags into the basic NK model: some (randomly selected) price-resetting firms set their price one period in advance, others set their price two periods in advance, and households choose their consumption two periods in advance. The system of structural equations can be reduced to a form of type (22) satisfying Assumptions 1-5 with $n = 2$, $\gamma = 1$, and $\delta = 2$. One of the two variables, the private sector's expectation of two-period-ahead inflation, is never observed by the central bank. So, the only non-empty set $J$ is a singleton, and Proposition 6 cannot be applied. But

Proposition 1 can be used to get an interest-rate rule ensuring determinacy for inside lags of any length. Propositions 3-5 can also be used to get an interest-rate rule that fully or partially controls the response of the economy to non-news and news shocks.

The model of Smets and Wouters (2007) is also a monetary-policy model with the short-term nominal interest rate as the policy instrument. However, it is significantly larger than the previous two models, as it incorporates many nominal and real frictions. Using its static structural equations, one can write its dynamic structural equations in a form of type (22) satisfying Assumptions 1-5 with $n = 6$, $\gamma = -1$, and $\delta = 5$. This system generically satisfies the condition stated in Proposition 1 for any non-empty set $J$. So, the central bank can ensure determinacy by making the interest rate react to any non-empty set of variables, and in particular to any single variable, in the presence of inside lags of any length.

The system of dynamic structural equations also generically satisfies the condition stated in Proposition 6 for all non-empty and non-singleton sets $J$ except one. So, the central bank can ensure determinacy with a non-superinertial interest-rate rule, in the presence of inside lags of any length, provided that she observes at least two variables. The only exception corresponds to the hypothetical case in which the central bank would observe only investment and the value of capital. The reason is that the investment Euler equation makes these two variables substitutable with each other for determinacy purposes. So, reacting to both variables, rather than only one, does not generate any degree of freedom that could be exploited to find a non-superinertial rule.

Unlike the previous three models, the model of Schmitt-Grohé and Uribe (1997) is a fiscal-policy model. The policymaker is a tax authority setting either the labor-income-tax rate or the income-tax rate. As it stands, the model does not satisfy Assumption 5, because the stock of public debt is a residual variable that follows a non-stationary process when the other endogenous variables follow arbitrary stationary processes. To overcome this difficulty (which will arise in most fiscal-policy models), I proceed as follows: I restrict the class of information sets for the tax authority to the sets that include the debt level (possibly with inside lags); I write the policy instrument as the sum of two terms, the first of which is an arbitrary term involving the (possibly lagged) debt level; and I treat the second term as the new instrument. In effect, this trick allows – but does not compel – the fiscal rule to be "locally Ricardian" in the sense of Woodford (2003, Chapter 4).

Using this trick and the static structural equations of the model, I write its dynamic structural equations in a form of type (22) satisfying Assumptions 1-5 with $n = 2$, $\gamma = 0$, and $\delta = 1$. This system generically satisfies the condition stated in Proposition 1 for all non-empty sets $J$ that include the debt level. So, with either policy instrument (the labor-income-tax rate or the income-tax rate), the tax authority can ensure determinacy in the presence of inside lags of any length, provided that she observes the (possibly lagged) debt level. The system also generically satisfies the condition stated in Proposition 6 for the only non-empty and non-singleton set $J$.

So, Proposition 6 can be used to get a non-superinertial tax-rate rule ensuring determinacy in the presence of inside lags of any length.

## 7 Concluding Remarks

Macroeconomic stabilization policy is notoriously subject to inside and outside lags. Can these lags prevent policymakers from ensuring determinacy? This paper has provided a negative answer to this question, in a broad class of dynamic rational-expectations models that allows for inside and outside lags of any length.

To establish this result, I have inverted the problem usually tackled in the literature: I have started from a targeted characteristic polynomial for the dynamic system (consisting of the structural equations and the policy-instrument rule), and I have derived a corresponding policy-instrument rule. This method has enabled me to establish much more general determinacy results with inside and outside lags than were previously established in sparse examples in the literature. Moreover, I have shown that beyond determinacy and for any lags, this method offers degrees of freedom that can be exploited both for implementation purposes (controlling the response of the economy to non-news and news shocks) and for robustness purposes (designing non-superinertial rules that may prove more robust under model uncertainty).

As in most of the literature on stabilization policy, I have focused throughout the paper on local-equilibrium determinacy in locally log-linearized models. At least in the context of interest-rate rules, however, there is usually no solid economic reason to assume away the existence of non-local equilibria, as argued by Cochrane (2011). The most common policy proposal to eliminate these equilibria, made initially by Benhabib, Schmitt-Grohé, and Uribe (2002), and discussed by Woodford (2003, Chapter 2), consists in switching from an interest-rate rule ensuring local-equilibrium determinacy to a money-growth rule when the economy goes *outside* a specified neighborhood of the steady state. The local analysis that I conduct in the paper fits naturally into this proposal, as it can be applied *inside* the neighborhood of the steady state.

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# Appendix

In this appendix, I prove Lemmas 1 and 2, and I apply my general results to four stabilization-policy models.

## A.1 Proof of Lemma 1

The structural equations (22) and the rule (24) form the following system:

$$\mathbb{E}_t\left\{\mathbf{\Delta}\left(L^{-1}\right)\left[\mathbf{A}\left(L\right)\mathbf{Z}_t + L^{-\gamma}\mathbf{B}\left(L\right)i_t\right]\right\} = \mathbf{0}, \quad \text{(A.1)}$$

$$F\left(L\right)i_t - L^h\mathbf{G}\left(L\right)\mathbf{Z}_t = 0. \quad \text{(A.2)}$$

Given that $F(0) \neq 0$ and $h \geq \gamma + 1$, I can use (A.2) to remove all the terms of type $\mathbb{E}_t\{i_{t+k}\}$ with $k \geq 0$ (if any) from (A.1), and thus rewrite (A.1) as

$$\mathbb{E}_t\left\{\mathbf{\Delta}\left(L^{-1}\right)\left[\mathbf{A}\left(0\right)\mathbf{Z}_t + \check{\mathbf{A}}\left(L\right)\mathbf{Z}_{t-1}\right]\right\} + \check{\mathbf{B}}\left(L\right)i_{t-1} = \mathbf{0},$$

where $\check{\mathbf{A}}(X) \in \mathbb{R}^{n\times n}[X]$ and $\check{\mathbf{B}}(X) \in \mathbb{R}^{n\times 1}[X]$. Next, given Assumption 1, I can re-express the system (A.1)-(A.2) in terms of $\tilde{\mathbf{Z}}_t \equiv \mathbf{A}(0)\mathbf{Z}_t$, instead of $\mathbf{Z}_t$:

$$\mathbb{E}_t\left\{\mathbf{\Delta}\left(L^{-1}\right)\left[\tilde{\mathbf{Z}}_t + \tilde{\mathbf{A}}\left(L\right)\tilde{\mathbf{Z}}_{t-1}\right]\right\} + \check{\mathbf{B}}\left(L\right)i_{t-1} = \mathbf{0}, \quad \text{(A.3)}$$

$$F\left(L\right)i_t - L^h\tilde{\mathbf{G}}\left(L\right)\tilde{\mathbf{Z}}_t = 0, \quad \text{(A.4)}$$

where $\tilde{\mathbf{A}}(X) \equiv \check{\mathbf{A}}(X)[\mathbf{A}(0)]^{-1}$ and $\tilde{\mathbf{G}}(X) \equiv \mathbf{G}(X)[\mathbf{A}(0)]^{-1}$.

For each $j \in \{1, ..., n\}$, let $\tilde{z}_{j,t}$ denote the $j^{th}$ element of $\tilde{\mathbf{Z}}_t$, and $\zeta_j$ the largest lag of $\tilde{z}_{j,t}$ in the system (A.3)-(A.4), i.e. the integer such that $\tilde{z}_{j,t-\zeta_j}$ appears in (A.3)-(A.4) and $\tilde{z}_{j,t-\zeta_j-k}$ does not appear for any $k \geq 1$ in (A.3)-(A.4). Assumption 4 implies that $\zeta_j \geq 0$ for all $j \in \{1, ..., n\}$. Moreover, Assumption 5 implies that $(\zeta_j, \delta_j) \neq (0, 0)$ for all $j \in \{1, ..., n\}$. Indeed, if there existed $j \in \{1, ..., n\}$ such that $(\zeta_j, \delta_j) = (0, 0)$, then one could easily rewrite the system in a block-recursive way, with $n$ equations not featuring the variable $\tilde{z}_{j,t}$ (at any past, current, or future date) and one equation residually determining $\tilde{z}_{j,t}$, so that Assumption 5 would not be satisfied.

Similarly, let $\zeta_{n+1}$ denote the largest lag of $i_t$ in the system (A.3)-(A.4), i.e. the integer such that $i_{t-\zeta_{n+1}}$ appears in (A.3)-(A.4) and $i_{t-\zeta_{n+1}-k}$ does not appear for any $k \geq 1$ in (A.3)-(A.4). Since $F(0) \neq 0$, we have $\zeta_{n+1} \geq 0$. I focus on the case in which $\zeta_{n+1} \geq 1$. In the alternative case where $\zeta_{n+1} = 0$, one could easily rewrite the system in a block-recursive way, with $n$ equations not featuring the variable $i_t$ (at any past, current, or future date) and one equation residually determining $i_t$, and one could then focus on the $n$-equation subsystem without the variable $i_t$.

Since $\zeta_j \geq 0$, $\delta_j \geq 0$, $(\zeta_j, \delta_j) \neq (0, 0)$ for all $j \in \{1, ..., n\}$, and $\zeta_{n+1} \geq 1$, I can easily (but tediously) rewrite the system (A.3)-(A.4) in the following Blanchard and Kahn's (1980) form:

$$\mathbb{E}_t\left\{\mathbf{X}_{t+1}\right\} = \mathbf{M}\mathbf{X}_t,$$

where

$$\mathbf{X}_t \equiv \begin{bmatrix} \mathbf{X}_t^1 \\ \vdots \\ \mathbf{X}_t^{n+1} \end{bmatrix}, \mathbf{X}_t^j \equiv \begin{bmatrix} \tilde{z}_{j,t+\delta_j-1} \\ \vdots \\ \tilde{z}_{j,t-\zeta_j} \end{bmatrix} \text{ for } j \in \{1,...,n\},\ \mathbf{X}_t^{n+1} \equiv \begin{bmatrix} i_{t-1} \\ \vdots \\ i_{t-\zeta_{n+1}} \end{bmatrix},$$

and where $\mathbf{M}$ is a square matrix with real-number elements. The non-predetermined variables of this system are the variables $\mathbb{E}_t\{\tilde{z}_{j,t+k_j}\}$ for all $j \in \{1,...,n\}$ such that $\delta_j \geq 1$ and all $k_j \in \{1,...,\delta_j\}$. Their number is $\delta \equiv \sum_{j=1}^{n} \delta_j$.

The characteristic polynomial of the system (A.1)-(A.2) is the same, up to a multiplicative factor of type $X^p$ with $p \in \mathbb{N}$, as the characteristic polynomial of the corresponding perfect-foresight deterministic system

$$\mathbf{M}_1(L) \begin{bmatrix} \mathbf{Z}_t \\ i_t \end{bmatrix} = \mathbf{0}$$

with

$$\mathbf{M}_1(X) \equiv \begin{bmatrix} X^{\max(\gamma,0)}\mathbf{A}(X) & X^{\max(0,-\gamma)}\mathbf{B}(X) \\ -X^h\mathbf{G}(X) & F(X) \end{bmatrix}.$$

It is also the same as the characteristic polynomial of the system (A.3)-(A.4), which in turn is the same, up to a multiplicative factor of type $X^p$ with $p \in \mathbb{N}$, as the characteristic polynomial of the corresponding perfect-foresight deterministic system

$$\mathbf{M}_2(L) \begin{bmatrix} \tilde{\mathbf{Z}}_t \\ i_t \end{bmatrix} = \mathbf{0} \tag{A.5}$$

with

$$\mathbf{M}_2(X) \equiv \begin{bmatrix} \mathbf{I}_n + X\tilde{\mathbf{A}}(X) & X\mathbf{\Delta}(X)\check{\mathbf{B}}(X) \\ -X^h\tilde{\mathbf{G}}(X) & F(X) \end{bmatrix},$$

where $\mathbf{I}_n$ denote the $n \times n$ identity matrix. Since $\det[\mathbf{M}_2(0)] = F(0) \neq 0$, I can use a standard result in time-series analysis (see, e.g., Hamilton, 1994, Chapter 10, Proposition 10.1) and get that the reciprocal polynomial of (A.5)'s characteristic polynomial is $\det[\mathbf{M}_2(X)]$. Now, adding a scalar multiple of one row to another row leaves the determinant of a matrix unchanged, so $\det[\mathbf{M}_2(X)] = \det[\mathbf{M}_1(X)]$. Moreover, Laplace's expansion implies that $\det[\mathbf{M}_1(X)]$ is equal to $X^{n\max(\gamma,0)}$ times the polynomial (25). Therefore, the reciprocal polynomial of (A.1)-(A.2)'s characteristic polynomial is the polynomial (25), up to a multiplicative factor of type $X^p$ with $p \in \mathbb{N}$. Since zero is not a root of (25), the multiplicative factor is one, and the reciprocal polynomial of (A.1)-(A.2)'s characteristic polynomial is the polynomial (25). Lemma 1 follows.

## A.2 Proof of Lemma 2

I proceed in three steps. In the first step, I show that Lemma 2 is implied by the following conjecture:

**Conjecture 1**: $\forall \tilde{Q}(X) \in \mathbb{R}[X]$, $\exists \tilde{P}(X) \in \mathbb{R}[X]$ *such that all the roots of* $X^{d(\tilde{P})+1}\tilde{Q}(X) + \tilde{P}(X)$ *are of modulus lower than one.*

To prove this implication, I start from an arbitrary $Q(X) \in \mathbb{R}[X]$ such that $Q(0) \neq 0$. Let $\tilde{Q}(X) \equiv X^{d(Q)}Q(X^{-1})$ denote the reciprocal polynomial of $Q(X)$. Conjecture 1 implies the existence of $\tilde{P}(X) \in \mathbb{R}[X]$ such that all the roots of $\tilde{O}(X) \equiv X^{d(\tilde{P})+1}\tilde{Q}(X) + \tilde{P}(X)$ are of modulus lower than one. Therefore, all the roots of $O(X) \equiv X^{d(\tilde{O})}\tilde{O}(X^{-1})$, the reciprocal polynomial of $\tilde{O}(X)$, are of modulus higher than one. Since $Q(0) \neq 0$, we have $d(\tilde{Q}) = d(Q)$ and $Q(X) = X^{d(\tilde{Q})}\tilde{Q}(X^{-1})$. We also have $\tilde{Q}(X) \neq 0$, and therefore $d(\tilde{O}) = d(\tilde{P}) + d(\tilde{Q}) + 1 = d(\tilde{P}) + d(Q) + 1$. As a consequence, $O(X) = Q(X) + X^{d(Q)+1}P(X)$, where $P(X) \equiv X^{d(\tilde{P})}\tilde{P}(X^{-1})$ denotes the reciprocal polynomial of $\tilde{P}(X)$. So, there exists $P(X)$ such that all the roots of $Q(X) + X^{d(Q)+1}P(X)$ are of modulus higher than one. Thus, Conjecture 1 implies Lemma 2.

In the second step, I show that Conjecture 1 is equivalent to another conjecture. Consider some arbitrary $\tilde{P}(X)$ and $\tilde{Q}(X)$ in $\mathbb{R}[X]$. I focus on the non-trivial case in which $\tilde{Q}(X) \neq 0$ and I assume, without any loss in generality, that the coefficient of $X^{d(\tilde{Q})}$ in $\tilde{Q}(X)$ is equal to one. Let $(\alpha_j)_{1\leq j\leq m} \in \mathbb{C}^m$ denote the roots of $\tilde{O}(X) \equiv X^{d(\tilde{P})+1}\tilde{Q}(X) + \tilde{P}(X)$ counted with their multiplicity. We have $\tilde{O}(X) = X^m + \sum_{k=1}^{m}(-1)^k \Sigma_k X^{m-k}$, where $\Sigma_k \equiv \sum_{1\leq j_1<j_2<...<j_k\leq m} \alpha_{j_1}\alpha_{j_2}...\alpha_{j_k}$ for all $k \in \{1,...,m\}$. Let $S_k \equiv \sum_{j=1}^{m} \alpha_j^k$ for all $k \in \{1,...,m\}$. For any $K \in \{1,...,m\}$, Newton's identities $S_k - \Sigma_1 S_{k-1} + \Sigma_2 S_{k-2} - ... + (-1)^k \Sigma_k k = 0$ for $k \in \{1,...,K\}$ give by recurrence $\Sigma_1$, ..., $\Sigma_K$ as functions of $(S_j)_{1\leq j\leq K}$ and, conversely, $S_1$, ..., $S_K$ as functions of $(\Sigma_j)_{1\leq j\leq K}$, where these functions are polynomial functions with real-number coefficients. Conjecture 1 is therefore equivalent to the following conjecture:

**Conjecture 2**: *$\forall K \in \mathbb{N}\setminus\{0\}$, $\forall (s_1,...,s_K) \in \mathbb{R}^K$, $\exists m \in \mathbb{N}\setminus\{0\}$ and $\exists (\alpha_1,...,\alpha_m) \in \mathbb{C}^m$ such that: (i) $m > K$, (ii) $\forall j \in \{1,...,m\}$, $|\alpha_j| < 1$, (iii) $\forall k \in \{1,...,K\}$, $\sum_{j=1}^{m} \alpha_j^k = s_k$, and (iv) all the coefficients of the polynomial $\prod_{j=1}^{m}(X - \alpha_j)$ are real numbers.*

In the third step, I prove Conjecture 2. Consider an arbitrary $K \in \mathbb{N}\setminus\{0\}$ and an arbitrary $(s_1,...,s_K) \in \mathbb{R}^K$. For any $k \in \{1,...,K\}$ and any finite set $A$ of complex numbers, let $S_k^A \equiv \sum_{a\in A} a^k$. For any finite sets of complex numbers $A$ and $B$, we have $S_k^{A\sqcup B} = S_k^A + S_k^B$, where $A \sqcup B$ denotes the union of $A$ and $B$ counting each element with its multiplicity (so that in particular the cardinality of $A \sqcup B$ is the sum of the cardinalities of $A$ and $B$). We also have that for any $\lambda \in \mathbb{C}$, if $B \equiv \{\lambda a | a \in A\}$, then $S_k^B = \lambda^k S_k^A$.

Now consider an arbitrary $k \in \{1,...,K\}$. Let $S_{k,k} \equiv s_k$ and $S_{k,j} \equiv 0$ for $j \in \{1,...,K\}\setminus\{k\}$. Let me define implicitly, by recurrence, $\Sigma_{k,j}$ for $j \in \{1,...,K\}$ by $S_{k,j} - \Sigma_{k,1}S_{k,j-1} + \Sigma_{k,2}S_{k,j-2} - ... + (-1)^j \Sigma_{k,j} j = 0$ for $j \in \{1,...,K\}$. Let $\tilde{O}_k(X) \equiv X^K - \Sigma_{k,1}X^{K-1} + \Sigma_{k,2}X^{K-2} - ... + (-1)^K \Sigma_{k,K}$. Finally, let $\gamma_{k,1}$, ..., $\gamma_{k,K}$ denote the roots of $\tilde{O}_k(X)$ counted with their multiplicity, and let $A_k \equiv \{\gamma_{k,1},...,\gamma_{k,K}\}$. Since all the coefficients of $\tilde{O}_k(X)$ are real numbers, we have $A_k = \{\overline{a} | a \in A_k\}$, where for any $a \in \mathbb{C}$, $\overline{a}$ denotes the complex conjugate of $a$. We also have, by construction, $S_k^{A_k} = s_k$ and $S_j^{A_k} = 0$ for $j \in \{1,...,K\}\setminus\{k\}$.

Let $r_k \in \mathbb{N}$ be such that $r_k > \max\{|\gamma_{k,j}| | 1 \leq j \leq K\}$. The set $B_k \equiv \{\gamma_{k,1}/r_k,...,\gamma_{k,K}/r_k\}$ is such that: (i) each of its elements is a complex number whose modulus is strictly lower than one;

(ii) $B_k = \{\bar{b} | b \in B_k\}$; and (iii) $S_k^{B_k} = s_k / r_k^k$ and $S_j^{B_k} = 0$ for $j \in \{1, ..., K\} \setminus \{k\}$. Therefore, the set $C_k$, defined as the union of $r_k^k$ times set $B_k$, is such that: (i) each of its elements is a complex number whose modulus is strictly lower than one; (ii) $C_k = \{\bar{c} | c \in C_k\}$; and (iii) $S_k^{C_k} = s_k$ and $S_j^{C_k} = 0$ for $j \in \{1, ..., K\} \setminus \{k\}$.

Finally, the set $C \equiv \bigsqcup_{1 \leq k \leq K} C_k$, whose cardinality is noted $m$ and whose elements are noted $\alpha_j$ for $j \in \{1, ..., m\}$, is such that: (i) $m \in \mathbb{N} \setminus \{0\}$ and $m = K \sum_{k=1}^{K} r_k^k > K$; (ii) $\forall j \in \{1, ..., m\}$, $\alpha_j \in \mathbb{C}$ and $|\alpha_j| < 1$; (iii) $\forall k \in \{1, ..., K\}$, $S_k^C = \sum_{j=1}^{m} \alpha_j^k = s_k$; and (iv) $C = \{\overline{\alpha} | \alpha \in C\}$, so that all the coefficients of the polynomial $\prod_{j=1}^{m} (X - \alpha_j)$ are real numbers. This result proves Conjecture 2 and, therefore, Conjecture 1 too. Lemma 2 follows.

### A.3 Application to the Basic NK Model

In the basic NK model without exogenous disturbances, at each date $t \in \mathbb{Z}$, the private sector sets output $y_t$ and inflation $\pi_t$ according to the following (locally log-linearized) IS equation and Phillips curve:

$$y_t = \mathbb{E}_t\{y_{t+1}\} - (1/\sigma)\left(i_t - \mathbb{E}_t\{\pi_{t+1}\}\right), \tag{A.6}$$

$$\pi_t = \beta \mathbb{E}_t\{\pi_{t+1}\} + \kappa y_t, \tag{A.7}$$

where $\sigma > 0$, $\beta \in (0, 1)$, and $\kappa > 0$ are three parameters.[1] The policymaker is a central bank setting the short-term nominal interest rate $i_t$. The system (A.6)-(A.7) can straightforwardly be written as (22) with $n = 2$, $\gamma = -1$,

$$\mathbf{Z}_t \equiv \begin{bmatrix} y_t \\ \pi_t \end{bmatrix}, \ \boldsymbol{\Delta}(X) \equiv \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}, \ \mathbf{A}(X) \equiv \begin{bmatrix} \sigma(1 - X) & 1 \\ \kappa X & \beta - X \end{bmatrix}, \text{ and } \mathbf{B}(X) \equiv \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

This system satisfies Assumption 1 ($\det \mathbf{A}(0) = \beta\sigma \neq 0$), Assumption 2 ($\mathbf{B}(0) = [\,-1 \quad 0]^T \neq \mathbf{0}$), Assumption 3 ($\Psi_1(X) = \beta - X \neq 0$ and $\Psi_2(X) = \kappa X \neq 0$), and Assumption 4 ($\det[\hat{\mathbf{A}}_0] = \sigma \neq 0$). It also satisfies Assumption 5, because the only way to rewrite it in an equivalent form of type (22) with a block-triangular $\mathbf{A}(X)$ would be to use the second structural equation (Phillips curve) to replace $y_t$ and $\mathbb{E}_t\{y_{t+1}\}$ in first structural equation (IS equation); the resulting system, however, would not satisfy Assumption 1.

Since $\Psi_3(X) = [X^2 - (1 + \beta + \kappa/\sigma)X + \beta]\sigma$, the system satisfies the condition stated in Proposition 1 for any non-empty set $J$: $D_{\bar{J}}(X)$ has no root in $\mathbb{D} \setminus \{0\}$, and hence fewer than $\delta = 2$ roots in $\mathbb{D} \setminus \{0\}$, for any $J \in \{\{1\}, \{2\}, \{1, 2\}\}$. So, in this model, the central bank can ensure determinacy by making the interest rate react to either output, or inflation, or both, in the presence of inside lags of any length.

Finally, the system satisfies the condition stated in Proposition 6 for the only non-empty and non-singleton set $J$, namely $J = \{1, 2\}$, since $D_{\{1,2\}}(X)$ has no root in $\mathbb{D} \setminus \{0\}$. So, for inside

[1] I use the same notations for the parameters as in Galí (2015). Woodford (2003) uses the same notations for $\beta$ and $\kappa$, but replaces $1/\sigma$ by $\sigma$.

lags of any length $\ell \in \mathbb{N}$, Proposition 6 implies the existence of a non-superinertial interest-rate rule ensuring determinacy and consistent with $I_t = \{\pi^{t-\ell}, y^{t-\ell}, i^{t-1}\}$.

### A.4 Application to the Model of Rotemberg and Woodford (1997, 1999)

In the model of Rotemberg and Woodford (1997, 1999) without exogenous disturbances, at each date $t \in \mathbb{Z}$, the private sector sets output $\hat{Y}_t$, the marginal utility of income $\hat{\lambda}_t$, inflation $\hat{\pi}_t$, and a relative-price index $\hat{X}_t$ according to the following (locally log-linearized) structural equations:

$$\hat{\lambda}_t = \mathbb{E}_t\{\hat{\lambda}_{t+1}\} + \hat{R}_t - \mathbb{E}_t\{\hat{\pi}_{t+1}\}, \tag{A.8}$$

$$\hat{Y}_t = -(1/\sigma)\mathbb{E}_{t-2}\{\hat{\lambda}_t\}, \tag{A.9}$$

$$\hat{\pi}_t = [1/(1+\psi)]\hat{X}_t + [\psi/(1+\psi)]\mathbb{E}_{t-2}\{\hat{\pi}_t\}, \tag{A.10}$$

$$\hat{X}_t = \beta\mathbb{E}_{t-1}\{\hat{X}_{t+1}\} + \kappa\hat{Y}_t - [\kappa/(\sigma+\omega)]\mathbb{E}_{t-1}\{\sigma\hat{Y}_t - \sigma\hat{Y}_{t+1} + \hat{R}_t - \hat{\pi}_{t+1}\}, \tag{A.11}$$

where $\sigma > 0$, $\psi > 0$, $\beta \in (0,1)$, $\kappa > 0$, and $\omega > 0$.[2] The policymaker is, again, a central bank setting the short-term nominal interest rate $\hat{R}_t$.

To rewrite the structural equations (A.8)-(A.11) in a form that does not involve any past expectation, I introduce the variables

$$\tilde{\tilde{Y}}_t \equiv \mathbb{E}_t\{\hat{Y}_{t+2}\}, \tag{A.12}$$

$$\tilde{\pi}_t \equiv \mathbb{E}_t\{\hat{\pi}_{t+1}\}, \tag{A.13}$$

$$\tilde{\tilde{\pi}}_t \equiv \mathbb{E}_t\{\hat{\pi}_{t+2}\}, \tag{A.14}$$

and I rewrite the system (A.8)-(A.14) as the following system:

$$\sigma\tilde{\tilde{Y}}_t = \sigma\mathbb{E}_t\{\tilde{\tilde{Y}}_{t+1}\} - \mathbb{E}_t\{\hat{R}_{t+2}\} + \mathbb{E}_t\{\tilde{\tilde{\pi}}_{t+1}\}, \tag{A.15}$$

$$\sigma\tilde{\tilde{Y}}_t = -\hat{\lambda}_t + \hat{R}_t - \tilde{\pi}_t + \mathbb{E}_t\{\hat{R}_{t+1}\} - \tilde{\tilde{\pi}}_t, \tag{A.16}$$

$$\tilde{\pi}_{t-1} = [1/(1+\psi)]\hat{X}_t + [\psi/(1+\psi)]\tilde{\tilde{\pi}}_{t-2}, \tag{A.17}$$

$$(1+\psi)\tilde{\pi}_t - \psi\tilde{\tilde{\pi}}_{t-1} = \beta\tilde{\tilde{\pi}}_t + [\kappa/(\sigma+\omega)]\left[\omega\tilde{\tilde{Y}}_{t-1} + \sigma\tilde{\tilde{Y}}_t - \mathbb{E}_t\{\hat{R}_{t+1}\} + \tilde{\tilde{\pi}}_t\right], \tag{A.18}$$

$$\tilde{\tilde{\pi}}_t = \mathbb{E}_t\{\tilde{\pi}_{t+1}\}, \tag{A.19}$$

$$\hat{Y}_t = \tilde{\tilde{Y}}_{t-2}, \tag{A.20}$$

$$\hat{\pi}_t = \tilde{\pi}_{t-1}. \tag{A.21}$$

The two systems, (A.8)-(A.14) and (A.15)-(A.21), are equivalent to each other. The former system implies the latter because (A.13)-(A.14) imply (A.19); (A.9) and (A.12) imply (A.20); (A.9)-(A.11) and (A.13) imply (A.21); and, in turn, (A.8)-(A.11) and (A.19)-(A.21) imply (A.15)-(A.18). Conversely, the latter system implies the former because (A.19)-(A.21) imply (A.12)-(A.14) and, in turn, (A.12)-(A.21) imply (A.8)-(A.11).

[2] I use the same notations for the variables and the parameters as Rotemberg and Woodford (1997). Rotemberg and Woodford (1999) use the same notations, except that they replace $\hat{\pi}_t$ by $\pi_t$, and $\psi$ by $(1-\psi)/\psi$. All variables are expressed in percentage deviation from their steady-state value.

Next, using (A.15) and (A.18), I rewrite (A.19) as

$$\tilde{\tilde{\pi}}_t = \beta \mathbb{E}_t\{\tilde{\tilde{\pi}}_{t+1}\} + \kappa \tilde{\tilde{Y}}_t. \tag{A.22}$$

The system consisting of (A.15)-(A.18) and (A.20)-(A.22) is block-recursive. The first block, consisting of (A.16)-(A.18) and (A.20)-(A.21), gives $(\hat{Y}_t, \hat{\lambda}_t, \hat{\pi}_t, \tilde{\pi}_t, \hat{X}_t)$ as a function of $(\tilde{\tilde{Y}}^t, \tilde{\tilde{\pi}}^t, \hat{R}_t, \mathbb{E}_t\{\hat{R}_{t+1}\}, \mathbb{E}_{t-1}\{\hat{R}_t\})$:

$$\begin{bmatrix} \hat{Y}_t & \hat{\lambda}_t & \hat{\pi}_t & \tilde{\pi}_t & \hat{X}_t \end{bmatrix}^T = \mathbf{M}(L) \begin{bmatrix} \tilde{\tilde{Y}}_t & \tilde{\tilde{\pi}}_t & \hat{R}_t & \mathbb{E}_t\{\hat{R}_{t+1}\} \end{bmatrix}^T,$$

where

$$\mathbf{M}(X) \equiv \frac{1}{\eta} \begin{bmatrix} \eta X^2 & 0 & 0 & 0 \\ -\sigma\eta - \kappa(\sigma + \omega X) & -\kappa - \eta - \beta(\sigma+\omega) - \psi(\sigma+\omega)X & \eta & \kappa + \eta \\ \kappa(\sigma+\omega X)X & [\kappa + (\sigma+\omega)(\beta + \psi X)]X & 0 & -\kappa X \\ \kappa(\sigma + \omega X) & \kappa + (\sigma+\omega)(\beta+\psi X) & 0 & -\kappa \\ \kappa(1+\psi)(\sigma+\omega X)X & (1+\psi)[\kappa + \beta(\sigma+\omega)]X & 0 & -\kappa(1+\psi)X \end{bmatrix}$$

with $\eta \equiv (1+\psi)(\sigma+\omega)$. The second block, consisting of (A.15) and (A.22), involves only (the present and expected future values of) the variables $\tilde{\tilde{Y}}_t$, $\tilde{\tilde{\pi}}_t$, and $\hat{R}_t$. I can rewrite this second block in the following form of type (22) with $n = 2$:

$$\mathbb{E}_t \left\{ \boldsymbol{\Delta}\left(L^{-1}\right) \left[ \mathbf{A}(L)\, \mathbf{Z}_t + L^{-\gamma} \mathbf{B}(L)\, \hat{R}_t \right] \right\} = \mathbf{0} \tag{A.23}$$

with $\gamma = 1$,

$$\mathbf{Z}_t \equiv \begin{bmatrix} \tilde{\tilde{Y}}_t \\ \tilde{\tilde{\pi}}_t \end{bmatrix}, \; \boldsymbol{\Delta}(X) \equiv \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}, \; \mathbf{A}(X) \equiv \begin{bmatrix} \sigma(1-X) & 1 \\ \kappa X & \beta - X \end{bmatrix}, \text{ and } \mathbf{B}(X) \equiv \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

The system (A.23) is, in fact, the same as the system of the basic NK model (discussed above), except that $y_t$, $\pi_t$, and $i_t$ have been replaced respectively by $\tilde{\tilde{Y}}_t$, $\tilde{\tilde{\pi}}_t$, and $\mathbb{E}_t\{\hat{R}_{t+2}\}$. Therefore, (A.23) satisfies Assumptions 1 to 5. The novelty is about the information set: the variable $\tilde{\tilde{\pi}}_t$ is never observed (as it is an expectation but not an action: $\tilde{\tilde{\pi}}_t \equiv \mathbb{E}_t\{\hat{\pi}_{t+2}\}$ may differ from $\hat{\pi}_{t+2}$), while the variable $\tilde{\tilde{Y}}_t$ is observed two periods later than $\hat{Y}_t$ (as it is not only an expectation, but also an action: $\tilde{\tilde{Y}}_t \equiv \mathbb{E}_t\{\hat{Y}_{t+2}\} = \hat{Y}_{t+2}$). So, the only possible set $J$ is $\{1\}$. For this set, the condition stated in Proposition 1 is satisfied, but not the condition stated in Proposition 6, as $J$ is a singleton. In this model with distributed outside lags, thus, Proposition 1 can be used to get an interest-rate rule ensuring determinacy, for inside lags of any length, but Proposition 6 cannot be used to get a non-superinertial interest-rate rule ensuring determinacy.

### A.5 Application to the Model of Smets and Wouters (2007)

In the model of Smets and Wouters (2007) without exogenous disturbances, at each date $t \in \mathbb{Z}$, the private sector sets output $y_t$, consumption $c_t$, investment $i_t$, the real value of capital $q_t$, the stock of installed capital $k_t$, the stock of utilized capital $k_t^s$, the capital-utilization rate $z_t$, the real rental rate of capital $r_t^k$, hours worked $l_t$, the price markup $\mu_t^p$, the inflation rate $\pi_t$, the wage markup $\mu_t^w$, and the real wage $w_t$ according to the following (locally log-linearized) structural equations:

$$
\begin{aligned}
y_t &= c_y c_t + i_y i_t + z_y z_t, && \text{(A.24)}\\
c_t &= c_1 c_{t-1} + (1-c_1)\,\mathbb{E}_t\{c_{t+1}\} + c_2\,(l_t - \mathbb{E}_t\{l_{t+1}\}) - c_3\,(r_t - \mathbb{E}_t\{\pi_{t+1}\})\,, && \text{(A.25)}\\
i_t &= i_1 i_{t-1} + (1-i_1)\,\mathbb{E}_t\{i_{t+1}\} + i_2 q_t, && \text{(A.26)}\\
q_t &= q_1 \mathbb{E}_t\{q_{t+1}\} + (1-q_1)\,\mathbb{E}_t\{r_{t+1}^k\} - (r_t - \mathbb{E}_t\{\pi_{t+1}\})\,, && \text{(A.27)}\\
y_t &= \phi_p\,[\alpha k_t^s + (1-\alpha)\,l_t]\,, && \text{(A.28)}\\
k_t^s &= k_{t-1} + z_t, && \text{(A.29)}\\
z_t &= z_1 r_t^k, && \text{(A.30)}\\
k_t &= k_1 k_{t-1} + (1-k_1)\,i_t, && \text{(A.31)}\\
\mu_t^p &= \alpha\,(k_t^s - l_t) - w_t, && \text{(A.32)}\\
\pi_t &= \pi_1 \pi_{t-1} + \pi_2 \mathbb{E}_t\{\pi_{t+1}\} - \pi_3 \mu_t^p, && \text{(A.33)}\\
r_t^k &= -\,(k_t^s - l_t) + w_t, && \text{(A.34)}\\
\mu_t^w &= w_t - \sigma_l l_t - (1-\lambda/\gamma)^{-1}\,[c_t - (\lambda/\gamma)\,c_{t-1}]\,, && \text{(A.35)}\\
w_t &= w_1 w_{t-1} + (1-w_1)\,\mathbb{E}_t\{w_{t+1} + \pi_{t+1}\} - w_2 \pi_t + w_3 \pi_{t-1} - w_4 \mu_t^w, && \text{(A.36)}
\end{aligned}
$$

where $c_y$, $i_y$, $z_y$, $c_1$, $c_2$, $c_3$, $i_1$, $i_2$, $q_1$, $\phi_p$, $\alpha$, $z_1$, $k_1$, $\pi_1$, $\pi_2$, $\pi_3$, $\sigma_l$, $\lambda$, $\gamma$, $w_1$, $w_2$, $w_3$, and $w_4$ are reduced-form parameters.[3] Because these reduced-form parameters are functions of a smaller number of structural parameters, they satisfy the following six equality constraints:

$$
i_1 = w_1,\; c_1 = \frac{\lambda}{\gamma+\lambda},\; w_2 = w_1 + \left(\frac{1-w_1}{w_1}\right) w_3,\; q_1 = \left(\frac{1-w_1}{w_1}\right) k_1,
$$

$$
\pi_2 = \left(\frac{1-w_1}{w_1}\right)\left[1 - \left(\frac{1-w_1}{w_1}\right)\pi_1\right],\text{ and } z_y = \left(\frac{i_y}{1-k_1}\right)\left(\frac{w_1}{1-w_1} - k_1\right).
$$

The policymaker is, again, a central bank setting the short-term nominal interest rate $r_t$.

I rewrite the system of structural equations (A.24)-(A.36) in a block-recursive way. More specifically, using (A.31), I rewrite (A.24), (A.28), (A.29), (A.30), (A.32), (A.34), and (A.35) as

$$
\begin{bmatrix} y_t & k_t & k_t^s & z_t & r_t^k & \mu_t^p & \mu_t^w \end{bmatrix}^T = \mathbf{M}(L)\,[c_t \quad i_t \quad w_t \quad l_t]^T\,, \tag{A.37}
$$

where

$$
\mathbf{M}(X) \equiv \frac{1}{\eta}\begin{bmatrix}
\alpha\phi_p c_y & \alpha\phi_p i_y & \alpha\phi_p z_1 z_y & [1-(1+\phi_p-\eta)\alpha]\phi_p \\
(1+z_1)k_1 c_y & (1+z_1)k_1 i_y + (1-k_1)\eta & (z_y-\alpha\phi_p)k_1 z_1 & k_1\eta - (1+z_1)k_1\phi_p \\
c_y & i_y & z_1 z_y & \eta-\phi_p \\
-z_1 c_y & -z_1 i_y & \alpha z_1 \phi_p & z_1\phi_p \\
-c_y & -i_y & \alpha\phi_p & \phi_p \\
\alpha c_y & \alpha i_y & -(1-\alpha z_1 z_y) & -(1+\phi_p-\eta)\alpha \\
\frac{-[1-(\lambda/\gamma)X]}{1-\lambda/\gamma} & 0 & 1 & -\sigma_l
\end{bmatrix}
$$

[3]I display the structural equations in their order of appearance in Smets and Wouters (2007), and I use the same notations as them for the endogenous variables and the reduced-form parameters. The only difference is that I have replaced $k_t$ by $k_t^s$ in one equation (Equation (11) in their paper, Equation (A.34) here), thus correcting a typo in their paper.

with $\eta \equiv \alpha\phi_p + z_1 z_y$. In turn, using (A.37), I rewrite (A.25), (A.26), (A.27), (A.31), (A.33), and (A.36) in a form of type (22) with $n = 6$:

$$\mathbb{E}_t \left\{ \mathbf{\Delta} \left( L^{-1} \right) \left[ \mathbf{A} \left( L \right) \mathbf{Z}_t + L^{-\gamma} \mathbf{B} \left( L \right) r_t \right] \right\} = \mathbf{0} \tag{A.38}$$

with $\mathbf{Z}_t \equiv [\, c_t \quad i_t \quad q_t \quad \pi_t \quad w_t \quad l_t \,]^T$, $\gamma = -1$, $\mathbf{B}(X) \equiv [\, -c_3 \quad 0 \quad -1 \quad 0 \quad 0 \quad 0 \,]^T$, $\mathbf{A}(X) \equiv [\, \mathbf{A}_1(X) \quad \mathbf{A}_2(X) \quad \mathbf{A}_3(X) \,]$,

$$\mathbf{A}_1(X) \equiv \begin{bmatrix} (1-c_1) - X + c_1 X^2 & 0 & 0 \\ 0 & (1-i_1) - X + i_1 X^2 & i_2 X \\ -(1-q_1) c_y / \eta & -(1-q_1) i_y / \eta & q_1 - X \\ (1+z_1) k_1 c_y (1 - k_1 X) & (1+z_1) k_1 i_y - [(1+z_1) k_1 i_y + (1-k_1)\eta] k_1 X & 0 \\ -\alpha \pi_3 c_y X & -\alpha \pi_3 i_y X & 0 \\ \frac{w_4}{1-\lambda/\gamma} [X - (\lambda/\gamma) X^2] & 0 & 0 \end{bmatrix},$$

$$\mathbf{A}_2(X) \equiv \begin{bmatrix} c_3 & 0 \\ 0 & 0 \\ 1 & (1-q_1)\alpha\phi_p/\eta \\ 0 & (z_y - \alpha\phi_p) k_1 z_1 (1 - k_1 X) \\ \eta(\pi_2 - X + \pi_1 X^2) & \pi_3 (\eta - \alpha z_1 z_y) X \\ (1+w_1) - w_2 X + w_3 X^2 & (1-w_1) - (1+w_4) X + w_1 X^2 \end{bmatrix},$$

$$\mathbf{A}_3(X) \equiv \begin{bmatrix} -c_2 + c_2 X \\ 0 \\ (1-q_1)\phi_p/\eta \\ -[(1+z_1)\phi_p - \eta] k_1 (1 - k_1 X) \\ \alpha \pi_3 \phi_p X \\ w_4 \sigma_l X \end{bmatrix}, \text{ and } \mathbf{\Delta}(X) \equiv \begin{bmatrix} X & 0 & 0 & 0 & 0 & 0 \\ 0 & X & 0 & 0 & 0 & 0 \\ 0 & 0 & X & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & X & 0 \\ 0 & 0 & 0 & 0 & 0 & X \end{bmatrix}.$$

It is straightforward to check that the system (A.38) satisfies Assumption 2. To show that this system generically satisfies Assumptions 1 and 4, I determine $\det[\mathbf{A}(0)]$ and $\det[\hat{\mathbf{A}}_0]$ as functions of the reduced-form parameters:

$$\begin{aligned} \det\left[\mathbf{A}(0)\right] &= \left[(1-c_1)\eta + (1+z_1)c_2 c_y - (1-c_1)(1+z_1)\phi_p\right](1-i_1)(1-w_1)\eta\pi_2 k_1 q_1, \\ \det[\hat{\mathbf{A}}_0] &= \left[\left(\eta - \alpha z_1 z_y\right)\pi_3 w_2 + \left(1+w_4\right)\eta\right]\left[\left(1+z_1\right)\phi_p - \left(1+z_1\right)c_2 c_y - \eta\right] k_1 \\ &\quad + \left\{\alpha\pi_3 w_2\left(\phi_p - c_2 c_y\right) - \left[\sigma_\ell + c_2/\left(1-\lambda/\gamma\right)\right] w_4 \eta\right\}\left(z_y - \alpha\phi_p\right) k_1 z_1, \end{aligned}$$

and I check that, despite the six equality constraints on the reduced-form parameters, these two expressions are non-zero except possibly for a zero-measure set of structural-parameter values. To show that the system (A.38) generically satisfies Assumption 3, I proceed as follows. Let $\Psi_{j,k}$, for any $j \in \{1, ..., n\}$ and any $k \in \{0, ..., d(\Psi_j)\}$, denote the coefficient of $X^k$ in $\Psi_j(X)$. I get $\Psi_{2,0} = \Psi_{4,0} = \Psi_{5,0} = 0$, but

$$\begin{aligned}
\Psi_{1,0} &= -[(1+z_1)\phi_p - \eta](1-i_1)(1-w_1)\eta k_1 q_1 \pi_2 c_3,\\
\Psi_{2,1} &= [(1-c_1)\eta + (1+z_1)c_2 c_y - (1-c_1)(1+z_1)\phi_p + (1-q_1)c_3 c_y](1-w_1)\eta k_1 i_2 \pi_2,\\
\Psi_{3,0} &= [(1-c_1)\eta + (1+z_1)c_2 c_y - (1-c_1)(1+z_1)\phi_p + (1-q_1)c_3 c_y](1-i_1)(1-w_1)\eta k_1 \pi_2,\\
\Psi_{4,1} &= -(1-i_1)(1-w_1)\alpha\eta k_1 q_1 c_3 \pi_3 c_y,\\
\Psi_{5,1} &= \{(1+z_1)\pi_2 w_4 c_y \sigma_l - (1+w_1)\alpha\pi_3 c_y + [(1+z_1)\phi_p - \eta]\pi_2 w_4/(1-\lambda/\gamma)\}(1-i_1)\eta k_1 q_1 c_3,\\
\Psi_{6,0} &= (1-i_1)(1-w_1)(1+z_1)\eta k_1 q_1 \pi_2 c_3 c_y.
\end{aligned}$$

It is easy to check that these coefficients are non-zero except possibly for a zero-measure set of structural-parameter values. So, $\Psi_j(X)$ is generically non-zero for all $j \in \{1, ..., n\}$. Finally, the system (A.38) satisfies Assumption 5 because $\mathbf{A}(X)$ is not block-triangular and cannot be made block-triangular. Indeed, as all the equations in (A.38) are dynamic (four of them are both forward- and backward-looking, one is purely forward-looking, and one purely backward-looking), almost all the ways to rewrite (A.38) in an equivalent, block-recursive way lead to systems that involve past expectations and therefore are not of type (22). The only way to rewrite (A.38) in an equivalent form of type (22) with a block-triangular $\mathbf{A}(X)$ would be to use the second line of (A.38) to replace $q_t$ and $\mathbb{E}_t\{q_{t+1}\}$ in its third line; the resulting system, however, would not satisfy Assumption 1.

I use the symbolic-computation software Mathematica to investigate the conditions stated in Propositions 1 and 6.[4] I find that $D_{\{j,n+1\}}(X)$ is of degree zero for any $j \in \{1, ..., n\}$, except possibly for a zero-measure set of structural-parameter values. So, the condition stated in Proposition 1 is generically satisfied for any non-empty set $J \subset \{1, ..., n\}$. In this model, thus, the central bank can ensure determinacy by making the interest rate react to any non-empty set of variables, and in particular to any single variable, in the presence of inside lags of any length.

For all non-empty and non-singleton sets $J$ except one, I find that $D_J(X)$ has generically no non-zero root, and therefore that the condition stated in Proposition 6 is generically satisfied. So, the central bank can always ensure determinacy with a non-superinertial interest-rate rule, in the presence of inside lags of any length, provided that she observes at least two endogenous variables. The only exception is for $J = \{2, 3\}$, which corresponds to the hypothetical case in which the central bank would observe only investment and the real value of capital ($I_t = \{i^{t-\ell}, q^{t-\ell}, r^{t-1}\}$). Indeed, I find that $D_{\{2,3\}}(X)$ is of degree 7, with $\Psi_2(X) = i_2 X D_{\{2,3\}}(X)$ and $\Psi_3(X) = [(1-i_1) - X + i_1 X^2] D_{\{2,3\}}(X)$. The reason for this exception is that the investment Euler equation (A.26) makes the sequence of past, current, and future investment levels ($i_{t-1}$, $i_t$, $i_{t+1}$) and the current real value of capital ($q_t$) substitutable with each other for determinacy purposes. So, reacting to both variables, rather than only one, does not generate any degree of freedom that could be exploited to find a non-superinertial rule.

[4] The code is available on my website.

To estimate their model, Smets and Wouters (2007) use data for $c_t$, $i_t$, $\pi_t$, $w_t$, $l_t$, but not $q_t$. So, some natural sets $J$ to consider are the subsets of $\{1, 2, 4, 5, 6\}$. What I have just shown is that, in the presence of inside lags of any length, determinacy can be ensured for any non-empty subset of $\{1, 2, 4, 5, 6\}$, and it can be ensured with a non-superinertial rule for any non-empty and non-singleton subset of $\{1, 2, 4, 5, 6\}$.

### A.6 Application to the Model of Schmitt-Grohé and Uribe (1997)

In the model of Schmitt-Grohé and Uribe (1997) without exogenous disturbances, at each date $t \in \mathbb{Z}$, the private sector sets output $y_t$, the capital stock $k_t$, hours worked $h_t$, consumption $c_t$, the (after-tax) rental rate of capital $u_t$, the (after-tax) wage $w_t$, and the stock of public debt $b_t$, according to the following structural equations, log-linearized in the neighborhood of the zero-debt steady state:

$$y_t = s_k k_{t-1} + (1 - s_k) h_t, \tag{A.39}$$

$$y_t = s_c c_t + s_i \delta^{-1} k_t - (1 - \delta) s_i \delta^{-1} k_{t-1}, \tag{A.40}$$

$$w_t = \sigma c_t + \gamma h_t, \tag{A.41}$$

$$c_t = \mathbb{E}_t\{c_{t+1}\} - [1 - \beta(1 - \delta)]\sigma^{-1}\mathbb{E}_t\{u_{t+1}\}, \tag{A.42}$$

$$u_t = -(1 - s_k)(k_{t-1} - h_t) - \omega\tau(1 - \tau)^{-1}\tau_t, \tag{A.43}$$

$$w_t = s_k(k_{t-1} - h_t) - \tau(1 - \tau)^{-1}\tau_t, \tag{A.44}$$

$$b_t = \beta^{-1} b_{t-1} - (1 - s_k + \omega s_k)\tau(y_t + \tau_t), \tag{A.45}$$

where $s_k \in (0, 1)$, $s_c \in (0, 1)$, $s_i \in (0, 1)$, $\delta \in (0, 1)$, $\sigma > 0$, $\gamma > 0$, $\beta \in (0, 1)$, $\tau \in (0, 1)$, and $\omega \in \{0, 1\}$.[5] The policymaker is a tax authority whose policy instrument $\tau_t$ is the labor-income-tax rate (when $\omega = 0$) or the income-tax rate (when $\omega = 1$).

As I explain in the main text, in order to satisfy Assumption 5, I restrict the class of information sets for the tax authority to the sets that include the debt level (possibly with inside lags), and I write the policy instrument as

$$\tau_t = \alpha b_{t-\ell} + \tilde{\tau}_t, \tag{A.46}$$

where $\ell \in \mathbb{N}$ and $\alpha \in \mathbb{R} \setminus \{0\}$. The coefficient $\alpha$ is arbitrary: I treat it as if it were a structural parameter, and I treat $\tilde{\tau}_t$ as the new policy instrument. Then, I rewrite the system (A.39)-(A.46) in a block-recursive way. More specifically, I rewrite (A.39)-(A.41), (A.43)-(A.44), and (A.46) as

$$[y_t \quad h_t \quad c_t \quad u_t \quad w_t \quad \tau_t]^T = \mathbf{M}(L)\,[k_t \quad b_t \quad \tilde{\tau}_t]^T\,, \tag{A.47}$$

[5] Most of Schmitt-Grohé and Uribe's (1997) analysis is conducted in continuous time; I refer here to the discrete-time analysis conducted in Section 4 and in the Appendix of their paper. I use the same notations as them for the variables and the parameters, with three exceptions: (i) I have replaced $k_{t+1}$ and $k_t$ by $k_t$ and $k_{t-1}$ respectively, as these variables are set at dates $t$ and $t - 1$ respectively; (ii) I have introduced the parameter $\sigma$ to allow for degrees of relative risk aversion different from one; and (iii) I have introduced the parameter $\omega$ to encompass the two alternative tax-policy instruments. All variables are expressed in percentage deviation from their steady-state value $-$ except public debt $b_t$, which is expressed as a fraction of steady-state output (since steady-state public debt is zero).

where $\mathbf{M}(X) \equiv [\,\mathbf{M}_1(X) \quad \mathbf{M}_2(X) \quad \mathbf{M}_3\,]$,

$$\mathbf{M}_1(X) \equiv \frac{1}{\delta\eta}\left[\begin{array}{c} (1-s_k)\,\sigma s_i + [(1+\gamma)\,\delta s_c s_k - (1-\delta)\,(1-s_k)\,\sigma s_i]\,X \\ \sigma s_i - [(\sigma - s_c)\,\delta s_k + (1-\delta)\,\sigma s_i]\,X \\ \varphi\sigma s_i/s_c + [(1+\gamma)\,\delta\sigma s_k - (1-\delta)\,\varphi\sigma s_i/s_c]\,X \\ (1-s_k)\,\sigma s_i - [(\gamma s_c + \sigma)\,\delta + (1-\delta)\,\sigma s_i]\,(1-s_k)\,X \\ (\gamma s_c + \varphi)\,\sigma s_i/s_c + [(\gamma s_c + \sigma)\,\delta s_k - (\gamma s_c + \varphi)\,(1-\delta)\,\sigma s_i/s_c]\,X \\ 0 \end{array}\right],$$

$$\mathbf{M}_2(X) \equiv \frac{-\alpha\tau X^{\ell}}{(1-\tau)\,\eta}\left[\begin{array}{c} (1-s_k)\,s_c \\ s_c \\ (1-s_k)\,\sigma \\ (1-s_k)\,s_c + \eta\omega \\ \gamma s_c + (1-s_k)\,\sigma \\ -(1-\tau)\,\eta/\tau \end{array}\right], \text{ and } \mathbf{M}_3 \equiv \frac{-\tau}{(1-\tau)\,\eta}\left[\begin{array}{c} (1-s_k)\,s_c \\ s_c \\ (1-s_k)\,\sigma \\ (1-s_k)\,s_c + \eta\omega \\ \gamma s_c + (1-s_k)\,\sigma \\ -(1-\tau)\,\eta/\tau \end{array}\right],$$

with $\eta \equiv (1+\gamma)\,s_c + (1-s_k)\,(\sigma - s_c)$ and $\varphi \equiv (1-s_k)\,\sigma - s_c$. In turn, using (A.47), I rewrite (A.42) and (A.45) in a form of type (22) with $n = 2$:

$$\mathbb{E}_t\left\{\mathbf{\Delta}\left(L^{-1}\right)\left[\mathbf{A}\,(L)\,\mathbf{Z}_t + L^{-\gamma}\mathbf{B}\,(L)\,\tilde{\tau}_t\right]\right\} = \mathbf{0} \qquad \text{(A.48)}$$

with $\gamma = 0$,

$$\mathbf{Z}_t \equiv \left[\begin{array}{c} k_t \\ b_t \end{array}\right],\ \mathbf{\Delta}(X) \equiv \left[\begin{array}{cc} X & 0 \\ 0 & 1 \end{array}\right],\ \mathbf{A}(X) \equiv \left[\begin{array}{cc} A_{11}(X) & A_{12}(X) \\ A_{21}(X) & A_{22}(X) \end{array}\right],\ \mathbf{B}(X) \equiv \left[\begin{array}{c} B_1(X) \\ B_2(X) \end{array}\right], \text{ and}$$

$$\begin{aligned}
A_{11}(X) &\equiv [\varphi\sigma - (1-s_k)\,\chi s_c]\,s_i/(\delta\eta s_c) + \left\{(1+\gamma)\,\delta\sigma^2 s_c s_k - (2-\delta)\,\varphi\sigma^2 s_i + [(\gamma s_c + \sigma)\,\delta\right. \\
&\quad \left. + (1-\delta)\,\sigma s_i]\,(1-s_k)\,\chi s_c\right\} X/(\delta\eta\sigma s_c) + [(1-\delta)\,\varphi s_i - (1+\gamma)\,\delta s_c s_k]\,\sigma X^2/(\delta\eta s_c), \\
A_{12}(X) &\equiv \left\{[(1-s_k)\,s_c + \eta\omega]\,\chi - (1-s_k)\,\sigma^2 + (1-s_k)\,\sigma^2 X\right\}\alpha\tau X^{\ell}/\left[(1-\tau)\,\eta\sigma\right], \\
A_{21}(X) &\equiv \left\{(1-s_k)\,\sigma s_i + [(1+\gamma)\,\delta s_c s_k - (1-\delta)\,(1-s_k)\,\sigma s_i]\,X\right\}(1-s_k+\omega s_k)\,\tau/\left(\delta\eta\right), \\
A_{22}(X) &\equiv 1 - \beta^{-1}X + [(1-\tau)\,\eta - (1-s_k)\,\tau s_c]\,(1-s_k+\omega s_k)\,\alpha\tau X^{\ell}/\left[(1-\tau)\,\eta\right], \\
B_1(X) &\equiv \left\{[(1-s_k)\,s_c + \eta\omega]\,\chi - (1-s_k)\,\sigma^2 + (1-s_k)\,\sigma^2 X\right\}\tau/\left[(1-\tau)\,\eta\sigma\right], \\
B_2(X) &\equiv [(1-\tau)\,\eta - (1-s_k)\,\tau s_c]\,(1-s_k+\omega s_k)\,\tau/\left[(1-\tau)\,\eta\right],
\end{aligned}$$

where $\chi \equiv 1 - \beta\,(1-\delta)$. It is straightforward to check that the system (A.48) satisfies Assumption 2. This system also satisfies generically Assumptions 1 and 4, because det$[\mathbf{A}(0)]$ and det$[\hat{\mathbf{A}}_0]$ are non-zero except possibly for a zero-measure set of parameter values:

$$\begin{aligned}
\det[\mathbf{A}(0)] &= [\varphi\sigma - (1-s_k)\,\chi s_c]\,s_i/\left(\delta\eta s_c\right) + \left\{[(1-s_k)\,\sigma - s_c]\,(1-\tau)\,\eta\sigma + (1-s_k)\,\sigma\tau s_c^2\right. \\
&\quad \left. - [(1-\tau)\,s_c + \omega\tau s_c]\,(1-s_k)\,\chi\eta\right\}(1-s_k+\omega s_k)\,\alpha\tau s_i \mathbb{1}_{\ell=0}/\left[(1-\tau)\,\delta\eta^2 s_c\right], \\
\det[\hat{\mathbf{A}}_0] &= \left\{(1+\gamma)\,\delta\sigma^2 s_c s_k - (2-\delta)\,\varphi\sigma^2 s_i + [(\gamma s_c + \sigma)\,\delta + (1-\delta)\,\sigma s_i]\,(1-s_k)\,\chi s_c\right\} \\
&\quad \{1 + [(1-\tau)\,\eta - (1-s_k)\,\tau s_c]\,(1-s_k+\omega s_k)\,\alpha\tau\mathbb{1}_{\ell=0}/\left[(1-\tau)\,\eta\right]\}/(\delta\eta\sigma s_c) \\
&\quad - \left\{(1-s_k)\,\sigma^2\mathbb{1}_{\ell=0} + \left[(1-s_k)\,\chi s_c + \eta\chi\omega - (1-s_k)\,\sigma^2\right]\mathbb{1}_{\ell=1}\right\} \\
&\quad (1-s_k+\omega s_k)\,(1-s_k)\,\alpha\tau^2 s_i/\left[(1-\tau)\,\delta\eta^2\right],
\end{aligned}$$

where $\mathbb{1}_{\ell=0}$ takes the value 1 if $\ell = 0$ and the value 0 otherwise. It is easy to check that the system (A.48) generically satisfies Assumption 3 too, i.e. that for any $\ell \in \mathbb{N}$, $\Psi_j(X) \neq 0$ for all $j \in \{1, ..., n\}$ except possibly for a zero-measure set of parameter values. Finally, the system (A.48) satisfies Assumption 5 because $\mathbf{A}(X)$ is not block-triangular and cannot be made block-triangular. More precisely, as the two equations in (A.48) are dynamic (the first one is both forward- and backward-looking, the second one is purely backward-looking), all the ways to rewrite (A.38) in an equivalent, block-recursive way lead to systems that involve past expectations and therefore are not of type (22).

Since I have restricted the analysis to information sets that include the debt level, the possible sets $J$ in Proposition 1 are $\{2\}$ and $\{1,2\}$. The condition stated in Proposition 1 is that $D_{\bar{J}}(X)$ should have at most $\delta = 1$ root in $\mathbb{D} \setminus \{0\}$. To show that this condition is generically met for any $J \in \{\{2\}, \{1,2\}\}$, I note that $\Psi_2(X)$ is generically of degree 2 and does not depend on $\alpha$, while $\Psi_3(X)$ can be written as $\Psi_3(X) = \Psi_{3,0}(X) + \alpha X^{\ell} \Psi_{3,1}(X)$, where $\Psi_{3,0}(X)$ and $\Psi_{3,1}(X)$ are generically of degrees 3 and 2 respectively and do not depend on $\alpha$. Therefore, if $\Psi_2(X)$ and $\Psi_3(X)$ have a common root for a non-zero-measure set of values for $\alpha$, then it must be a common root of $\Psi_2(X)$, $\Psi_{3,0}(X)$, and $\Psi_{3,1}(X)$. Since $\Psi_2(X)$ and $\Psi_{3,1}(X)$ are both of degree 2, it is easy to check that they have no common root, except possibly for a zero-measure set of parameter values. Therefore, $\Psi_2(X)$ and $\Psi_3(X)$ have generically no common root, that is to say that $D_{\bar{J}}(X)$ has generically no root for $J = \{2\}$, and hence no root either for $J = \{1,2\}$. As a consequence, the system (A.48) meets the condition stated in Proposition 1, except possibly for a zero-measure set of parameter values. In this model, thus, with either policy instrument (the labor-income-tax rate or the income-tax rate), the tax authority can always ensure determinacy by reacting to either the capital stock, or the debt level, or both, in the presence of inside lags of any length.[6]

The only non-empty and non-singleton set $J$ is $\{1,2\}$. The condition stated in Proposition 6 is, thus, that $D_{\{1,2\}}(X)$ should have no root in $\mathbb{D} \setminus \{0\}$. This condition is generically satisfied. To establish this result, I note that $\Psi_2(X)$ is generically of degree 2 and does not depend on $\alpha$, while $\Psi_1(X)$ can be written as $\Psi_1(X) = \Psi_{1,0}(X) + \alpha X^{\ell} \Psi_{1,1}(X)$, where $\Psi_{1,0}(X)$ and $\Psi_{1,1}(X)$ are generically of degrees 2 and 1 respectively and do not depend on $\alpha$. Therefore, if $\Psi_1(X)$ and $\Psi_2(X)$ have a common root for a non-zero-measure set of values for $\alpha$, then it must be a common root of $\Psi_{1,0}(X)$, $\Psi_{1,1}(X)$, and $\Psi_2(X)$. Since $\Psi_{1,1}(X)$ is of degree 1 and $\Psi_{1,0}(X)$ and $\Psi_2(X)$ of degree 2, it is easy to check that they have no common root, except possibly for a zero-measure set of parameter values. Therefore, $\Psi_1(X)$ and $\Psi_2(X)$ have generically no common root, that is to say that $D_{\{1,2\}}(X)$ has generically no root. As a consequence, the system (A.48) meets the condition stated in Proposition 6, except possibly for a zero-measure set of parameter values. In this model, thus, with either policy instrument (the labor-income-tax rate or the income-tax rate), the tax authority can always ensure determinacy with a non-superinertial rule reacting to

[6] If the tax authority does not observe the capital stock, but observes output and hours worked, then she can infer the capital stock using the production function (A.39).

the capital stock and the debt level, in the presence of inside lags of any length.