Stabilization Policy and Lags

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Abstract: Macroeconomic stabilization policy is notoriously subject to inside lags (which delay the reaction of policy to the state of the economy) and outside lags (which delay the effects of policy on the economy). In a broad class of dynamic rational-expectations models, I show that under a weak condition, neither inside lags nor outside lags of any length restrict the ability of the policymaker to ensure local-equilibrium determinacy and to control the anticipation and convergence rates, no matter how many different variables the policymaker observes. To establish this result, I invert the problem usually addressed in the literature: I start from a targeted characteristic polynomial, and I derive a corresponding policy-instrument rule. For any lags, this method offers some degrees of freedom that can be exploited to design rules with additional properties; I illustrate this possibility by designing non-superinertial rules, which the literature suggests may be more robust under model uncertainty.

Keywords: stabilization policy, inside lags, outside lags, local-equilibrium determinacy, anticipation rates, convergence rates, non-superinertial rules.

JEL codes: E32, E52.

1 Introduction

One of the main problems faced by macroeconomic stabilization policy is the existence of lags. Economists distinguish between two kinds of lags. Recognition, decision, and implementation lags, called "inside lags," delay the reaction of policy to the state of the economy. Transmission lags, called "outside lags," delay the effects of policy on the economy. Inside and outside lags are not equally problematic for all policies: for instance, monetary policy is thought to have a shorter inside lag and a longer outside lag than fiscal policy (Mankiw, 2019, Chapter 16). But all policies are subject to lags, to some extent or another.

The existence of lags has been acknowledged for a long time. Friedman (1961), for instance, famously emphasized long and variable outside lags for monetary policy. The implications of lags for stabilization policy were studied some time ago in backward-looking models without

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rational expectations (e.g., Fisher and Cooper, 1973). Little is known, however, about their implications in today's forward-looking rational-expectations models. The latter models differ fundamentally from the former in two key ways. First, they raise some distinctive issues that do matter for stabilization policy, like sunspot-driven fluctuations or the effects of news shocks. Second, in these models, unlike in the former, inside and outside lags are not equivalent to each other: inside lags prevent the policymaker from reacting to current or recent endogenous variables set by the private sector, while outside lags make current endogenous variables depend only on past variables and past expectations, so that they are predetermined and cannot be affected (neither directly nor indirectly) by current or recent policy shocks.

As inside and outside lags both put the policymaker behind the curve, each in its own way, to what extent do they restrict her ability to "control the economy" in dynamic rational-expectations models?[1] In this paper, I focus on four aspects of "controlling the economy." First, can these lags prevent the policymaker from ensuring local-equilibrium determinacy (i.e. existence and uniqueness of a stationary solution to the locally log-linearized model), and thus from ruling out sunspot-driven fluctuations, which are typically detrimental to welfare? Second, do they restrict her ability to control the "anticipation rates," which are the roots of the characteristic polynomial lying *outside* the unit circle of the complex plane, and which measure the speed at which the current impact of news about more and more distant future events vanishes? Or, third, her ability to control the "convergence rates," which are the roots of the characteristic polynomial *inside* the unit circle, and which measure the speed at which the economy goes back to steady state?[2] Fourth, can they prevent her from achieving any of the above when she observes (the history of) only a restricted set of endogenous variables? The main contribution of this paper is to provide a negative answer to these four questions in a broad class of dynamic rational-expectations models.

Most of the literature (reviewed below) focuses on the issue of determinacy. It addresses this issue only in some sparse examples, and provides little insight into whether and how the results could be extended beyond these examples. The approach is typically the following: (i) consider a specific discrete-time rational-expectations model, with outside lags of a specific length (of length zero if the focus is on inside lags); (ii) consider a specific parametric family of policy-instrument rules, consistent with inside lags of a specific length (of length zero if the focus is on outside lags); and (iii) derive the characteristic polynomial of the resulting system, and find under what inequality conditions on the model's parameters and the rule's coefficients the roots of this polynomial satisfy Blanchard and Kahn's (1980) root-counting condition for determinacy. These inequality conditions are obtained analytically for very simple models and rules; for more complex models and rules, however, the results are typically numerical, and hold only for some

[1] For convenience, throughout the paper, I refer to the policymaker with the female pronoun "she."

[2] I borrow the terms "anticipation rate" and "convergence rate" from Ljungqvist and Sargent (2018). The term "anticipation rate" is also used by other authors (e.g. Mertens and Ravn, 2010, and Leeper et al., 2013), following (an earlier version of) Ljungqvist and Sargent (2018).

specific calibrations.

In this paper, I invert the problem: for any given model, I choose a characteristic polynomial, and I derive a corresponding policy-instrument rule, i.e. a rule such that the system composed of the structural equations and this rule has this characteristic polynomial. I choose a characteristic polynomial that has as many roots outside the unit circle as there are non-predetermined variables in the system, so as to meet Blanchard and Kahn's (1980) root-counting condition. I choose not only the number of roots outside the unit circle, but also their values, i.e. the anticipation rates. I also choose the number of roots inside the unit circle and their values, i.e. the convergence rates. I do that for a broad class of dynamic rational-expectations models, for any value of their structural parameters, and for inside and outside lags of any length. I thus get the general result that inside and outside lags, no matter how long they are, do not restrict the ability of the policymaker to ensure determinacy and to control the anticipation and convergence rates, provided that the choice of the policymaker is not arbitrarily restricted to a specific parametric family of policy-instrument rules.

Moreover, I allow for the possibility that the policymaker observes (the history of) only a subset of endogenous variables. For any subset of variables, thus, I design a policy-instrument rule that involves only this subset of variables (in addition to being consistent with inside lags and leading to the targeted characteristic polynomial). It is easy to think of some variables that are likely not to be (directly) observed by the policymaker – e.g., Lagrange multipliers of private agents' optimization problems. As I illustrate in the main text, however, unobserved variables may also result from a certain kind of outside lags, which I call "distributed outside lags," and which make the structural equations involve expectations formed at different past dates. Under such lags, to rewrite the dynamic system in Blanchard and Kahn's (1980) form, which involves only current expectations, one needs to treat these expectations as new variables that are not (directly) observed by the policymaker.

I design the policy-instrument rule under a certain condition. This condition is about the model and the set of observed variables, not about the targeted characteristic polynomial, nor about the length of inside lags. I provide two reasons to think that this condition is likely to be met in practice. First, I show that it is necessarily met for any model having at least one stationary solution under a policy-instrument peg and any non-empty set of observed variables. Second, I show that it is met in three off-the-shelf monetary- or fiscal-policy models, for any values of their structural parameters (except possibly a zero-measure subset), and any non-empty set of observed variables. One of these models (Svensson, 1997, and Ball, 1999) has no stationary solutions under a peg for all structural-parameter values, while it is unclear whether or not the other two models (Smets and Wouters, 2007, and Schmitt-Grohé and Uribe, 1997) have at least one stationary solution under a peg for all structural-parameter values.

The method that I use (starting from a targeted characteristic polynomial, and deriving a corresponding policy-instrument rule) offers some degrees of freedom that can be exploited to design rules with additional properties. I illustrate this possibility by designing a policy-instrument rule that not only is consistent with inside lags and the set of observed variables, ensures determinacy, and controls the anticipation and convergence rates, but also is not "superinertial." A superinertial rule is a rule that would make the policy instrument explode over time if the variables set by the private sector were taken out of the rule.[3] As I document below, both numerical and theoretical results in the literature suggest that superinertial rules can easily lead to non-existence of a local equilibrium in backward-looking models. These results provide a motivation for adopting a non-superinertial rule when there is a non-zero probability that the true model is backward-looking, even if this probability is arbitrarily small. To be clear, the design of rules whose properties are robust across alternative models is mostly beyond the scope of this paper; I make here only a small step in this direction by identifying and illustrating some degrees of freedom that can be exploited for robustness purposes.

I design these non-superinertial rules under a certain condition about the model and the set of observed variables. This condition is stronger than the one mentioned above, but, I argue, still likely to be met in practice. I show indeed that it is met in four well known monetary- or fiscal-policy models, for any values of their structural parameters (except possibly a zero-measure subset), and for most non-empty and non-singleton sets of observed variables. These four models are the three mentioned above (which do always, or might sometimes, have no stationary solutions under a peg), plus the basic New Keynesian model (Woodford, 2003, Galí, 2015) (which always has an infinity of stationary solutions under a peg).

A few remarks may serve to put my contribution in the context of the literature. Early examples of papers studying determinacy conditions with *inside* lags, either analytically in a simple framework or numerically, include Rotemberg and Woodford (1999), Benhabib et al. (2001), Bullard and Mitra (2002), Carlstrom and Fuerst (2002), and Benhabib (2004) for monetary policy, and Schmitt-Grohé and Uribe (1997) for fiscal policy. McCallum (1999) argues that monetary-policy rules need to take inside lags into account to be operational, and Benhabib (2004) argues that the length of these lags may be long – e.g., up to sixty periods if inflation changes twice daily and the interest rate is set according to inflation lagged thirty days. Early examples of papers studying determinacy conditions with *outside* lags, again either analytically in a simple framework or numerically, include Rotemberg and Woodford (1999), Woodford (2003, Chapter 5), and Svensson and Woodford (2005).

As I explained above, the key difference with all these papers is that I do not start from a parametric family of rules and study determinacy conditions, either analytically or numerically. Instead, I start from a targeted characteristic polynomial, and I derive analytically a corresponding rule. This method enables me to establish much more general determinacy results with inside and outside lags than were previously established in these sparse examples. It also

[3]The term "superinertial" was coined by Woodford (1999).

enables me to establish general results about anticipation and convergence rates, which is an issue that cannot be easily addressed with the standard approach of starting from a parametric family of rules.

In Loisel (2022), I manage to get general determinacy results within the standard approach: I establish simple, necessary or sufficient conditions for determinacy under a generic policy-instrument rule in a generic model. These conditions are directly about the coefficients and horizons of the rule, and lead to new principles for stabilization policy. They have some important implications for (in)determinacy in the presence of inside lags. One of these implications, in particular, is that in any model that does not deliver determinacy under a policy-instrument peg, under any non-superinertial rule consistent with inside lags of length $\ell \in \mathbb{N}$, making ℓ go to infinity while keeping the coefficients of the rule constant (i.e. lagging more and more the endogenous variables in the rule without modifying their coefficients) necessarily leads to indeterminacy.

In the present paper, I show that despite this result, inside lags do not prevent the policymaker from ensuring determinacy, even with a non-superinertial rule. Simply, the coefficients of the rule, including those that are implicitly zero, should be adjusted as $\ell \to +\infty$ for the rule to keep ensuring determinacy. Beyond this determinacy result, moreover, I show that the policymaker keep also full control over the anticipation and convergence rates; and I establish these determinacy and control results not only in the presence of inside lags, but also in the presence of outside lags.

Two papers depart from the standard approach (of starting from a parametric family of rules) and design stabilization-policy rules in a more or less general framework. First, Giannoni and Woodford (2017), building on their earlier work (reported in Giannoni and Woodford, 2002, and Woodford, 2003, Chapter 8), analytically design "target criteria" that implement the optimal state-contingent path (for a given objective function) as the unique local equilibrium in a broad class of models. Second, in Loisel (2021), I study whether and how the policymaker can implement, as the unique local equilibrium, a given state-contingent path in a class of univariate models (i.e. models with only one variable set by the private sector). Unlike the present paper, however, these two papers do not allow for inside lags, nor for outside lags. In addition, they do not pay attention to the anticipation rates (as they do not consider news shocks).

Superinertial rules have been studied and discussed mostly in the monetary-policy literature. Two main results stand out in this literature. First, optimal interest-rate rules are often found (analytically or numerically) to be superinertial in forward-looking models (Rotemberg and Woodford, 1999; Woodford, 1999, 2003, Chapter 8; Levin, Wieland and Williams, 1999, 2003; Giannoni and Woodford, 2002, 2003, 2005). Second, superinertial interest-rate rules are often found (numerically) to lead to non-existence of a local equilibrium in backward-looking models (Rudebusch and Svensson, 1999; Taylor, 1999a, 1999b; Levin and Williams, 2003). In

Loisel (2022), I show that superinertial policy-instrument rules with a sufficiently small reaction to endogenous variables necessarily lead to non-existence of a local equilibrium in backward-looking models. These numerical and theoretical results provide a motivation for adopting a non-superinertial rule when there is a non-zero probability that the true model is backward-looking, even if this probability is arbitrarily small.

The policy-instrument rules that I design in this paper are typically more complex than those commonly considered in the literature, like Taylor's (1993) interest-rate rule, as they can involve an arbitrarily large number of arguments. This greater complexity does not matter at all in the rational-expectations paradigm. Moreover, it may even make these rules closer to actual decision procedures used by real-world policymakers, which amount to complex rules involving many inputs. However, it also makes these rules less easily communicable to the private sector in practice. For this reason, these rules should be viewed as complex reaction functions that, as argued by Svensson (2003, 2011), need not be communicated to the private sector. The policymaker could communicate about the targeted anticipation and convergence rates or, more generally, the targeted path for the economy; the private sector needs to know the existence of a corresponding rule, but the rule itself could remain behind the scenes.

Finally, throughout the paper, I restrict attention to local equilibria in locally log-linearized models, like most of the related literature (but unlike, e.g., Benhabib et al., 2001, 2002, 2003). This focus may not be that much of a limitation if non-local equilibria can be ruled out with the type of escape clause considered in, e.g., Benhabib et al. (2002); whether they can or cannot is, however, subject to debate (see, e.g., Cochrane, 2011, 2022).

The rest of the paper is organized as follows. Section 2 illustrates some of the main results of the paper in a basic New Keynesian framework. Section 3 derives the general results for inside lags. Section 4 extends the analysis of Section 3 to non-superinertial rules. Section 5 extends the analysis of Sections 3-4 to outside lags. I then conclude and provide a technical appendix.

2 A basic New Keynesian illustration

In this illustrative section, I derive some of the main results of the paper in a simple and well known context: the basic New Keynesian model, with a rule making the interest rate react to inflation. The analysis is a special case of the more general analysis conducted in the next sections.

2.1 Model and class of rules

I refer the reader to Woodford (2003) and Galí (2015) for a detailed presentation of the basic New Keynesian model. In this model, at each date $t \in \mathbb{Z}$, the private sector sets inflation π_t and output y_t according to the following (locally log-linearized) IS equation and Phillips curve:

$$\begin{aligned} y_t &= \mathbb{E}_t\{y_{t+1}\} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t\{\pi_{t+1}\}\right), & (1) \\ \pi_t &= \beta\mathbb{E}_t\{\pi_{t+1}\} + \kappa y_t, & (2) \end{aligned}$$

where $\mathbb{E}_t\{.\}$ denotes the date-t rational-expectations operator, and $\sigma > 0$, $\beta \in (0,1)$, and $\kappa > 0$ are three parameters. I abstract from exogenous shocks for simplicity, as I do not need them to make my points.

The policymaker is a central bank setting the short-term nominal interest rate i_t. I consider inside lags of (any) length $\ell \in \mathbb{N}$; so, when $\ell \geq 1$, the central bank cannot set i_t as a function of (π_t, y_t), ..., $(\pi_{t-\ell+1}, y_{t-\ell+1})$. For simplicity, I focus on interest-rate rules that do not involve output. I consider, thus, the class of (log-linearized) interest-rate rules of type

$$\rho(L)i_t = \phi(L)\pi_{t-\ell}, \tag{3}$$

where $\rho(z) \in \mathbb{R}[z]$ with $\rho(0) \neq 0$, $\phi(z) \in \mathbb{R}[z]$, and L is the lag operator.[4] These rules allow for any degree of interest-rate inertia (captured by the polynomial $\rho(z)$).

2.2 Determinacy and control

Using the Phillips curve (2) to replace y_t and y_{t+1} in the IS equation (1), I get

$$\mathbb{E}_t\left\{Q(L)\pi_{t+2}\right\} + i_t = 0, \tag{4}$$

where $Q(z) := (\sigma/\kappa)[\beta - (1 + \beta + \kappa/\sigma)z + z^2] \in \mathbb{R}[z]$. The dynamic system (1)-(3) has the same number of non-predetermined variables and the same characteristic polynomial as the reduced dynamic system (3)-(4). The latter system has two non-predetermined variables (namely, $\mathbb{E}_t\left\{\pi_{t+1}\right\}$ and $\mathbb{E}_t\left\{\pi_{t+2}\right\}$). The *reciprocal* polynomial of its characteristic polynomial is[5]

$$P(z) := Q(z)\rho(z) + z^{\ell+2}\phi(z).$$

So, this system satisfies Blanchard and Kahn's (1980) root-counting condition for determinacy if and only if $P(z)$ has exactly two roots inside the unit circle $\mathcal{C}$ of the complex plane (i.e. as many roots inside $\mathcal{C}$ as there are non-predetermined variables in the system). In what follows, I will design some rules of type (3) that not only make $P(z)$ have exactly two roots inside $\mathcal{C}$, but also, more generally, make $P(z)$ coincide with any given targeted polynomial $P^*(z)$ having exactly two roots inside $\mathcal{C}$. These rules will, thus, make the anticipation and convergence rates take any given targeted values. The anticipation and convergence rates, as defined in Ljungqvist and Sargent (2018), are the roots of the characteristic polynomial respectively outside $\mathcal{C}$ and

[4]Throughout the paper, $\mathbb{R}[z]$ denotes the set of polynomials in z with real-number coefficients.

[5]For any $\Pi(z) \in \mathbb{R}[z]$, the reciprocal polynomial of $\Pi(z)$ is $z^{\deg(\Pi)}\Pi(z^{-1})$. I work with the reciprocal polynomial of the characteristic polynomial, rather than with the characteristic polynomial itself, as the former is more convenient to use than the latter in conjunction with the lag operator.

inside $\mathcal{C}$, or equivalently the roots of its reciprocal polynomial $P(z)$ respectively inside $\mathcal{C}$ and outside $\mathcal{C}$. As clear from Blanchard and Kahn's (1980, p. 1308) analysis, the anticipation rates characterize the speed at which the current impact of news about more and more distant future events vanishes (for any news shocks that would be introduced into the model), while the convergence rates characterize the speed at which the economy goes back to steady state (when not starting from the steady state).

More specifically, I consider any targeted polynomial $P^*(z) \in \mathbb{R}[z]$ such that

$$\text{(i) } P^*(0) \neq 0, \text{ and (ii) } p^* = 2, \tag{5}$$

where $p^* := \#\{z \in \mathbb{C} | P^*(z) = 0, |z| < 1\}$ denotes the number of roots of $P^*(z)$ inside $\mathcal{C}$ (counting multiplicity). Condition (5) restricts the targeted polynomial $P^*(z)$: (i) not to have zero as a root (simply because reciprocal polynomials cannot have zero as a root); (ii) to have exactly two roots inside $\mathcal{C}$ (in order to satisfy Blanchard and Kahn's (1980) root-counting condition, as discussed above).

For any targeted polynomial $P^*(z)$ satisfying (5), I seek to design some rules of type (3) with the following properties:

$$\text{(i) } \rho(0) \neq 0, \text{ (ii) } P(z) = P^*(z), \text{ (iii) } \phi(z) \neq 0, \text{ and}$$
$$\text{(iv) } \rho(z) \text{ and } \phi(z) \text{ have no common roots inside } \mathcal{C}. \tag{6}$$

Property (6.i) ensures that the rule does involve the current policy instrument i_t. Property (6.ii) implies that the dynamic system has the targeted anticipation and convergence rates and satisfies Blanchard and Kahn's (1980) root-counting condition for determinacy. Properties (6.iii)-(6.iv) imply that the system also satisfies Blanchard and Kahn's (1980) *no-decoupling* condition for determinacy.[6]

More specifically, these last two properties preclude two different forms of system decoupling. Property (6.iii) ensures that the dynamics of i_t are not decoupled from the dynamics of π_t in the rule (so that the rule is, in this sense, a *feedback* rule). Property (6.iv) ensures that the rule by itself does not make (π_t, i_t) explode over time. If $\rho(z)$ and $\phi(z)$ had a common root inside $\mathcal{C}$, say a real number $r \in (-1, 1) \setminus \{0\}$, then the rule could be rewritten as $(1 - L/r)v_t = 0$ with $v_t := \tilde{\rho}(L)i_t - \tilde{\phi}(L)\pi_{t-\ell}$, where $\tilde{\rho}(z) := \rho(z)/(1 - z/r) \in \mathbb{R}[z]$ and $\tilde{\phi}(z) := \phi(z)/(1 - z/r) \in \mathbb{R}[z]$. Therefore, the rule would generate explosive dynamics for the variable v_t, and, hence, also for π_t and/or i_t. In essence, the system would have an "unstable eigenvalue" $(1/r)$ whose eigenvector would be a predetermined variable (v_t).

The main tool that I will use to design rules of type (3) satisfying (6) is:

[6] The "no-decoupling condition" requires that the system should not be "decoupled" in the sense of Sims (2007). It is formulated as a matrix-rank condition in Blanchard and Kahn (1980, p. 1308), and is often called the "rank condition" in the literature. Sims' (2007) bare-bones example of a system meeting the root-counting condition but not the no-decoupling condition is $x_t = 1.1x_{t-1} + \varepsilon_t$ and $\mathbb{E}_t\{y_{t+1}\} = 0.9y_t + \nu_t$.

Lemma 1 (Bézout's identity): *For any $m \in \mathbb{N} \setminus \{0\}$ and any $[\Pi_j(z)]_{0 \leq j \leq m} \in \mathbb{R}[z]^{m+1}$, there exists $[\mathcal{B}_j(z)]_{0 \leq j \leq m} \in \mathbb{R}[z]^{m+1}$ such that $\sum_{0 \leq j \leq m} \Pi_j(z)\mathcal{B}_j(z) = D(z)$, where $D(z) := \gcd\{[\Pi_j(z)]_{0 \leq j \leq m}\}$.*[7]

Proof: The standard proof for the case $m = 1$ is based on the Euclidean algorithm; see, e.g., Prasolov (2004, p. 47) for this proof. The extension to the case $m \geq 2$ is straightforward by induction. ■

I will use Lemma 1 with $m = 1$ in the present section, and with $m \geq 1$ in the next sections. For now, I get the following proposition, where the term "arithmetically derivable rule" refers to a rule that can be derived from the structural parameters β, κ, σ, and the targeted polynomial $P^*(z)$ with only a finite number of arithmetic operations (addition, subtraction, multiplication, division):

Proposition 1 (Determinacy and control with inside lags in the basic New Keynesian model): *Consider the basic New Keynesian model (1)-(2). For any inside-lags length $\ell \in \mathbb{N}$ and any targeted polynomial $P^*(z) \in \mathbb{R}[z]$ satisfying (5), there exist some arithmetically derivable rules of type (3) ensuring determinacy and making $P(z)$, the reciprocal polynomial of the characteristic polynomial, coincide with $P^*(z)$.*

Proof: I apply Lemma 1 to $m = 1$, $\Pi_0(z) = Q(z)$, and $\Pi_1(z) = z^{\ell+2}$. Since these polynomials have no common roots, there exists $(\mathcal{B}_0(z), \mathcal{B}_1(z)) \in \mathbb{R}[z]^2$ such that

$$Q(z)\mathcal{B}_0(z) + z^{\ell+2}\mathcal{B}_1(z) = 1. \tag{7}$$

Multiplying the left- and right-hand sides of (7) by $P^*(z)$ leads to

$$Q(z)\mathcal{B}_0(z)P^*(z) + z^{\ell+2}\mathcal{B}_1(z)P^*(z) = P^*(z).$$

Choosing $\rho(z) = \mathcal{B}_0(z)P^*(z)$ and $\phi(z) = \mathcal{B}_1(z)P^*(z)$ would violate Condition (6.iv), because it would make both $\rho(z)$ and $\phi(z)$ multiples of $P^*(z)$, which has $p^* = 2$ roots inside $\mathcal{C}$. To overcome this difficulty, I introduce an arbitrary polynomial $\mathcal{K}(z) \in \mathbb{R}[z]$ and rewrite the previous equation as

$$Q(z)\left[\mathcal{B}_0(z)P^*(z) + \mathcal{K}(z)z^{\ell+2}\right] + z^{\ell+2}\left[\mathcal{B}_1(z)P^*(z) - \mathcal{K}(z)Q(z)\right] = P^*(z). \tag{8}$$

Thus, except for a zero-measure set of polynomials $\mathcal{K}(z)$, the rules of type (3) with

$$\begin{aligned} \rho(z) &= \mathcal{B}_0(z)P^*(z) + \mathcal{K}(z)z^{\ell+2} \\ \phi(z) &= \mathcal{B}_1(z)P^*(z) - \mathcal{K}(z)Q(z) \end{aligned}$$

[7] Throughout the paper, the operator gcd{.}, applied to a set of polynomials, denotes their greatest common divisor, which is defined up to a multiplicative non-zero real-number scalar.

satisfy Conditions (6.ii)-(6.iv). They also satisfy Condition (6.i) because $\mathcal{B}_0(0) \neq 0$ (as follows from (7) with $z = 0$) and $P^*(0) \neq 0$ (due to (5.i)). Finally, these rules are arithmetically derivable because the different steps of their design process (in particular getting $\mathcal{B}_0(z)$ and $\mathcal{B}_1(z)$ with the Euclidean algorithm) involve only a finite number of arithmetic operations. ■

Thus, although they put the central bank behind the curve, inside lags do not restrict her ability to ensure determinacy and control the anticipation and convergence rates. Key to this result is the assumption that her choice set is not arbitrarily restricted to a specific parametric family of interest-rate rules; instead, she is allowed to choose any rule of type (3); these rules have a finite but unbounded number of parameters.

Because the rules described in Proposition 1 are arithmetically derivable, their coefficients can be explicitly expressed as rational functions (i.e. fractions of polynomial functions) of the structural parameters and the coefficients of $P^*(z)$, and, therefore, also as rational functions of the structural parameters, the anticipation rates, and the convergence rates. Rational functions are particularly easy to manipulate analytically. For instance, their derivatives can be easily computed − with the help of a symbolic-computation software − to determine how the coefficients of the rules respond to an arbitrarily small change in the value of the structural parameters, the anticipation rates, or the convergence rates.

2.3 Non-superinertial rules

For any targeted polynomial $P^*(z)$, there exists an infinity of rules having the properties described in Proposition 1. This infinity is captured by the arbitrary polynomial $\mathcal{K}(z)$ in the proof of Proposition 1: adding $\mathcal{K}(L)L^{\ell+2}[Q(L)\pi_{t+2} + i_t]$ to (the left- or right-hand side of) any rule of type (3) leaves the characteristic polynomial unchanged. This result is due to the fact that the structural equations (1)-(2), which imply (4), make the term $Q(L)\pi_{t+2} + i_t$ equal to zero in expectation at date t. So, adding this term (taken at any date) to the rule has no effect on the characteristic polynomial.

I now exploit these degrees of freedom to design some interest-rate rules that not only ensure determinacy and control the anticipation and convergence rates, but also are not superinertial. As I explain in the Introduction, both numerical and theoretical results in the literature suggest that non-superinertial rules may be more robust than superinertial rules under model uncertainty.

A rule of type (3) is said to be superinertial when $\rho(z)$ has at least one root inside $\mathcal{C}$. So, for any targeted polynomial $P^*(z)$ satisfying (5), I seek to design some rules of type (3) that not only have Properties (6.i)-(6.iv), but also are such that

$$\rho(z) \text{ has no roots inside } \mathcal{C}. \tag{9}$$

Note that this last property implies Property (6.iv).

To design these rules, I first establish the following lemma:

Lemma 2: *For any $U(z) \in \mathbb{R}[z]$ such that $U(0) \neq 0$, there exists $V(z) \in \mathbb{R}[z]$ such that $U(z) + z^{\deg(U)+1}V(z)$ has no roots inside $\mathcal{C}$.*

Proof: See Appendix A.1. In essence, the proof in Appendix A.1 first uses Newton's identities to rewrite the constraint on the coefficients of z^0, ..., $z^{\deg(U)}$ in the polynomial $U(z)+z^{\deg(U)+1}V(z)$ as a constraint on the sums of the k^{th} powers of the roots of this polynomial. It then designs a polynomial that satisfies the latter constraint and has no roots inside $\mathcal{C}$. ■

Using Lemma 2, I then get the following proposition:

Proposition 2 (Extension of Proposition 1 to non-superinertial rules): *Consider the basic New Keynesian model (1)-(2). For any inside-lags length $\ell \in \mathbb{N}$ and any targeted polynomial $P^*(z) \in \mathbb{R}[z]$ satisfying (5), there exist some non-superinertial rules of type (3) ensuring determinacy and making $P(z)$, the reciprocal polynomial of the characteristic polynomial, coincide with $P^*(z)$.*

Proof: I start from the proof of Proposition 1. Using (7) and $\deg(Q) = 2$, I get

$$\deg(\mathcal{B}_0) = \deg(\mathcal{B}_1) + \ell. \tag{10}$$

Lemma 2, applied to $U(z) = \mathcal{B}_0(z)P^*(z)$, implies the existence of $V(z) \in \mathbb{R}[z]$ such that $\mathcal{B}_0(z)P^*(z)+z^{\deg(\mathcal{B}_0)+\deg(P^*)+1}V(z)$ has no roots inside $\mathcal{C}$. The expression $z^{\deg(\mathcal{B}_1)+\deg(P^*)-1}V(z)$ is a polynomial, i.e. it has no negative exponents of z, because $\deg(P^*) \geq 2$ (due to (5.ii)). So, I can choose $\mathcal{K}(z) = z^{\deg(\mathcal{B}_1)+\deg(P^*)-1}V(z)$, or equivalently $\mathcal{K}(z) = z^{\deg(\mathcal{B}_0)+\deg(P^*)-\ell-1}V(z)$ (as follows from (10)). Replacing the first $\mathcal{K}(z)$ in (8) by the latter expression, and the second one by the former expression, I get

$$Q(z)\left[\mathcal{B}_0(z)P^*(z) + z^{\deg(\mathcal{B}_0)+\deg(P^*)+1}V(z)\right] + z^{\ell+2}\left[\mathcal{B}_1(z)P^*(z) - z^{\deg(\mathcal{B}_1)+\deg(P^*)-1}Q(z)V(z)\right] = P^*(z). \tag{11}$$

Thus, the rule of type (3) with

$$\begin{aligned}
\rho(z) &= \mathcal{B}_0(z)P^*(z) + z^{\deg(\mathcal{B}_0)+\deg(P^*)+1}V(z) \\
\phi(z) &= \mathcal{B}_1(z)P^*(z) - z^{\deg(\mathcal{B}_1)+\deg(P^*)-1}Q(z)V(z)
\end{aligned}$$

satisfies Conditions (6.ii) and (9). It also satisfies Condition (6.iv), since (9) implies (6.iv). Finally, it satisfies Condition (6.i) (because $\mathcal{B}_0(0) \neq 0$ and $P^*(0) \neq 0$) and Condition (6.iii) (because $\deg(Q) = 2$). The polynomial $V(z)$ in this proof can be replaced with any polynomial $\tilde{V}(z)$ that is sufficiently close to $V(z)$ for $\mathcal{B}_0(z)P^*(z) + z^{\deg(\mathcal{B}_0)+\deg(P^*)+1}\tilde{V}(z)$ to keep no roots inside $\mathcal{C}$; so, there is not just one, but an infinity of rules of type (3) satisfying Conditions (6) and (9). ■

Note that, unlike Proposition 1, Proposition 2 is silent about whether the rules are arithmetically derivable or not. The reason is that the proof of Proposition 2 uses Lemma 2 to design the rules, and the proof of Lemma 2 does not derive the polynomial $V(z)$ arithmetically.

2.4 Non-distributed outside lags

I now introduce outside lags into the basic New Keynesian model, and I extend the previous results to the resulting model. In models with outside lags, the private sector takes its decisions in advance, so current variables depend on *past* expectations, rather than *current* expectations. As a result, current variables are predetermined and could not be affected (neither directly nor indirectly) by current or recent policy shocks, if such shocks were introduced into the policy-instrument rule.

I focus here on a certain kind of outside lags, which I call "non-distributed outside lags." I leave the other kind of outside lags (which I call "distributed outside lags") for Section 5. In the presence of non-distributed outside lags of length $\ell' \in \mathbb{N}$, the private sector takes all its decisions a single fixed number ℓ' of periods in advance. So, the structural equations express current variables as a function of expectations formed at date $t - \ell'$. More specifically, the IS equation (1) and the Phillips curve (2) become respectively

$$
\begin{aligned}
y_t &= \mathbb{E}_{t-\ell'}\{y_{t+1}\} - \frac{1}{\sigma}\left(\mathbb{E}_{t-\ell'}\{i_t\} - \mathbb{E}_{t-\ell'}\{\pi_{t+1}\}\right), && (12)\\
\pi_t &= \beta\mathbb{E}_{t-\ell'}\{\pi_{t+1}\} + \kappa\mathbb{E}_{t-\ell'}\{y_t\}. && (13)
\end{aligned}
$$

When $\ell' = 0$, the structural equations (12)-(13) are the same as (1)-(2).

I consider the class of rules of type (3), and hence I allow for inside lags of length ℓ. To get rid of the past-expectation terms in (12)-(13), I introduce the variables

$$
\text{(i) } \tilde{y}_t := \mathbb{E}_t\{y_{t+\ell'}\}, \text{ (ii) } \tilde{\pi}_t := \mathbb{E}_t\{\pi_{t+\ell'}\}, \text{ (iii) } \tilde{i}_t := \mathbb{E}_t\{i_{t+\ell'}\}, \qquad (14)
$$

and I rewrite the system composed of (3) and (12)-(14) as the following system:

$$
\begin{aligned}
&\tilde{y}_t = \mathbb{E}_t\{\tilde{y}_{t+1}\} - \frac{1}{\sigma}\left(\mathbb{E}_t\{\tilde{i}_t\} - \mathbb{E}_t\{\tilde{\pi}_{t+1}\}\right), && (15)\\
&\tilde{\pi}_t = \beta\mathbb{E}_t\{\tilde{\pi}_{t+1}\} + \kappa\mathbb{E}_t\{\tilde{y}_t\}, && (16)\\
&\rho(L)\tilde{i}_t = \phi(L)\tilde{\pi}_{t-\ell}, && (17)\\
&\text{(i) } y_t = \tilde{y}_{t-\ell'}, \text{ (ii) } \pi_t = \tilde{\pi}_{t-\ell'}, \text{ and (iii) } i_t = \tilde{i}_{t-\ell'}. && (18)
\end{aligned}
$$

The two systems are equivalent to each other. To see why the former system implies the latter, note that the IS equation (12) implies that y_t is determined at date $t - \ell'$, i.e. $y_t = \mathbb{E}_{t-\ell'}\{y_t\}$, which is (18.i). Similarly, the Phillips curve (13) implies that π_t is determined at date $t - \ell'$, i.e. $\pi_t = \mathbb{E}_{t-\ell'}\{\pi_t\}$, which is (18.ii). In turn, if π_t is determined at date $t-\ell'$, then the rule (3) implies that i_t is determined at date $t-\ell-\ell'$, and hence that i_t is *already* determined at date $t-\ell'$, i.e. $i_t = \mathbb{E}_{t-\ell'}\{i_t\}$, which is (18.iii). Finally, replacing (y_t, π_t, i_t) with $(\tilde{y}_{t-\ell'}, \tilde{\pi}_{t-\ell'}, \tilde{i}_{t-\ell'})$ in (3) and (12)-(13) gives (15)-(17) at date $t-\ell'$, and therefore (15)-(17) at date t as well.

To see why, conversely, the latter system implies the former, note that (18.i) implies that y_t is determined at date $t-\ell'$, and therefore that $y_{t+\ell'}$ is determined at date t, i.e. $y_{t+\ell'} = \mathbb{E}_t\{y_{t+\ell'}\}$, which gives (14.i). Similarly for π_t and i_t, (18.ii)-(18.iii) imply (14.ii)-(14.iii). Finally, replacing $(\tilde{y}_t, \tilde{\pi}_t, \tilde{i}_t)$ with $(y_{t+\ell'}, \pi_{t+\ell'}, i_{t+\ell'})$ in (15)-(17) gives (3) and (12)-(13) at date $t+\ell'$, and therefore (3) and (12)-(13) at date t as well.

The system (15)-(18) is block-recursive: the sub-system (15)-(17) determines $(\tilde{y}_t, \tilde{\pi}_t, \tilde{i}_t)$, uniquely or not; and, for any value of $(\tilde{y}_t, \tilde{\pi}_t, \tilde{i}_t)$, the sub-system (18) determines (y_t, π_t, i_t) uniquely.

So, there is a unique solution in (y_t, π_t, i_t) to the system composed of (3) and (12)-(13) if and only if there is a unique solution in $(\tilde{y}_t, \tilde{\pi}_t, \tilde{i}_t)$ to the system (15)-(17). In turn, there is a unique solution in $(\tilde{y}_t, \tilde{\pi}_t, \tilde{i}_t)$ to the system (15)-(17) if and only if there is a unique solution in (y_t, π_t, i_t) to the system (1)-(3), because these two systems are identical to each other. So, we are back to the situation without outside lags (and with inside lags). Thus, non-distributed outside lags do not affect any of the results of the previous sections, no matter the length of these lags:

Proposition 3 (Extension of Propositions 1-2 to non-distributed outside lags): *Propositions 1-2 still hold if "the basic New Keynesian model (1)-(2)" is replaced by "the model (12)-(13) with any outside-lags length $\ell' \in \mathbb{N}$" in these propositions.*

The reason for this neutrality is the following. At date t, the central bank eventually observes $y_{t-\ell}$ and $\pi_{t-\ell}$ (due to inside lags), which were actually decided at date $t-\ell-\ell'$ (due to non-distributed outside lags). But $y_{t-\ell}$ and $\pi_{t-\ell}$ were decided at date $t-\ell-\ell'$ on the basis of expectations whose horizon is ℓ'-periods longer than without outside lags; and all variables until date $t-\ell$ were already known with perfect foresight at date $t-\ell-\ell'$. So, the situation is essentially the same as without outside lags ($\ell' = 0$), and all the results still hold.

3 Determinacy and control with inside lags

In this section, I show, in a broad class of models, that inside lags of any length do not restrict the ability of the policymaker to ensure determinacy, nor to control anticipation and convergence rates, under a weak condition on the model and the set of observed variables. The analysis in this section is essentially a generalization of the analysis in Subsections 2.1-2.2.

3.1 Class of models

I start by describing the class of discrete-time rational-expectations models that I consider (which is essentially the same as in Loisel, 2022). At each date $t \in \mathbb{Z}$, the private sector sets an n-dimension vector of endogenous variables $\mathbf{X}_t$ according to the following (locally log-linearized) structural equations:

$$\mathbb{E}_t \left\{ \mathbf{\Delta}\left(L^{-1}\right)\left[\mathbf{A}\left(L\right)\mathbf{X}_t + L^{-\gamma}\mathbf{B}\left(L\right)i_t\right]\right\} = \mathbf{0}, \tag{19}$$

where i_t denotes the policy instrument at date t. I abstract again from exogenous shocks for simplicity, as I do not need them to make my points. These structural equations are parameterized by $n \in \mathbb{N} \setminus \{0\}$, $\gamma \in \mathbb{N}$, $\mathbf{A}(z) \in \mathbb{R}^{n \times n}[z]$, $\mathbf{B}(z) \in \mathbb{R}^{n \times 1}[z]$, and $\mathbf{\Delta}(z) := \text{diag}(z^{\delta_1}, ..., z^{\delta_n}) \in \mathbb{R}^{n \times n}[z]$, where $(\delta_1, ..., \delta_n) \in \mathbb{N}^n$.[8] I assume that $\det[\mathbf{A}(0)] \neq 0$ (without any loss in generality if the structural equations are independent of each other) and that $\mathbf{B}(z) \neq \mathbf{0}$ (for the policy instrument to have some effect on the endogenous variables set by the private sector). I also assume that

$$\forall j \in \{1, ..., n\}, \;\; W_j(z) := \det \begin{bmatrix} \mathbf{A}(z) & \mathbf{B}(z) \\ \mathbf{e}_j^\top & 0 \end{bmatrix} \neq 0,$$

where $\mathbf{e}_j$ denotes the $n \times 1$ vector whose j^{th} element is one and whose other elements are all zero. This assumption is not restrictive either, for the following reason. If it were not satisfied, i.e. if there existed $j \in \{1, ..., n\}$ such that $W_j(z) = 0$, then there would exist a linear combination of the structural equations that would involve only elements of $\{\mathbb{E}_t\{\mathbf{e}_j^\top \mathbf{X}_{t+k}\} | k \in \mathbb{Z}\}$; in that case, the variable $\mathbf{e}_j^\top \mathbf{X}_t$ should then be considered as exogenous, not endogenous, and could be removed from the system.

It will be useful, later in this section, to distinguish between three kinds of models, depending on their (in)determinacy properties under a policy-instrument peg. Under a peg ($i_t = 0$), the number of non-predetermined variables is $\delta := \sum_{j=1}^n \delta_j$, as I show in Loisel (2022). Moreover, the reciprocal polynomial of the characteristic polynomial is $Q(z) := \det[\mathbf{A}(z)]$. So, the number of "unstable eigenvalues" of the dynamic system is equal to $q := \#\{z \in \mathbb{C} | Q(z) = 0, |z| < 1\}$, the number of roots of $Q(z)$ inside $\mathcal{C}$ (counting multiplicity). For simplicity, I make the following two "regularity assumptions:" $Q(z)$ has no roots on $\mathcal{C}$, and Blanchard and Kahn's (1980) *no-decoupling* condition is met under a peg. So, using Blanchard and Kahn's (1980) *root-counting* condition, I get that, under a peg, models with $q \leq \delta - 1$ have an infinity of stationary solutions, those with $q = \delta$ have a unique stationary solution, and those with $q \geq \delta + 1$ have no stationary solutions.

[8]Throughout the paper, letters in bold denote vectors and matrices that have potentially more than one element. $\mathbf{0}$ denotes a vector or a matrix whose elements are all equal to zero and whose dimensions depend on the specific context in which it is used. For any $(n_1, n_2) \in (\mathbb{N} \setminus \{0\})^2$, $\mathbb{R}^{n_1 \times n_2}[z]$ denotes the set of polynomials in z whose coefficients are $n_1 \times n_2$ matrices with real-number elements.

3.2 Class of rules

I consider inside lags of (any) length $\ell \in \mathbb{N}$. The policymaker, thus, cannot set the policy instrument as a function of endogenous variables set by the private sector less than ℓ periods earlier. I consider the class of (log-linearized) policy-instrument rules of type

$$\rho(L) i_t = \boldsymbol{\phi}(L) \mathbf{X}_{t-\max(\ell,\gamma+1)}, \tag{20}$$

where $\rho(z) \in \mathbb{R}[z]$ with $\rho(0) \neq 0$, and $\boldsymbol{\phi}(z) \in \mathbb{R}^{1\times n}[z]$. The right-hand side of these rules excludes not only $\{\mathbf{X}_{t-k} | 0 \leq k \leq \ell\}$, consistently with inside lags, but also $\{\mathbf{X}_{t-k} | \ell + 1 \leq k \leq \gamma + 1\}$ when $\gamma \geq \ell$, in order to facilitate the forthcoming analysis (by making the number of non-predetermined variables independent of the rule). These rules also allow for any degree of policy-instrument inertia (captured by the polynomial $\rho(z)$).

I allow for the possibility that the policymaker may never observe some endogenous variables, and hence cannot set the policy instrument as a function of these variables (not even with inside lags). Let $J \subseteq \{1, ..., n\}$ denote the set of integers j such that the variable $\mathbf{e}_j^\top \mathbf{X}_t$ is observed (with inside lags) by the policymaker. If $J \neq \{1, ..., n\}$, then I require that

$$\forall j \in \{1, ..., n\} \setminus J, \;\; \boldsymbol{\phi}(z) \mathbf{e}_j = 0. \tag{21}$$

With a slight abuse of language, I will refer to J as the "set of observed variables" or the "observation set." My motivation for allowing for $J \neq \{1, ..., n\}$ is twofold. First, some endogenous variables in macroeconomic models may not have natural empirical counterparts, like Lagrange multipliers of private agents' optimization problems for instance. Second, and more to the point, the presence of unobserved variables may result from a certain kind of outside lags, as I will illustrate in Section 5; so, I am preparing the ground for the analysis under outside lags.

3.3 Determinacy and control

I start by establishing a useful preliminary result:

Lemma 3: *Consider a model of type (19), a non-empty observation set $J \subseteq \{1, ..., n\}$, and a rule of type (20) satisfying the observation constraint (21). Then, the dynamic system composed of (19) and (20) has exactly δ non-predetermined variables, and the reciprocal polynomial of its characteristic polynomial is*

$$P(z) := Q(z)\rho(z) - z^{\max(1,\ell-\gamma)} \sum\nolimits_{j \in J} W_j(z) \boldsymbol{\phi}(z) \mathbf{e}_j.$$

Proof: See Appendix A.2. ■

Lemma 3 implies that the dynamic system composed of (19) and (20) satisfies Blanchard and Kahn's (1980) root-counting condition if and only if $P(z)$ has exactly δ roots inside $\mathcal{C}$ (i.e. as many roots inside $\mathcal{C}$ as there are non-predetermined variables in the system). In what follows, I will design some rules of type (20) that not only make $P(z)$ have exactly δ roots inside $\mathcal{C}$, but also, more generally, make $P(z)$ coincide with any given targeted polynomial $P^*(z)$ having exactly δ roots inside $\mathcal{C}$. These rules will, thus, make the anticipation and convergence rates take any given targeted values.

For any non-empty observation set $J \subseteq \{1, ..., n\}$, let $D_J(z) := \gcd\{Q(z), [W_j(z)]_{j \in J}\}$. Under any rule of type (20) satisfying (21), $P(z)$ is a multiple of $D_J(z)$, as clear from Lemma 3. So, for $P(z)$ to have exactly δ roots inside $\mathcal{C}$, we need

$$d_J \leq \delta, \tag{22}$$

where $d_J := \#\{z \in \mathbb{C} | D_J(z) = 0, |z| < 1\}$ denotes the number of roots of $D_J(z)$ inside $\mathcal{C}$ (counting multiplicity). Under Condition (22), I can consider any targeted polynomial $P^*(z) \in \mathbb{R}[z]$ such that

$$\text{(i) } P^*(0) \neq 0, \text{ (ii) } P^*(z)/D_J(z) \in \mathbb{R}[z], \text{ and (iii) } p^* = \delta, \tag{23}$$

where again $p^* := \#\{z \in \mathbb{C} | P^*(z) = 0, |z| < 1\}$ denotes the number of roots of $P^*(z)$ inside $\mathcal{C}$ (counting multiplicity). Condition (23) restricts the targeted polynomial $P^*(z)$: (i) not to have zero as a root (simply because reciprocal polynomials cannot have zero as a root); (ii) to be a multiple of $D_J(z)$ (for the reason I have just discussed); (iii) to have exactly δ roots inside $\mathcal{C}$ (in order to satisfy Blanchard and Kahn's (1980) root-counting condition).

I get the following proposition, which generalizes Proposition 1 to a broad class of models and to (almost) any observation set:

Proposition 4 (Determinacy and control with inside lags in a broad class of models): *Consider a model of type (19). For any inside-lags length $\ell \in \mathbb{N}$, any non-empty observation set $J \subseteq \{1, ..., n\}$ satisfying (22), and any targeted polynomial $P^*(z) \in \mathbb{R}[z]$ satisfying (23), there exist some arithmetically derivable rules of type (20) satisfying the observation constraint (21), ensuring determinacy, and making $P(z)$, the reciprocal polynomial of the characteristic polynomial, coincide with $P^*(z)$.*

Proof: See Appendix A.3. The proof in Appendix A.3 is essentially a generalization of the proof of Proposition 1, using Lemma 1 with $m \geq 1$. ■

Proposition 4 says that even in the presence of inside lags of any length, the policymaker can still ensure determinacy and still control the anticipation and convergence rates, no matter how many different variables she observes, provided that Condition (22) is met. Again, the key assumption behind this result is that her choice set is not arbitrarily restricted to a specific parametric family of policy-instrument rules; instead, she is allowed to choose any rule of type (20) satisfying the observation constraint (21); these rules have a finite but unbounded number of parameters. And again, because the rules described in Proposition 4 are arithmetically derivable, their coefficients can be explicitly expressed as rational functions of the structural parameters and the targeted anticipation and convergence rates.

3.4 About Condition (22)

Proposition 4 applies only if the model and the observation set satisfy Condition (22). How restrictive is this condition? Very little, I argue, for two reasons.

First, I show that this condition is necessarily met for any model having at least one stationary solution under a policy-instrument peg and for any non-empty set of observed variables. The property of having an infinity of solutions under a peg (i.e. $q \leq \delta - 1$) has been called the "Sargent-Wallace property" by Giannoni and Woodford (2002) and Woodford (2003, Chapter 8), after Sargent and Wallace (1975). So, I call the property of having at least one stationary solution under a peg (i.e. $q \leq \delta$) a "weak Sargent-Wallace property." Since $D_J(z)$ is, by definition, a divisor of $Q(z)$, we have $d_J \leq q$ and, therefore, $d_J \leq \delta$ in any model having the weak Sargent-Wallace property. I thus obtain the following proposition:

Proposition 5 (A weak Sargent-Wallace property implying Condition (22)): *All models of type (19) with $q \leq \delta$ satisfy Condition (22) for any non-empty observation set $J \subseteq \{1, ..., n\}$.*

In the monetary-policy literature, for instance, the basic New Keynesian model (considered in the previous section) satisfies $q \leq \delta - 1$ for all structural-parameter values, and hence falls generically into the scope of Proposition 5. There is a general presumption that extended New Keynesian models also typically satisfy $q \leq \delta - 1$ (Cochrane, 2011, 2022). Given the relative complexity of these models, however, it is usually unclear whether or not they satisfy the weak Sargent-Wallace property $q \leq \delta$ for all structural-parameter values.

The second reason to view Condition (22) as very little restrictive is the following. In the Online Appendix, I consider three off-the-shelf monetary- or fiscal-policy models. One of them, the model of Svensson (1997) and Ball (1999), satisfies $q \geq \delta + 1$ for all structural-parameter values, and hence is generically outside the scope of Proposition 5. The other two models are those of Smets and Wouters (2007) and Schmitt-Grohé and Uribe (1997); given their relative complexity, it is unclear whether or not they fall into the scope of Proposition 5 for all structural-parameter values.

I first show that these three models can be written in a form of type (19). Then, for each of these three models, I show that for any non-empty observation set J and for all structural-parameter values (except possibly a zero-measure subset of values), not only Condition (22) is met, but in fact $\deg(D_J) = 0$; that is to say, not only $D_J(z)$ has at most δ roots inside $\mathcal{C}$, but in fact $D_J(z)$

has no roots at all (neither inside nor outside $\mathcal{C}$). I establish this result analytically in the model of Svensson (1997) and Ball (1999), and with the help of the symbolic-computation software Mathematica in the models of Smets and Wouters (2007) and Schmitt-Grohé and Uribe (1997).

Both this finding and Proposition 5 suggest, in my view, that Condition (22) is typically met for any non-empty observation set J in existing stabilization-policy models.

4 Non-superinertial rules

In this section, I extend the results of the previous section to non-superinertial rules. More specifically, I show, in the same broad class of models, that inside lags of any length do not restrict the ability of the policymaker to ensure determinacy and control the anticipation and convergence rates with a non-superinertial rule, under a certain condition on the model and the observation set. The analysis in this section is essentially a generalization of the analysis in Subsection 2.3. As I explain in the Introduction, my motivation for considering non-superinertial rules stems from both numerical and theoretical results in the literature, which suggest that non-superinertial rules may be more robust than superinertial rules under model uncertainty.

4.1 Determinacy and control

I consider exactly the same classes of models and rules as in the previous section. For any model and any targeted anticipation and convergence rates, there exists an infinity of policy-instrument rules having the properties described in Proposition 4. I now exploit these degrees of freedom to design some rules that not only have the properties described in Proposition 4, but also are not superinertial. A rule of type (20) is said to be superinertial when $\rho(z)$ has at least one root inside $\mathcal{C}$. So, I seek to design rules of type (20) satisfying (9).

Let $\chi \in \mathbb{N}$ denote the multiplicity of zero as a root of $\gcd\{[W_j(z)]_{j\in J}\}$ (with $\gcd\{[W_j(z)]_{j\in J}\} = W_j(z)$ if J is the singleton $\{j\}$), and let $G_J(z) := z^{-\chi}\gcd\{[W_j(z)]_{j\in J}\}$. To extend Proposition 4 to non-superinertial rules, I assume that

$$\deg(G_J) = 0. \tag{24}$$

Condition (24) is stronger than Condition (22) in the previous section. Indeed, Condition (24) implies $\deg(D_J) = 0$, which in turn implies $d_J = 0$, which in turn implies Condition (22). So, under Condition (24), I can consider any targeted polynomial $P^*(z) \in \mathbb{R}[z]$ such that

$$\text{(i) } P^*(0) \neq 0, \text{ and (ii) } p^* = \delta. \tag{25}$$

Condition (25) is just a simplified version of Condition (23), taking into account the fact that $\deg(D_J) = 0$ under Condition (24).

I thus get the following proposition:

Proposition 6 (Extension of Proposition 4 to non-superinertial rules): *Consider a model of type (19). For any inside-lags length $\ell \in \mathbb{N}$, any non-empty observation set $J \subseteq \{1, ..., n\}$ satisfying (24), and any targeted polynomial $P^*(z) \in \mathbb{R}[z]$ satisfying (25), there exist some non-superinertial rules of type (20) satisfying the observation constraint (21), ensuring determinacy, and making $P(z)$, the reciprocal polynomial of the characteristic polynomial, coincide with $P^*(z)$.*

Proof: See Appendix A.4. The proof in Appendix A.4 is essentially a generalization of the proof of Proposition 2. Like the proof of Proposition 2, it uses Lemma 2. Unlike the proofs of Propositions 1-2 and 4, however, it applies Lemma 1 not only once (to $Q(z)$ and $[-z^{\max(1,\ell-\gamma)}W_j(z)]_{j\in J}$), but twice (also to $[W_j(z)]_{j\in J}$); I explain why below. ■

Proposition 6 says that even in the presence of inside lags of any length, the policymaker can still ensure determinacy and still control the anticipation and convergence rates with a non-superinertial rule, provided that Condition (24) is met.

4.2 About Condition (24)

Proposition 6 applies only if the model and the observation set satisfy Condition (24). The role of Condition (24) is the following. Under this condition, the structural equations imply a relationship between the policy instrument and the observed variable(s) that crucially involves the policy instrument only at a *single* date, like the relationship (4) in Section 2. This relationship can be obtained by applying Lemma 1 to $[W_j(z)]_{j\in J}$, as I do in Appendix A.4. Adding this relationship (without the expectation operators) to a policy-instrument rule leaves the characteristic polynomial unchanged. The fact that the relationship involves the policy instrument only at a single date implies that Lemma 2 can then be applied to exploit these degrees of freedom and find a non-superinertial rule, as in Subsection 2.3.

As I wrote above, Condition (24) is stronger than Condition (22), which is its counterpart in Proposition 4. In particular, unlike Condition (22), Condition (24) is not implied by the weak Sargent-Wallace property of Proposition 5. To get a sense of how restrictive Condition (24) is, and to gain further insight into this condition, I investigate, in the Online Appendix, whether it is met in four off-the-shelf monetary- or fiscal-policy models, depending on the observation set J and the structural-parameter values. One of these models is the basic New Keynesian model (Woodford, 2003, Galí, 2015); the other models are the three models considered in the previous section, i.e. those of Svensson (1997) and Ball (1999), Smets and Wouters (2007), and Schmitt-Grohé and Uribe (1997).

I show that in all these models, for all structural-parameter values (except possibly a zero-measure subset of values), Condition (24) is *violated* for all but two *singleton* observation sets, and it is *satisfied* for all but six *non-empty and non-singleton* observation sets. I establish this result analytically in the basic New Keynesian model and the model of Svensson (1997) and Ball (1999), and with the help of the symbolic-computation software Mathematica in the models of Smets and Wouters (2007) and Schmitt-Grohé and Uribe (1997).

For singleton observation sets, the two exceptions correspond to the case in which the central bank reacts only to inflation in the basic New Keynesian model and in the model of Svensson (1997) and Ball (1999). In the basic New Keynesian model, we have seen in Section 2 that the structural equations (1)-(2) imply the relationship (4) between the interest rate and inflation, which involves the interest rate only at a single date. A similar relationship can be obtained in the model of Svensson (1997) and Ball (1999). In these two models, however, the structural equations do not imply any relationship between the interest rate and output that would involve the interest rate only at a single date. Similarly, in the other two models, the structural equations do not imply any relationship between the policy instrument and any single variable that would involve the policy instrument only at a single date.

For non-empty and non-singleton observation sets, the six exceptions are six two-element sets in Smets and Wouters' (2007) model. One of them, for instance, corresponds to the case in which the central bank would react only to investment and the value of capital in this model. The reason is that, in this model, the investment Euler equation is a relationship between only these two variables; it does not involve any other variable set by the private sector, nor the policy instrument (which is the interest rate). The other structural equations can be combined together to get another relationship involving these two variables and no other variable set by the private sector. This second relationship, however, involves the interest rate at different dates. As a result, no combinations of these two relationships involve the interest rate at a single date.

For any couple of observed variables other than these six in Smets and Wouters' (2007) model, and for any couple of observed variables in the other models, Condition (24) is satisfied. In the basic New Keynesian model, for instance, the (consumption) Euler equation (1) is a relationship between inflation, output, and the interest rate that involves the interest rate only at a single date. Similarly, for any set of three or more observed variables in any model that has a sufficient number of variables, Condition (24) is satisfied.

5 Outside lags

In this section, I extend the results of the previous two sections to models with outside lags. In these models, current variables depend on *past* expectations (rather than *current* expectations, as previously). I show how to rewrite the dynamic system (composed of the model and the rule) as an equivalent system involving only current expectations, to which the results of the previous sections can then be applied. I distinguish between two kinds of outside lags: non-distributed outside lags, which involve expectations formed at a single past horizon, and distributed outside lags, which involve expectations formed at several horizons.[9] I start with the former, and end with the latter.

5.1 Non-distributed outside lags

Extending the results of the previous two sections to models with non-distributed outside lags amounts to generalizing the analysis in Subsection 2.4. This generalization is straightforward. In the presence of non-distributed outside lags of length $\ell' \in \mathbb{N} \setminus \{0\}$, the private sector decides on its actions a single fixed number of periods ℓ' in advance, so the structural equations (19) become

$$\mathbb{E}_{t-\ell'}\left\{\mathbf{\Delta}\left(L^{-1}\right)\left[\mathbf{A}\left(L\right)\mathbf{X}_t + L^{-\gamma}\mathbf{B}\left(L\right)i_t\right]\right\} + \mathbf{A}_{\Delta}\left(\mathbf{X}_t - \mathbb{E}_{t-\ell'}\left\{\mathbf{X}_t\right\}\right) = \mathbf{0}, \tag{26}$$

where $\mathbf{A}_{\Delta}$ denotes the coefficient of z^0 in the "Laurent polynomial" $\mathbf{\Delta}(z^{-1})\mathbf{A}(z)$, with $\det(\mathbf{A}_{\Delta}) \neq 0$. These equations express the current endogenous variables $\mathbf{X}_t$ as a function of expectations formed at date $t-\ell'$. In this sense, $\mathbf{X}_t$ is predetermined at date $t-\ell'$, and could not be affected by current or recent policy shocks (if such shocks were to be introduced into the policy-instrument rule). When $\ell' = 0$, the equation (26) is the same as (19).

I consider the class of rules of type (20), and hence I allow for inside lags of length ℓ. To get rid of the past-expectation terms in (26), I introduce the variables

$$\text{(i) } \tilde{\mathbf{X}}_t := \mathbb{E}_t\{\mathbf{X}_{t+\ell'}\}, \text{ (ii) } \tilde{i}_t := \mathbb{E}_t\{i_{t+\ell'}\}, \tag{27}$$

and I rewrite the system composed of (20) and (26)-(27) as the following system:

$$\mathbb{E}_t\left\{\mathbf{\Delta}\left(L^{-1}\right)\left[\mathbf{A}\left(L\right)\tilde{\mathbf{X}}_t + L^{-\gamma}\mathbf{B}\left(L\right)\tilde{i}_t\right]\right\} = \mathbf{0}, \tag{28}$$

$$\rho(L)\tilde{i}_t = \boldsymbol{\phi}(L)\tilde{\mathbf{X}}_{t-\max(\ell,\gamma+1)}, \tag{29}$$

$$\text{(i) } \mathbf{X}_t = \tilde{\mathbf{X}}_{t-\ell'}, \text{ and (ii) } i_t = \tilde{i}_{t-\ell'}. \tag{30}$$

The two systems are equivalent to each other. To see why the former system implies the latter, note that the structural equations (26) imply that $\mathbf{X}_t$ is determined at date $t-\ell'$, i.e. $\mathbf{X}_t = \mathbb{E}_{t-\ell'}\{\mathbf{X}_t\}$, which is (30.i). In turn, if $\mathbf{X}_t$ is determined at date $t-\ell'$, then the rule (20) implies that i_t is determined at date $t-\max(\ell,\gamma+1)-\ell'$, and hence that i_t is *already* determined at date $t-\ell'$, i.e. $i_t = \mathbb{E}_{t-\ell'}\{i_t\}$, which is (30.ii). Finally, replacing $(\mathbf{X}_t, i_t)$ with $(\tilde{\mathbf{X}}_{t-\ell'}, \tilde{i}_{t-\ell'})$ in (20) and (26) gives (28)-(29) at date $t-\ell'$, and therefore also (28)-(29) at date t.

To see why, conversely, the latter system implies the former, note that (30.i) implies that $\mathbf{X}_t$ is determined at date $t-\ell'$, and therefore that $\mathbf{X}_{t+\ell'}$ is determined at date t, i.e. $\mathbf{X}_{t+\ell'} = \mathbb{E}_t\{\mathbf{X}_{t+\ell'}\}$, which gives (27.i). Similarly for i_t, (30.ii) implies (27.ii). Finally, replacing $(\tilde{\mathbf{X}}_t, \tilde{i}_t)$

[9] I focus on *finitely* distributed outside lags, which involve a finite number of past-expectation terms. Infinitely distributed outside lags, such as those considered in Mankiw and Reis (2002), raise specific difficulties whose study is beyond the scope of this paper.

with $(\mathbf{X}_{t+\ell'}, i_{t+\ell'})$ in (28)-(29) gives (20) and (26) at date $t+\ell'$, and therefore also (20) and (26) at date t.

The system (28)-(30) is block-recursive: the sub-system (28)-(29) determines $(\tilde{\mathbf{X}}_t, \tilde{i}_t)$, uniquely or not; and, for any value of $(\tilde{\mathbf{X}}_t, \tilde{i}_t)$, the sub-system (30) determines $(\mathbf{X}_t, i_t)$ uniquely.

So, there is a unique solution in $(\mathbf{X}_t, i_t)$ to the system composed of (20) and (26) if and only if there is a unique solution in $(\tilde{\mathbf{X}}_t, \tilde{i}_t)$ to the system (28)-(29). In turn, there is a unique solution in $(\tilde{\mathbf{X}}_t, \tilde{i}_t)$ to the system (28)-(29) if and only if there is a unique solution in $(\mathbf{X}_t, i_t)$ to the system composed of (19)-(20), because these two systems are identical to each other. So, we are back to the situation without outside lags (and with inside lags). Thus, non-distributed outside lags do not affect any of the results of the previous sections, no matter the length of these lags:

Proposition 7 (Extension of Propositions 4-6 to non-distributed outside lags): *Propositions 4-6 still hold if "model of type (19)" is replaced by "model of type (26) with any outside-lags length $\ell' \in \mathbb{N}$" in these propositions.*

The reason for this neutrality exactly the same as in Subsection 2.4.

5.2 Distributed outside lags

I now turn to distributed outside lags. I start with two simple examples of such lags. The goal of these examples is to illustrate how the dynamic system (composed of the model and the rule) can be rewritten as a system involving only current expectations, simply by introducing new variables and increasing the dimension of the system. The new variables are the private sector's expectations whose past values affect current variables; they are not directly observed by the policymaker; in this sense, they are "latent" variables.

The first example is the following. Starting from the basic New Keynesian model (considered in Section 2), I relax the assumption that all firms decide on their price contemporaneously. Instead, I assume that only a fraction $\omega \in (0,1)$ of them decide on their price contemporaneously, while the remaining fraction $1-\omega$ decide on their price one period in advance. As a result, the Phillips curve (2) becomes

$$\pi_t = \omega \left(\beta \mathbb{E}_t\{\pi_{t+1}\} + \kappa y_t \right) + (1-\omega)\, \mathbb{E}_{t-1}\left\{ \beta \pi_{t+1} + \kappa y_t \right\}. \tag{31}$$

The IS equation (1) is unchanged. In this example, outside lags do not entirely cancel the contemporaneous effect of policy shocks on inflation (if such shocks were introduced into the interest-rate rule); they just dampen this effect, all the more so as ω is small.

To rewrite the structural equations (1) and (31) in a form of type (19), I first need to get rid of the past-expectation term in (31). To do so, I introduce the variable

$$x_t := \mathbb{E}_t \left\{ \beta \pi_{t+2} + \kappa y_{t+1} \right\}, \tag{32}$$

and I rewrite (31) as

$$\pi_t = \omega\left(\beta \mathbb{E}_t\{\pi_{t+1}\} + \kappa y_t\right) + (1-\omega)\, x_{t-1}. \tag{33}$$

Next, in order to satisfy the assumption $\det[\mathbf{A}(0)] \neq 0$ (which systems of type (19) are required to satisfy), I apply the operator $\mathbb{E}_t\{.\}$ to the left- and right-hand sides of (33) taken at date $t+1$, and I use (32), to get

$$\mathbb{E}_t\{\pi_{t+1}\} = x_t. \tag{34}$$

I then use (34) to replace $\mathbb{E}_t\{\pi_{t+1}\}$ with x_t in (33):

$$\pi_t = \omega\left(\beta x_t + \kappa y_t\right) + (1-\omega)\, x_{t-1}. \tag{35}$$

The system composed of (1) and (31)-(32) is equivalent to the system composed of (1) and (34)-(35). The latter system is of type (19) with $n = 3$, $\mathbf{X}_t := [\,y_t \quad \pi_t \quad x_t\,]^\top$, $\mathbf{\Delta}(z) := \mathrm{diag}(z, 1, z)$, $\gamma = 0$,

$$\mathbf{A}(z) := \begin{bmatrix} 1-z & \frac{1}{\sigma} & 0 \\ \kappa\omega & -1 & \beta\omega + (1-\omega)z \\ 0 & 1 & -z \end{bmatrix}, \text{ and } \mathbf{B}(z) := \begin{bmatrix} \frac{-z}{\sigma} \\ 0 \\ 0 \end{bmatrix}.$$

This system satisfies the three assumptions made in Subsection 3.1: (i) $\det[\mathbf{A}(0)] = -\beta\omega \neq 0$; (ii) $\mathbf{B}(z) \neq \mathbf{0}$; and (iii) $W_1(z) = (\omega/\sigma)z(\beta - z) \neq 0$, $W_2(z) = (\kappa\omega/\sigma)z^2 \neq 0$, and $W_3(z) = -(\kappa\omega/\sigma)z \neq 0$. So, I can apply Propositions 4-6 to this system, for observation sets that do not include the unobserved latent variable x_t (i.e. for observation sets J such that $3 \notin J$).

The second example is the mirror image of the first one, with simple distributed outside lags this time for households rather than firms. More specifically, starting from the basic New Keynesian model, I now assume that only a fraction $\omega \in (0,1)$ of households decide on their actions contemporaneously, while the remaining fraction $1-\omega$ decide on their actions one period in advance. As a result, the IS equation (1) becomes

$$y_t = \omega\left[\mathbb{E}_t\{y_{t+1}\} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t\{\pi_{t+1}\}\right)\right] + (1-\omega)\,\mathbb{E}_{t-1}\left\{y_{t+1} - \frac{1}{\sigma}\left(i_t - \pi_{t+1}\right)\right\}. \tag{36}$$

The Phillips curve (2) is unchanged. To rewrite the structural equations (2) and (36) in a form of type (19), I proceed in essentially the same way as previously (introducing this time the latent variable $x_t := \mathbb{E}_t\{y_{t+2} + (1/\sigma)\pi_{t+2}\}$). There is, however, one importance difference with the previous example. This difference is due to the presence of a $\mathbb{E}_{t-1}\{i_t\}$ term in (36). To get rid of this term, I consider a class of rules that express the current interest rate as a function of only past variables (even in the absence of inside lags), so that I can replace $\mathbb{E}_{t-1}\{i_t\}$ with i_t. I relegate the details of this rewriting to Appendix A.5; the outcome is the system is of type (19) with $n = 3$, $\mathbf{X}_t := [\,y_t \quad \pi_t \quad x_t\,]^\top$, $\mathbf{\Delta}(z) := \mathrm{diag}(z, z, 1)$, $\gamma = 1$,

$$\mathbf{A}(z) := \begin{bmatrix} \kappa z & \beta - z & 0 \\ 1 & 0 & -z \\ 1 + \frac{\kappa\omega}{\beta\sigma} & \frac{-\omega}{\beta\sigma} & -\omega - (1-\omega)z \end{bmatrix}, \text{ and } \mathbf{B}(z) := \begin{bmatrix} 0 \\ \frac{z}{\sigma} \\ \frac{\omega}{\sigma} + \frac{z}{\sigma} \end{bmatrix}.$$

This system satisfies the three assumptions made in Subsection 3.1: (i) $\det[\mathbf{A}(0)] = \beta\omega \neq 0$; (ii) $\mathbf{B}(z) \neq \mathbf{0}$; and (iii) $W_1(z) = (\omega/\sigma)z^2(z-\beta) \neq 0$, $W_2(z) = -(\kappa\omega/\sigma)z^3 \neq 0$, and $W_3(z) =$

$(\omega/\sigma)[(1+\kappa/\sigma)z-\beta] \neq 0$. Moreover, since $\gamma = 1$ in this system, the corresponding rules of type (20) express the current interest rate as a function of only past variables (even in the absence of inside lags), as assumed above. So, as previously, I can apply Propositions 4-6 to this system, for observation sets that do not include the unobserved latent variable x_t (i.e. for observation sets $J \in \{\{1\},\{2\},\{1,2\}\}$).

These simple examples illustrate how a dynamic system with distributed outside lags (involving expectations formed at different dates) can be rewritten as a system of type (19)-(20) (involving only current expectations) with some unobserved latent variables. In the latter system, the observation sets J that can be considered are those that exclude latent variables. So, Propositions 4-6 can straightforwardly be extended to models with distributed outside lags as follows:

Proposition 8 (Extension of Propositions 4-6 to distributed outside lags): *Consider a model with distributed outside lags, which can be written in a form of type (19) with a set $\tilde{J}$ of latent variables. Then Propositions 4-6 still hold for this model and can be applied to any observation set J such that $J \cap \tilde{J} = \emptyset$.*

I now apply Propositions 4 and 6 to the two examples above, for each observation set $J \in \{\{1\},\{2\},\{1,2\}\}$. In both examples, we have $|Q(z)| = \omega|z^2 - (1+\beta+\kappa/\sigma)z + \beta|$. The roots of $W_1(z)$ and $W_2(z)$ are 0 and β; none of them is a root of $Q(z)$, since $|Q(0)| = \beta\sigma \neq 0$ and $|Q(\beta)| = \beta\kappa \neq 0$. So, for any observation set $J \in \{\{1\},\{2\},\{1,2\}\}$, $D_J(z)$ has no roots at all, and hence Condition (22) is met. Moreover, Condition (24) is not met for $J = \{1\}$ (since $W_1(\beta) = 0$); it is met for $J = \{2\}$ (since the only root of $W_2(z)$ is 0); and it is met for $J = \{1,2\}$ (since $W_1(z)$ and $W_2(z)$ have no roots in common except 0). Thus, in both examples, the central bank can ensure determinacy and control the anticipation and convergence rates in the presence of inside lags of any length, with an interest-rate rule involving only inflation, or only output, or both inflation and output (Proposition 4); and she can do so with a non-superinertial interest-rate rule involving only inflation or both inflation and output (Proposition 6).

6 Conclusion

Macroeconomic stabilization policy is notoriously subject to inside and outside lags. Can these lags prevent the policymaker from ensuring determinacy and controlling the anticipation and convergence rates, for a given set of observed variables? This paper has provided a negative answer to this question, in a broad class of dynamic rational-expectations models that allows for inside and outside lags of any length. To establish this result, I have inverted the problem usually addressed in the literature: I have started from a targeted characteristic polynomial, and I have derived a corresponding policy-instrument rule. This approach has enabled me to establish much more general results about determinacy and control with inside and outside lags than were previously established in some sparse examples in the literature. For any lags, moreover, the method I have used offers some degrees of freedom that can be exploited to design rules with additional properties; I have illustrated this possibility by designing non-superinertial rules, which the literature suggests may be more robust under model uncertainty.

Determinacy, anticipation rates, and convergence rates are important features of economic dynamics, but they are certainly not the only ones. Another important feature is, of course, the contemporaneous impact of shocks on the economy, and we know that inside and outside lags do restrict the ability of the policymaker to control this feature. In this paper, I have abstracted from shocks for two main reasons. First, addressing the issue of how best to respond to shocks in the presence of lags would require to take a stand on the (ideally model-dependent) objective of the policymaker, making it harder to establish general results. Second, to the extent that the policymaker observes the shocks (with inside lags), the issue of how best to respond to shocks in the presence of lags is independent of the issues I have addressed in the paper (since adding any exogenous term to the policy-instrument rules I have designed in this paper does not affect any of the results about determinacy and the anticipation and convergence rates). I leave a more complete analysis of the implications of inside and outside lags for stabilization policy for future research.

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Appendix

In this appendix, I prove Lemmas 2 and 3, and Propositions 4 and 6. I also show, in the second example with distributed outside lags, how to rewrite the structural equations in a form of type (19).

A.1 Proof of Lemma 2

I proceed in three steps. In the first step, I show that Lemma 2 is implied by the following lemma:

Lemma 2A: $\forall \tilde{U}(z) \in \mathbb{R}[z]$, *$\exists \tilde{V}(z) \in \mathbb{R}[z]$ such that $z^{\deg(\tilde{V})+1}\tilde{U}(z) + \tilde{V}(z)$ has no roots outside $\mathcal{C}$.*

To prove this implication, I start from an arbitrary $U(z) \in \mathbb{R}[z]$ such that $U(0) \neq 0$. Let $\tilde{U}(z) := z^{\deg(U)}U(z^{-1})$ denote the reciprocal polynomial of $U(z)$. Lemma 2A implies the existence of $\tilde{V}(z) \in \mathbb{R}[z]$ such that $\tilde{O}(z) := z^{\deg(\tilde{V})+1}\tilde{U}(z) + \tilde{V}(z)$ has no roots outside $\mathcal{C}$. Therefore, $O(z) := z^{\deg(\tilde{O})}\tilde{O}(z^{-1})$, the reciprocal polynomial of $\tilde{O}(z)$, has no roots inside $\mathcal{C}$. Since $U(0) \neq 0$, we have $\deg(\tilde{U}) = \deg(U)$ and $U(z) = z^{\deg(\tilde{U})}\tilde{U}(z^{-1})$. As a consequence, $O(z) = U(z) + z^{\deg(U)+1}V(z)$, where $V(z) := z^{\deg(\tilde{V})}\tilde{V}(z^{-1})$ is the reciprocal polynomial of $\tilde{V}(z)$. So, there exists $V(z)$ such that $U(z) + z^{\deg(U)+1}V(z)$ has no roots inside $\mathcal{C}$. Thus, Lemma 2A implies Lemma 2.

In the second step, I show that Lemma 2A is equivalent to another lemma. Consider some arbitrary $\tilde{U}(z)$ and $\tilde{V}(z)$ in $\mathbb{R}[z]$. I focus on the non-trivial case in which $\tilde{U}(z) \neq 0$ and I assume, without any loss in generality, that the coefficient of $z^{\deg(\tilde{U})}$ in $\tilde{U}(z)$ is equal to one. Let $(\alpha_j)_{1\leq j\leq m} \in \mathbb{C}^m$ denote the roots of $\tilde{O}(z) := z^{\deg(\tilde{V})+1}\tilde{U}(z)+\tilde{V}(z)$ counted with their multiplicity. We have $\tilde{O}(z) = z^m + \sum_{k=1}^{m}(-1)^k\Sigma_k z^{m-k}$, where $\Sigma_k := \sum_{1\leq j_1<j_2<...<j_k\leq m}\alpha_{j_1}\alpha_{j_2}...\alpha_{j_k}$ for all $k \in \{1, ..., m\}$. Let $S_k := \sum_{j=1}^{m}\alpha_j^k$ for all $k \in \{1, ..., m\}$. For any $K \in \{1, ..., m\}$, Newton's identities $S_k - \Sigma_1 S_{k-1} + \Sigma_2 S_{k-2} - ... + (-1)^k\Sigma_k k = 0$ for $k \in \{1, ..., K\}$ give by recurrence Σ_1, ..., Σ_K as functions of $(S_j)_{1\leq j\leq K}$ and, conversely, S_1, ..., S_K as functions of $(\Sigma_j)_{1\leq j\leq K}$; moreover, these functions are polynomial functions with real-number coefficients. Lemma 2A is therefore equivalent to the following lemma:

Lemma 2B: *$\forall K \in \mathbb{N}\setminus\{0\}$, $\forall(s_1, ..., s_K) \in \mathbb{R}^K$, $\exists m \in \mathbb{N}\setminus\{0\}$ and $\exists(\alpha_1, ..., \alpha_m) \in \mathbb{C}^m$ such that: (i) $m > K$, (ii) $\forall j \in \{1, ..., m\}$, $|\alpha_j| < 1$, (iii) $\forall k \in \{1, ..., K\}$, $\sum_{j=1}^{m}\alpha_j^k = s_k$, and (iv) all the coefficients of the polynomial $\prod_{j=1}^{m}(z-\alpha_j)$ are real numbers.*

In the third step, I prove Lemma 2B. Consider an arbitrary $K \in \mathbb{N}\setminus\{0\}$ and some arbitrary $(s_1, ..., s_K) \in \mathbb{R}^K$. For any $k \in \{1, ..., K\}$ and any finite set A of complex numbers, let $S_k^A := \sum_{a\in A} a^k$. For any finite sets of complex numbers A and B, we have $S_k^{A\sqcup B} = S_k^A + S_k^B$, where $A \sqcup B$ denotes the union of A and B counting each element with its multiplicity (so that in particular $\#(A\sqcup B) = \#A + \#B$). We also have that for any $\lambda \in \mathbb{C}$, if $B := \{\lambda a | a \in A\}$, then $S_k^B = \lambda^k S_k^A$.

Consider an arbitrary $k \in \{1, ..., K\}$. Let $S_{k,k} := s_k$ and $S_{k,j} := 0$ for $j \in \{1, ..., K\}\setminus\{k\}$. Let me define implicitly, by recurrence, $\Sigma_{k,j}$ for $j \in \{1, ..., K\}$ by $S_{k,j} - \Sigma_{k,1}S_{k,j-1} + \Sigma_{k,2}S_{k,j-2} - ... + (-1)^j\Sigma_{k,j}j = 0$ for $j \in \{1, ..., K\}$. Let $\tilde{O}_k(z) := z^K - \Sigma_{k,1}z^{K-1} + \Sigma_{k,2}z^{K-2} - ... + (-1)^K\Sigma_{k,K}$. Finally, let $\gamma_{k,1}$, ..., $\gamma_{k,K}$ denote the roots of $\tilde{O}_k(z)$ counted with their multiplicity, and let $A_k := \{\gamma_{k,1}, ..., \gamma_{k,K}\}$. Since all the coefficients of $\tilde{O}_k(z)$ are real numbers, we have $A_k = \{\overline{a} | a \in A_k\}$, where for any $a \in \mathbb{C}$, $\overline{a}$ denotes the complex conjugate of a. We also have, by construction, $S_k^{A_k} = s_k$ and $S_j^{A_k} = 0$ for $j \in \{1, ..., K\}\setminus\{k\}$.

Let $r_k \in \mathbb{N}$ be such that $r_k > \max\{|\gamma_{k,j}|\,|1 \leq j \leq K\}$. The set $B_k := \{\gamma_{k,1}/r_k, ..., \gamma_{k,K}/r_k\}$ is such that: (i) each of its elements is a complex number whose modulus is strictly lower than one; (ii) $B_k = \{\overline{b} | b \in B_k\}$; and (iii) $S_k^{B_k} = s_k/r_k^k$ and $S_j^{B_k} = 0$ for $j \in \{1, ..., K\}\setminus\{k\}$. Therefore, the set C_k, defined as the union of r_k^k times set B_k, is such that: (i) each of its elements is a complex number whose modulus is strictly lower than one; (ii) $C_k = \{\overline{c} | c \in C_k\}$; and (iii) $S_k^{C_k} = s_k$ and $S_j^{C_k} = 0$ for $j \in \{1, ..., K\}\setminus\{k\}$.

Finally, the set $C := \bigsqcup_{1\leq k\leq K} C_k$, whose cardinality is noted m and whose elements are noted α_j for $j \in \{1, ..., m\}$, is such that: (i) $m \in \mathbb{N}\setminus\{0\}$ and $m = K\sum_{k=1}^{K} r_k^k > K$; (ii) $\forall j \in \{1, ..., m\}$,

$\alpha_j \in \mathbb{C}$ and $|\alpha_j| < 1$; (iii) $\forall k \in \{1,...,K\}$, $S_k^C = \sum_{j=1}^m \alpha_j^k = s_k$; and (iv) $C = \{\overline{\alpha}|\alpha \in C\}$, so that all the coefficients of the polynomial $\prod_{j=1}^m (z-\alpha_j)$ are real numbers. This result proves Lemma 2B and, therefore, Lemma 2A too. Lemma 2 follows.

A.2 Proof of Lemma 3

For $\rho(z) = 1$, Lemma 3 directly follows from Lemma 1 in Loisel (2022), applied to $\phi\mathbf{V}(z) = \boldsymbol{\phi}(z)$ and $h = -\max(\ell, \gamma+1)$. Using the notations in Loisel (2022), I get $h - m \le -(\gamma+1) - (\omega-\gamma) = -\omega - 1 < 0$; therefore, the number of non-predetermined variables is δ, and the reciprocal polynomial of the characteristic polynomial is $Q(z) - \phi W(z) z^{\max(1,\ell-\gamma)}$. Using the Laplace expansion, I rewrite this polynomial as $Q(z) - z^{\max(1,\ell-\gamma)} \sum_{j\in J} W_j(z)\boldsymbol{\phi}(z)\mathbf{e}_j$.

For an arbitrary $\rho(z) \in \mathbb{R}[z]$ satisfying $\rho(0) \neq 0$ and having no roots on $\mathcal{C}$, Lemma 3 directly follows from Proposition 9 and its proof in Loisel (2022).

A.3 Proof of Proposition 4

The proof of Proposition 4 is essentially a generalization of the proof of Proposition 1. For any non-empty $J \subseteq \{1,...,n\}$ satisfying (22) and any $P^*(z)$ satisfying (23), I design some rules of type (20) satisfying the observation constraint (21) as well as

$$\text{(i) } \rho(0) \neq 0, \ \text{(ii) } P(z) = P^*(z), \ \text{(iii) } \boldsymbol{\phi}(z) \neq \mathbf{0}, \ \text{and} \\ \text{(iv) } \rho(z) \text{ and } \boldsymbol{\phi}(z) \text{ have no common roots inside } \mathcal{C}, \quad \text{(A.1)}$$

where (A.1) is a straightforward generalization of (6) (in the sense that (A.1) simply replaces $\phi(z)$ with $\boldsymbol{\phi}(z)$ in (6)). Since $Q(0) = \det[\mathbf{A}(0)] \neq 0$, we have $D_J(z) := \gcd\{Q(z), [W_j(z)]_{j\in J}\} = \gcd\{Q(z), [-z^{\max(1,\ell-\gamma)} W_j(z)]_{j\in J}\}$. Therefore, Lemma 1 implies the existence of $\mathcal{B}_j(z) \in \mathbb{R}[z]$ for $j \in \{0\} \cup J$ such that

$$Q(z)\mathcal{B}_0(z) - z^{\max(1,\ell-\gamma)} \sum_{j\in J} W_j(z)\mathcal{B}_j(z) = D_J(z). \quad \text{(A.2)}$$

Multiplying the left- and right-hand sides of (A.2) by $\tilde{P}^*(z) := P^*(z)/D_J(z)$ leads to

$$Q(z)\mathcal{B}_0(z)\tilde{P}^*(z) - z^{\max(1,\ell-\gamma)} \sum_{j\in J} W_j(z)\mathcal{B}_j(z)\tilde{P}^*(z) = P^*(z).$$

In order to satisfy Blanchard and Kahn's (1980) *no-decoupling* condition, I introduce some arbitrary polynomials $\mathcal{K}_j(z) \in \mathbb{R}[z]$ for $j \in J$, and I rewrite the previous equation as

$$Q(z)\left[\mathcal{B}_0(z)\tilde{P}^*(z) + z^{\max(1,\ell-\gamma)} \sum_{j\in J} \mathcal{K}_j(z)W_j(z)\right] - \\ z^{\max(1,\ell-\gamma)} \sum_{j\in J} W_j(z)\left[\mathcal{B}_j(z)\tilde{P}^*(z) - \mathcal{K}_j(z)Q(z)\right] = P^*(z). \quad \text{(A.3)}$$

Thus, except for a zero-measure set of polynomials $[\mathcal{K}_j(z)]_{j\in J}$, the rules of type (20) with

$$\begin{aligned} \rho(z) &= \mathcal{B}_0(z)\tilde{P}^*(z) + z^{\max(1,\ell-\gamma)} \sum_{j\in J} \mathcal{K}_j(z)W_j(z) \\ \boldsymbol{\phi}(z) &= \sum_{j\in J} \left[\mathcal{B}_j(z)\tilde{P}^*(z) - \mathcal{K}_j(z)Q(z)\right]\mathbf{e}_j^\top \end{aligned}$$

satisfy Conditions (21), (A.1.ii) (given Lemma 3), and (A.1.iii)-(A.1.iv). They also satisfy Condition (A.1.i) because $\mathcal{B}_0(0) \neq 0$ (as follows from (A.2), $Q(0) \neq 0$, and $D_J(0) \neq 0$) and $\tilde{P}^*(0) \neq 0$ (due to (23.i)). Finally, these rules are arithmetically derivable because the different steps of their design process (in particular getting $\mathcal{B}_j(z) \in \mathbb{R}[z]$ for $j \in \{0\} \cup J$ with the Euclidean algorithm) involve only a finite number of arithmetic operations. Proposition 4 follows.

A.4 Proof of Proposition 6

Condition (24) implies $\gcd\{[W_j(z)]_{j \in J}\} = z^\chi$. So, Lemma 1 implies the existence of $\tilde{\mathcal{B}}_j(z) \in \mathbb{R}[z]$ for $j \in J$ such that

$$\sum\nolimits_{j \in J} W_j(z)\tilde{\mathcal{B}}_j(z) = z^\chi. \tag{A.4}$$

Multiplying the left- and right-hand sides of (A.4) by an arbitrary polynomial $\tilde{\mathcal{K}}(z) \in \mathbb{R}[z]$ leads to

$$\sum\nolimits_{j \in J} W_j(z)\tilde{\mathcal{B}}_j(z)\tilde{\mathcal{K}}(z) = z^\chi \tilde{\mathcal{K}}(z). \tag{A.5}$$

Condition (24) and $Q(0) \neq 0$ together imply $D_J(z) = 1$, which in turn implies Condition (22). So, under Condition (24), Proposition 4 and its proof (in Appendix A.3) remain valid, with $D_J(z) = 1$ and, therefore, with $\tilde{P}^*(z) = P^*(z)$ too. Replacing $\tilde{P}^*(z)$ with $P^*(z)$ and choosing $\mathcal{K}_j(z) = \tilde{\mathcal{B}}_j(z)\tilde{\mathcal{K}}(z)$ for $j \in J$ in (A.3) leads to

$$\begin{aligned} Q(z)\left[\mathcal{B}_0(z)P^*(z) + z^{\max(1,\ell-\gamma)+\chi}\tilde{\mathcal{K}}(z)\right] - \\ z^{\max(1,\ell-\gamma)} \sum\nolimits_{j \in J} W_j(z)\left[\mathcal{B}_j(z)P^*(z) - \tilde{\mathcal{B}}_j(z)\tilde{\mathcal{K}}(z)Q(z)\right] = P^*(z). \end{aligned} \tag{A.6}$$

Let $h := \max(1, \ell - \gamma) + \chi - \deg(\mathcal{B}_0) - \deg(P^*) - 1 \in \mathbb{Z}$. I apply Lemma 2 to $U(z) = \mathcal{B}_0(z)P^*(z) + \mathbb{1}_{h \geq 0} z^{\max(1,\ell-\gamma)+\chi}$, where $\mathbb{1}_{h \geq 0}$ takes the value 1 if $h \geq 0$ and the value 0 if $h \leq -1$. I obtain the existence of $V(z) \in \mathbb{R}[z]$ such that

$$\rho_V(z) := \mathcal{B}_0(z)P^*(z) + \mathbb{1}_{h \geq 0} z^{\max(1,\ell-\gamma)+\chi} + z^{\deg(\mathcal{B}_0)+\deg(P^*)+1+\max(0,h+1)}V(z)$$

has no roots inside $\mathcal{C}$. Let $\tilde{V}(z)$ be an arbitrary polynomial that is sufficiently close to $V(z)$ for $\rho_{\tilde{V}}(z)$ to have no roots inside $\mathcal{C}$. Choosing $\tilde{\mathcal{K}}(z) = \mathbb{1}_{h \geq 0} + z^{\max(1,-h)}\tilde{V}(z)$ in (A.6) then leads to

$$Q(z)\rho_{\tilde{V}}(z) - z^{\max(1,\ell-\gamma)} \sum\nolimits_{j \in J} W_j(z)\boldsymbol{\phi}_{\tilde{V}}(z)\mathbf{e}_j = P^*(z),$$

where

$$\boldsymbol{\phi}_{\tilde{V}}(z) := \sum\nolimits_{j \in J} \left\{\mathcal{B}_j(z)P^*(z) - \tilde{\mathcal{B}}_j(z)Q(z)\left[\mathbb{1}_{h \geq 0} + z^{\max(1,-h)}\tilde{V}(z)\right]\right\}\mathbf{e}_j^\top.$$

Thus, the rule of type (20) with $\rho(z) = \rho_{\tilde{V}}(z)$ and $\boldsymbol{\phi}(z) = \boldsymbol{\phi}_{\tilde{V}}(z)$ satisfies Conditions (9), (21), (A.1.ii) (given Lemma 3), and (A.1.iv) (since (9) implies (A.1.iv)). It also satisfies Condition (A.1.i) (because $\mathcal{B}_0(0) \neq 0$ and $P^*(0) \neq 0$). Finally, it satisfies Condition (A.1.iii) except possibly for a zero-measure set of polynomials $\tilde{V}(z)$. Proposition 6 follows.

A.5 Second example with distributed outside lags

To rewrite the structural equations (2) and (36) in a form of type (19), I first need to get rid of past-expectation terms in (36). To do so, I introduce the latent variable

$$x_t := \mathbb{E}_t \left\{ y_{t+2} + \frac{1}{\sigma} \pi_{t+2} \right\}, \tag{A.7}$$

and I consider a class of rules that express the current interest rate as a function of only past variables (even in the absence of inside lags), implying

$$\mathbb{E}_{t-1}\{i_t\} = i_t. \tag{A.8}$$

Using (A.7)-(A.8), I rewrite (36) as

$$y_t = \omega \left(\mathbb{E}_t\{y_{t+1}\} + \frac{1}{\sigma} \mathbb{E}_t\{\pi_{t+1}\} \right) + (1-\omega)\, x_{t-1} - \frac{1}{\sigma} i_t. \tag{A.9}$$

Next, in order to satisfy the assumption $\det[\mathbf{A}(0)] \neq 0$ (which systems of type (19) are required to satisfy), I apply the operator $\mathbb{E}_t\{.\}$ to the left- and right-hand sides of (A.9) taken at date $t+1$, and I use (A.7)-(A.8), to get

$$\mathbb{E}_t\{y_{t+1}\} = x_t - \frac{1}{\sigma} i_{t+1}. \tag{A.10}$$

I then use (2) and (A.10) to replace $\mathbb{E}_t\{y_{t+1}\}$ and $\mathbb{E}_t\{\pi_{t+1}\}$ in (A.9), and get

$$\left(1 + \frac{\kappa\omega}{\beta\sigma}\right) y_t = \frac{\omega}{\beta\sigma} \pi_t + \omega x_t + (1-\omega) x_{t-1} - \frac{\omega}{\sigma} i_{t+1} - \frac{1}{\sigma} i_t. \tag{A.11}$$

The system composed of (2),(36), and (A.7) is equivalent to the system composed of (2) and (A.10)-(A.11). The latter system is the system of type (19) described in the main text.

Online Appendix

In this online appendix, I consider, in turn, four off-the-shelf stabilization-policy models: the basic New Keynesian model (Woodford, 2003, Galí, 2015), and the models of Svensson (1997) and Ball (1999), Smets and Wouters (2007), and Schmitt-Grohé and Uribe (1997). The first three models are monetary-policy models; the last one is a fiscal-policy model.

I write each of these models in a form of type (19). I establish two results that hold for all structural-parameter values (except possibly a zero-measure subset of values). First, for each non-empty observation set J in each model, not only Condition (22) is met, but in fact $\deg(D_J) = 0$; that is to say, not only $D_J(z)$ has at most δ roots inside $\mathcal{C}$, but in fact $D_J(z)$ has no roots at all (neither inside nor outside $\mathcal{C}$). Second, over all the models, Condition (24) is *violated* for all but two *singleton* observation sets, and it is *satisfied* for all but six *non-empty and non-singleton* observation sets. I establish these results analytically in the simplest models, and with the help of the symbolic-computation software Mathematica in the more complex models.[1]

O.1 Basic New Keynesian model

In the basic New Keynesian model (without exogenous shocks), at each date $t \in \mathbb{Z}$, the private sector sets output y_t and inflation π_t according to the following (locally log-linearized) IS equation and Phillips curve:

$$\begin{aligned} y_t &= \mathbb{E}_t\{y_{t+1}\} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t\{\pi_{t+1}\}\right), && \text{(O.1)} \\ \pi_t &= \beta\mathbb{E}_t\{\pi_{t+1}\} + \kappa y_t, && \text{(O.2)} \end{aligned}$$

where $\sigma > 0$, $\beta \in (0,1)$, and $\kappa > 0$ are three parameters. The policymaker is a central bank setting the short-term nominal interest rate i_t. The IS equation and Phillips curve above can straightforwardly be written in a form of type (19) with $n = 2$, $\mathbf{X}_t := [y_t \quad \pi_t]^\top$, $\boldsymbol{\Delta}(z) := \mathrm{diag}(z, z)$, $\gamma = 0$,

$$\mathbf{A}(z) := \begin{bmatrix} \sigma(1-z) & 1 \\ z & (\beta - z)/\kappa \end{bmatrix}, \text{ and } \mathbf{B}(z) := \begin{bmatrix} -z \\ 0 \end{bmatrix}.$$

This system satisfies the three assumptions made in the main text: (i) $\det[\mathbf{A}(0)] = \beta\sigma/\kappa \neq 0$; (ii) $\mathbf{B}(z) \neq \mathbf{0}$; and (iii) $W_1(z) = z(\beta - z)/\kappa \neq 0$ and $W_2(z) = -z^2 \neq 0$.

The roots of $W_1(z)$ and $W_2(z)$ are 0 and β. None of them is a root of $Q(z) = (\sigma/\kappa)[z^2 - (1 + \beta + \kappa/\sigma)z + \beta]$, since $Q(0) = \beta\sigma/\kappa \neq 0$ and $Q(\beta) = -\beta \neq 0$. So, for any observation set $J \in \{\{1\}, \{2\}, \{1,2\}\}$, $D_J(z)$ has no roots at all, and hence Condition (22) is met. Moreover, Condition (24) is not met for $J = \{1\}$ (since $W_1(\beta) = 0$); it is met for $J = \{2\}$ (since the only root of $W_2(z)$ is 0); and it is met for $J = \{1, 2\}$ (since $W_1(z)$ and $W_2(z)$ have no roots in common except 0).

[1] The code is available on my website.

O.2 Model of Svensson (1997) and Ball (1999)

In the model of Svensson (1997) and Ball (1999) (without exogenous shocks), at each date $t \in \mathbb{Z}$, the private sector sets output y_t and inflation π_t according to the following IS equation and Phillips curve:

$$
\begin{aligned}
y_t &= \lambda y_{t-1} - \mu \left(i_{t-1} - \pi_{t-1} \right), \\
\pi_t &= \pi_{t-1} + \chi y_{t-1},
\end{aligned}
$$

where $\lambda \in (0,1)$, $\mu > 0$, and $\chi > 0$ are three parameters. The policymaker is, again, a central bank setting the short-term nominal interest rate i_t. The IS equation and Phillips curve above can straightforwardly be written in a form of type (19) with $n = 2$, $\mathbf{X}_t := [y_t \quad \pi_t]^\top$, $\mathbf{\Delta}(z) := \text{diag}(1,1)$, $\gamma = 0$,

$$
\mathbf{A}(z) := \begin{bmatrix} 1 - \lambda z & -\mu z \\ -\chi z & 1 - z \end{bmatrix}, \text{ and } \mathbf{B}(z) := \begin{bmatrix} \mu z \\ 0 \end{bmatrix}.
$$

This system satisfies the three assumptions made in the main text: (i) $\det[\mathbf{A}(0)] = 1 \neq 0$; (ii) $\mathbf{B}(z) \neq \mathbf{0}$; and (iii) $W_1(z) = -\mu z(1-z) \neq 0$ and $W_2(z) = -\mu \chi z^2 \neq 0$.

The roots of $W_1(z)$ and $W_2(z)$ are 0 and 1. None of them is a root of $Q(z) = (\lambda - \mu\chi) z^2 - (1+\lambda) z + 1$, since $Q(0) = 1 \neq 0$ and $Q(1) = -\mu\chi \neq 0$. So, for any observation set $J \in \{\{1\},\{2\},\{1,2\}\}$, $D_J(z)$ has no roots at all, and hence Condition (22) is met. Moreover, Condition (24) is not met for $J = \{1\}$ (since $W_1(1) = 0$); it is met for $J = \{2\}$ (since the only root of $W_2(z)$ is 0); and it is met for $J = \{1,2\}$ (since $W_1(z)$ and $W_2(z)$ have no roots in common except 0).

O.3 Model of Smets and Wouters (2007)

In the model of Smets and Wouters (2007) (without exogenous shocks), at each date $t \in \mathbb{Z}$, the private sector sets output y_t, consumption c_t, investment i_t, the real value of capital q_t, the stock of installed capital k_t, the stock of utilized capital k_t^s, the capital-utilization rate z_t, the real rental rate of capital r_t^k, hours worked l_t, the price markup μ_t^p, the inflation rate π_t, the wage markup μ_t^w, and the real wage w_t according to the following (locally log-linearized) structural equations:

$$
\begin{aligned}
y_t &= c_y c_t + i_y i_t + z_y z_t, && \text{(O.3)}\\
c_t &= c_1 c_{t-1} + (1-c_1)\,\mathbb{E}_t\{c_{t+1}\} + c_2\,(l_t - \mathbb{E}_t\{l_{t+1}\}) - c_3\,(r_t - \mathbb{E}_t\{\pi_{t+1}\}), && \text{(O.4)}\\
i_t &= i_1 i_{t-1} + (1-i_1)\,\mathbb{E}_t\{i_{t+1}\} + i_2 q_t, && \text{(O.5)}\\
q_t &= q_1 \mathbb{E}_t\{q_{t+1}\} + (1-q_1)\,\mathbb{E}_t\{r^k_{t+1}\} - (r_t - \mathbb{E}_t\{\pi_{t+1}\}), && \text{(O.6)}\\
y_t &= \phi_p\,[\alpha k^s_t + (1-\alpha)\,l_t], && \text{(O.7)}\\
k^s_t &= k_{t-1} + z_t, && \text{(O.8)}\\
z_t &= z_1 r^k_t, && \text{(O.9)}\\
k_t &= k_1 k_{t-1} + (1-k_1)\,i_t, && \text{(O.10)}\\
\mu^p_t &= \alpha\,(k^s_t - l_t) - w_t, && \text{(O.11)}\\
\pi_t &= \pi_1 \pi_{t-1} + \pi_2 \mathbb{E}_t\{\pi_{t+1}\} - \pi_3 \mu^p_t, && \text{(O.12)}\\
r^k_t &= -\,(k^s_t - l_t) + w_t, && \text{(O.13)}\\
\mu^w_t &= w_t - \sigma_l l_t - (1-\lambda/\gamma)^{-1}\,[c_t - (\lambda/\gamma)\,c_{t-1}], && \text{(O.14)}\\
w_t &= w_1 w_{t-1} + (1-w_1)\,\mathbb{E}_t\{w_{t+1} + \pi_{t+1}\} - w_2 \pi_t + w_3 \pi_{t-1} - w_4 \mu^w_t, && \text{(O.15)}
\end{aligned}
$$

where c_y, i_y, z_y, c_1, c_2, c_3, i_1, i_2, q_1, ϕ_p, α, z_1, k_1, π_1, π_2, π_3, σ_l, λ, γ, w_1, w_2, w_3, and w_4 are reduced-form parameters.[2] Because these reduced-form parameters are functions of a smaller number of structural parameters, they satisfy the following six equality constraints:

$$
i_1 = w_1,\; c_1 = \frac{\lambda}{\gamma+\lambda},\; w_2 = w_1 + \left(\frac{1-w_1}{w_1}\right) w_3,\; q_1 = \left(\frac{1-w_1}{w_1}\right) k_1,
$$

$$
\pi_2 = \left(\frac{1-w_1}{w_1}\right)\left[1 - \left(\frac{1-w_1}{w_1}\right)\pi_1\right],\text{ and } z_y = \left(\frac{i_y}{1-k_1}\right)\left(\frac{w_1}{1-w_1} - k_1\right).
$$

The policymaker is, again, a central bank setting the short-term nominal interest rate r_t.

I rewrite the system of structural equations (O.3)-(O.15) in a block-recursive way. More specifically, using (O.10), I rewrite (O.3), (O.7), (O.8), (O.9), (O.11), (O.13), and (O.14) as

$$
\begin{bmatrix} y_t & k_t & k^s_t & z_t & r^k_t & \mu^p_t & \mu^w_t \end{bmatrix}^\top = \mathbf{A}_1(L)\,[\,c_t \quad i_t \quad w_t \quad l_t\,]^\top, \tag{O.16}
$$

where

$$
\mathbf{A}_1(z) := \frac{1}{\eta}\begin{bmatrix}
\alpha\phi_p c_y & \alpha\phi_p i_y & \alpha\phi_p z_1 z_y & [1-(1+\phi_p-\eta)\alpha]\phi_p \\
(1+z_1)k_1 c_y & (1+z_1)k_1 i_y + (1-k_1)\eta & (z_y - \alpha\phi_p)k_1 z_1 & k_1\eta - (1+z_1)k_1\phi_p \\
c_y & i_y & z_1 z_y & \eta - \phi_p \\
-z_1 c_y & -z_1 i_y & \alpha z_1 \phi_p & z_1\phi_p \\
-c_y & -i_y & \alpha\phi_p & \phi_p \\
\alpha c_y & \alpha i_y & -(1-\alpha z_1 z_y) & -(1+\phi_p-\eta)\alpha \\
\frac{-[1-(\lambda/\gamma)z]}{1-\lambda/\gamma} & 0 & 1 & -\sigma_l
\end{bmatrix}
$$

[2]I display the structural equations in their order of appearance in Smets and Wouters (2007), and I use the same notations as them for the endogenous variables and the reduced-form parameters. The only difference is that I have replaced k_t by k^s_t in one equation (Equation (11) in their paper, Equation (O.13) here), thus correcting a typo in their paper.

with $\eta := \alpha\phi_p + z_1 z_y$. In turn, using (O.16), I rewrite (O.3)-(O.15) in a form of type (19) with $n = 13$, $\mathbf{X}_t := \begin{bmatrix} y_t & k_t & k_t^s & z_t & r_t^k & \mu_t^p & \mu_t^w & c_t & i_t & w_t & l_t & q_t & \pi_t \end{bmatrix}^\top$, $\mathbf{\Delta}(z) := \mathrm{diag}(\mathbf{I}_7, \mathbf{I}_3 z, \mathbf{I}_1, \mathbf{I}_2 z)$, and $\gamma = 0$:

$$\mathbb{E}_t \left\{ \mathbf{\Delta}\left(L^{-1}\right) \left[\mathbf{A}\left(L\right)\mathbf{X}_t + \mathbf{B}\left(L\right) r_t\right] \right\} = \mathbf{0}, \tag{O.17}$$

where

$$\mathbf{A}(z) := \begin{bmatrix} \underset{(7\times7)}{\mathbf{I}_7} & \underset{(7\times4)}{-\mathbf{A}_1(z)} & & \underset{(7\times2)}{\mathbf{0}} \\ \underset{(6\times7)}{\mathbf{0}} & \underset{(6\times2)}{\mathbf{A}_2(z)} & \underset{(6\times2)}{\mathbf{A}_3(z)} & \underset{(6\times2)}{\mathbf{A}_4(z)} \end{bmatrix}, \quad \mathbf{A}_4(z) := \begin{bmatrix} 0 & c_3 \\ i_2 z & 0 \\ q_1 - z & 1 \\ 0 & 0 \\ 0 & \eta(\pi_2 - z + \pi_1 z^2) \\ 0 & (1+w_1) - w_2 z + w_3 z^2 \end{bmatrix},$$

$$\mathbf{A}_2(z) := \begin{bmatrix} (1-c_1) - z + c_1 z^2 & 0 \\ 0 & (1-i_1) - z + i_1 z^2 \\ -(1-q_1)c_y/\eta & -(1-q_1)i_y/\eta \\ (1+z_1)k_1 c_y(1-k_1 z) & (1+z_1)k_1 i_y - [(1+z_1)k_1 i_y + (1-k_1)\eta]k_1 z \\ -\alpha\pi_3 c_y z & -\alpha\pi_3 i_y z \\ \frac{w_4}{1-\lambda/\gamma}[z - (\lambda/\gamma)z^2] & 0 \end{bmatrix},$$

$$\mathbf{A}_3(z) := \begin{bmatrix} 0 & -c_2 + c_2 z \\ 0 & 0 \\ (1-q_1)\alpha\phi_p/\eta & (1-q_1)\phi_p/\eta \\ (z_y - \alpha\phi_p)k_1 z_1(1-k_1 z) & -[(1+z_1)\phi_p - \eta]k_1(1-k_1 z) \\ \pi_3(\eta - \alpha z_1 z_y)z & \alpha\pi_3\phi_p z \\ (1-w_1) - (1+w_4)z + w_1 z^2 & w_4\sigma_l z \end{bmatrix},$$

$$\text{and} \quad \mathbf{B}(z) := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c_3 z & 0 & -z & 0 & 0 & 0 \end{bmatrix}^\top.$$

I check that the system (O.17) satisfies the three assumptions made in the main text. First,

$$\det[\mathbf{A}(0)] = [(1-c_1)\eta + (1+z_1)c_2 c_y - (1-c_1)(1+z_1)\phi_p]\,(1-i_1)(1-w_1)\eta\pi_2 k_1 q_1$$

is non-zero except possibly for a zero-measure set of structural-parameter values (taking into account the six equality constraints on the reduced-form parameters). Second, $\mathbf{B}(z) \neq \mathbf{0}$. Third, I use the symbolic-computation software Mathematica to check that $W_j(z) \neq 0$ for all $j \in \{1, ..., 13\}$ except possibly for a zero-measure set of structural-parameter values (taking again into account the six equality constraints on the reduced-form parameters).

I also use Mathematica to investigate whether Conditions (22) and (24) are met. I find that $D_J(z)$ has generically no roots at all, and hence Condition (22) is generically met, for any observation set $J \in \{\{1\}, ..., \{13\}\}$, and hence also more generally for any non-empty observation set $J \subseteq \{1, ..., 13\}$ (taking again into account the six equality constraints on the reduced-form parameters). I also find that Condition (24) is generically violated for all singleton sets $J \in \{\{1\}, ..., \{13\}\}$, and that it is met for all non-empty and non-singleton sets $J \subseteq \{1, ..., 13\}$ except six sets (taking again into account the six equality constraints on the reduced-form parameters).

The six exceptions are for $J \in \{\{2,9\},\{2,12\},\{4,5\},\{4,12\},\{5,12\},\{9,12\}\}$. Consider, for instance, the set $J = \{9,12\}$. This set corresponds to the case in which the only endogenous variables in the rule are investment and the real value of capital. I find that $G_{\{9,12\}}(z)$ is of degree 7, with $W_9(z) = i_2 z G_{\{9,12\}}(z)$ and $W_{12}(z) = [(1-i_1) - z + i_1 z^2] G_{\{9,12\}}(z)$. As I explain in the main text, the reason for this exception is that the investment Euler equation (O.5), which corresponds to the ninth line in (O.17), involves only investment and the real value of capital; it does not involve any other variable set by the private sector, nor the interest rate. The other structural equations, i.e. the other lines in (O.17), can be combined together to get another relationship involving these two variables and no other variable set by the private sector. This second relationship, however, involves the interest rate at different dates. As a result, no combinations of these two relationships involve the interest rate at a single date, and Condition (24) is violated.

The reason why Condition (24) is violated for the sets $J \in \{\{2,9\},\{2,12\},\{4,5\}\}$ is essentially the same as for the set $J = \{9,12\}$. The role played by the structural equation (O.5) for $J = \{9,12\}$ is played by (O.10) for $J = \{2,9\}$, by a combination of (O.5) and (O.10) for $J = \{2,12\}$, and by (O.9) for $J = \{4,5\}$.

O.4 Model of Schmitt-Grohé and Uribe (1997)

In the model of Schmitt-Grohé and Uribe (1997) (without exogenous shocks), at each date $t \in \mathbb{Z}$, the private sector sets output y_t, the capital stock k_t, hours worked h_t, consumption c_t, the (after-tax) rental rate of capital u_t, the (after-tax) wage w_t, and the stock of public debt b_t, according to the following structural equations, log-linearized in the neighborhood of the zero-debt steady state:

$$y_t = s_k k_{t-1} + (1-s_k) h_t, \tag{O.18}$$

$$y_t = s_c c_t + s_i \delta^{-1} k_t - (1-\delta) s_i \delta^{-1} k_{t-1}, \tag{O.19}$$

$$w_t = \sigma c_t + \gamma h_t, \tag{O.20}$$

$$c_t = \mathbb{E}_t\{c_{t+1}\} - [1-\beta(1-\delta)]\sigma^{-1}\mathbb{E}_t\{u_{t+1}\}, \tag{O.21}$$

$$u_t = -(1-s_k)(k_{t-1} - h_t) - \omega\tau(1-\tau)^{-1}\tau_t, \tag{O.22}$$

$$w_t = s_k(k_{t-1} - h_t) - \tau(1-\tau)^{-1}\tau_t, \tag{O.23}$$

$$b_t = \beta^{-1} b_{t-1} - (1 - s_k + \omega s_k)\tau(y_t + \tau_t), \tag{O.24}$$

where $s_k \in (0,1)$, $s_c \in (0,1)$, $s_i \in (0,1)$, $\delta \in (0,1)$, $\sigma > 0$, $\gamma > 0$, $\beta \in (0,1)$, $\tau \in (0,1)$, and $\omega \in \{0,1\}$.[3] The policymaker is a tax authority whose policy instrument τ_t is the labor-income-tax rate (when $\omega = 0$) or the income-tax rate (when $\omega = 1$).

[3] Most of Schmitt-Grohé and Uribe's (1997) analysis is conducted in continuous time; I refer here to the discrete-time analysis conducted in Section 4 and in the Appendix of their paper. I use the same notations as them for the variables and the parameters, with three exceptions: (i) I have replaced k_{t+1} and k_t by k_t and k_{t-1} respectively, as these variables are set at dates t and $t-1$ respectively; (ii) I have introduced the parameter σ to allow for degrees of relative risk aversion different from one; and (iii) I have introduced the parameter ω to

I rewrite the system (O.18)-(O.24) in a block-recursive way. More specifically, I rewrite (O.18)-(O.20) and (O.22)-(O.23) as

$$\begin{bmatrix} y_t & h_t & c_t & u_t & w_t \end{bmatrix}^\top = \mathbf{A}_1(L)k_t + \mathbf{B}_1\tau_t, \tag{O.25}$$

where

$$\mathbf{A}_1(z) := \frac{1}{\delta\eta}\begin{bmatrix} (1-s_k)\,\sigma s_i + [(1+\gamma)\,\delta s_c s_k - (1-\delta)\,(1-s_k)\,\sigma s_i]\,z \\ \sigma s_i - [(\sigma - s_c)\,\delta s_k + (1-\delta)\,\sigma s_i]\,z \\ \varphi\sigma s_i/s_c + [(1+\gamma)\,\delta\sigma s_k - (1-\delta)\,\varphi\sigma s_i/s_c]\,z \\ (1-s_k)\,\sigma s_i - [(\gamma s_c+\sigma)\,\delta + (1-\delta)\,\sigma s_i]\,(1-s_k)\,z \\ (\gamma s_c+\varphi)\,\sigma s_i/s_c + [(\gamma s_c+\sigma)\,\delta s_k - (\gamma s_c+\varphi)\,(1-\delta)\,\sigma s_i/s_c]\,z \end{bmatrix}$$

$$\text{and } \mathbf{B}_1 := \frac{-\tau}{(1-\tau)\,\eta}\begin{bmatrix} (1-s_k)\,s_c \\ s_c \\ (1-s_k)\,\sigma \\ (1-s_k)\,s_c + \eta\omega \\ \gamma s_c + (1-s_k)\,\sigma \end{bmatrix},$$

with $\eta := (1+\gamma)\,s_c + (1-s_k)\,(\sigma - s_c)$ and $\varphi := (1-s_k)\,\sigma - s_c$. In turn, using (O.25), I rewrite (O.18)-(O.24) in a form of type (19) with $n=7$, $\mathbf{X}_t := \begin{bmatrix} y_t & h_t & c_t & u_t & w_t & k_t & b_t \end{bmatrix}^\top$, $\mathbf{\Delta}(z) := \text{diag}(\mathbf{I}_5, \mathbf{I}_1 z, \mathbf{I}_1)$, and $\gamma = 0$:

$$\mathbb{E}_t\left\{\mathbf{\Delta}\left(L^{-1}\right)[\mathbf{A}\,(L)\,\mathbf{X}_t + \mathbf{B}\,(L)\,\tau_t]\right\} = \mathbf{0}, \tag{O.26}$$

where

$$\mathbf{A}(z) := \begin{bmatrix} \mathbf{I}_6 & -\mathbf{A}_1(z) & \mathbf{0} \\ \mathbf{0} & A_2(z) & 0 \\ \mathbf{0} & A_3(z) & A_4(z) \end{bmatrix},\ \mathbf{B}(z) := \begin{bmatrix} -\mathbf{B}_1 \\ B_2(z) \\ B_3 \end{bmatrix},$$

$$\begin{aligned} A_2(z) &:= [\varphi\sigma - (1-s_k)\,\chi s_c]\,s_i/(\delta\eta s_c) + \left\{(1+\gamma)\,\delta\sigma^2 s_c s_k - (2-\delta)\,\varphi\sigma^2 s_i + [(\gamma s_c+\sigma)\,\delta \right. \\ &\quad \left. + (1-\delta)\,\sigma s_i]\,(1-s_k)\,\chi s_c\right\} z/(\delta\eta\sigma s_c) + [(1-\delta)\,\varphi s_i - (1+\gamma)\,\delta s_c s_k]\,\sigma z^2/(\delta\eta s_c), \\ A_3(z) &:= \{(1-s_k)\,\sigma s_i + [(1+\gamma)\,\delta s_c s_k - (1-\delta)\,(1-s_k)\,\sigma s_i]\,z\}\,(1-s_k+\omega s_k)\,\tau/\left(\delta\eta\right), \\ A_4(z) &:= 1 - \beta^{-1}z, \\ B_2(z) &:= \left\{[(1-s_k)\,s_c + \eta\omega]\,\chi - (1-s_k)\,\sigma^2 + (1-s_k)\,\sigma^2 z\right\}\tau/\left[(1-\tau)\,\eta\sigma\right], \\ \text{and } B_3 &:= [(1-\tau)\,\eta - (1-s_k)\,\tau s_c]\,(1-s_k+\omega s_k)\,\tau/\left[(1-\tau)\,\eta\right], \end{aligned}$$

with $\chi := 1 - \beta\,(1-\delta)$.

I check that the system (O.26) satisfies the three assumptions made in the main text. First,

$$\det[\mathbf{A}(0)] = [\varphi\sigma - (1-s_k)\,\chi s_c]\,\frac{s_i}{\delta\eta s_c}$$

is non-zero except for a zero-measure set of structural-parameter values. Second, $\mathbf{B}(z) \neq \mathbf{0}$. Third, I use the symbolic-computation software Mathematica to check that $W_j(z) \neq 0$ for all $j \in \{1, ..., 7\}$ except possibly for a zero-measure set of structural-parameter values.

encompass the two alternative tax-policy instruments. All variables are expressed in percentage deviation from their steady-state value – except public debt b_t, which is expressed as a fraction of steady-state output (since steady-state public debt is zero).

For the dynamic system composed of the structural equations (O.26) and the policy-instrument rule to satisfy Blanchard and Kahn's (1980) *no-decoupling* condition, we need the rule to make the policy instrument react to the debt level (possibly with inside lags); otherwise, the debt level b_t would be a residual variable appearing only in the structural equation (O.24), and it would therefore explode over time (at the rate $1/\beta$) for any arbitrary stationary processes of output y_t and the policy instrument τ_t. So, I focus on observation sets that include the debt level, i.e. on sets J such that $7 \in J$.

I use Mathematica to investigate whether Conditions (22) and (24) are met for such observation sets. I find that $D_{\{7\}}(z)$ has generically no roots at all, and hence that Condition (22) is generically met for $J = \{7\}$; as a consequence, this condition is also generically met for any observation set $J \subseteq \{1, ..., 7\}$ such that $7 \in J$. I also find that $G_{\{7\}}(z)$ has generically at least one root, and hence that Condition (24) is generically violated for $J = \{7\}$. Finally, I find that $G_{\{j,7\}}(z)$ has generically no roots at all for any $j \in \{1, ..., 6\}$, and hence that Condition (24) is generically met for all non-singleton sets $J \subseteq \{1, ..., 7\}$ such that $7 \in J$.

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