Revisiting Speculative Hyperinflations in Monetary Models: A Rejoinder

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Abstract: In this note, we present a formal proof of Obstfeld and Rogoff's (1983, 2021) claim that their fractional-currency-backing scheme eliminates the inflationary equilibria in the money-in-utility model. To do so, we fully articulate the optimization problem of consumers who have the option to redeem (part or all of) their cash for a small amount of goods at any date.

Keywords: speculative hyperinflation, fractional currency backing, money-in-utility model.

JEL codes: E31, E42.

1 Introduction

Monetary models with a pure fiat currency often have equilibria in which money becomes worthless (in units of goods) at some finite date, or asymptotically. This theoretical possibility (that a currency with no intrinsic value may be worthless in equilibrium) is not surprising; and one may argue that any sensible model with pure fiat money should share this implication. In many applications, however, it seems natural to set aside this theoretical possibility, and this is often justified by invoking a fractional-currency-backing scheme proposed by Wallace (1981) and Obstfeld and Rogoff (1983). In this rejoinder, we present a formal proof of Obstfeld and Rogoff's (1983) claim that their fractional-currency-backing scheme eliminates the inflationary equilibria in their money-in-utility (MIU) model.

Obstfeld and Rogoff (1983) show that the MIU model with a constant money supply has an equilibrium with a constant price level, as well as a countable infinity of inflationary equilibria in which the price level reaches infinity at some finite date T. They argue that a backstop mechanism eliminates inflationary equilibria if the government stands ready to redeem the money

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stock in exchange for a small amount of goods − which may be secured through taxation, or may represent (say) gold reserves held by the central bank.

Cochrane (2011), however, argues that the backstop mechanism cannot eliminate the inflationary equilibria. His analysis suggests that the backstop mechanism only modifies the inflationary equilibrium paths in a minor way: it leads to equilibria in which consumers exercise the option to exchange their money holdings for goods at date $T-1$, just before money becomes worthless (and the price level reaches infinity). In a response to Cochrane's criticism, Obstfeld and Rogoff (2021) argue that the currency backing implies an arbitrage condition that puts a cap on the equilibrium price level. Cochrane (2023) questions the validity of this response (which is motivated intuitively but not based on the model's equilibrium conditions); he claims that "Obstfeld and Rogoff deviate from the definition of Walrasian equilibrium" (Cochrane, 2023, online appendix, page 638).

In this rejoinder, we modify Obstfeld and Rogoff's (2021) analysis in two ways. First and more importantly, we fully articulate the optimization problem of consumers who have the option to redeem (part or all of) their cash for a small amount of goods at any date. Second, we make, for convenience, the following more minor changes: (i) we take the consumption good, rather than fiat money, as our numeraire; (ii) we replace their nominal bonds with real bonds in our setup; and (iii) we relax their Inada condition on the marginal utility of money for zero money. Changes (i)-(ii) do not modify the substance of Obstfeld and Rogoff's point; they serve to circumvent the need (for us) to define a Walrasian equilibrium in which the price level and the nominal interest rate are infinite. Change (iii) serves to show that our results do not rest on an infinite marginal utility of money (for zero money), and that our results extend to cases in which, absent the backstop mechanism, money may become worthless asymptotically rather than at a finite date.

Our more detailed derivations lead to a conclusion similar to the one reached by Obstfeld and Rogoff (1983, 2021): as long as the amount of goods offered by the backstop mechanism is sufficiently small for an equilibrium with a constant price level to exist, the backstop mechanism eliminates all inflationary equilibria.

2 Model

Following Obstfeld and Rogoff (2021) and Cochrane (2023), we consider a simple MIU model with an endowment of goods and separable preferences. The representative household has a constant endowment $y > 0$ at each date $t \in \mathbb{N}$. The nominal money stock remains constant over time, equal to its initial value $M > 0$, unless households exercise their option to redeem (part or all of) their cash balances. If the option is exercised, the government uses a lump-sum tax and offers ϵ/M units of the consumption good per unit of currency tendered, where $\epsilon > 0$. Our benchmark model assumes that the option to trade some money for goods can be exercised at the end of each period, after getting utility from cash balances during the period; we consider the alternative assumption − that the option can be exercised at the beginning of each period − in the online appendix. We also consider non-separable preferences in the online appendix.

The representative household chooses beginning-of-period money balances M_t, end-of-period money balances $\tilde{M}_t$, consumption c_t, and real-bond holdings b_t for all $t \geq 0$ to maximize

$$\sum_{t=0}^{+\infty} \beta^t \left[u\left(c_t\right) + v\left(q_t M_t\right) \right]$$

subject to

$$y + \frac{\epsilon}{M}\left(M_t - \tilde{M}_t\right) + R_{t-1} b_{t-1} + q_t \tilde{M}_{t-1} - c_t - b_t - q_t M_t - \tau_t \geq 0, \tag{1}$$

$$M_t - \tilde{M}_t \geq 0, \tag{2}$$

$$\tilde{M}_t \geq 0, \tag{3}$$

where $\beta \in (0,1)$, $b_{-1} \equiv 0$, and $\tilde{M}_{-1} \equiv M$. The utility functions u and v are strictly increasing, strictly concave, and continuously differentiable. Bonds are in zero net supply and pay a gross real interest rate R_t. The variables q_t and τ_t denote, respectively, the price of money in units of the consumption good and lump-sum taxes. Households can use the government guarantee to exchange money for goods, but not to exchange goods for money; so, their choices are subject to the inequality constraint (2), as well as the non-negativity constraint (3).

We do not impose Inada conditions on u and v.[1] Instead, we only impose the restriction

$$v'\left(0\right) > u'\left(y\right). \tag{4}$$

As we show in the online appendix, relaxing the usual Inada condition $\lim_{x\to 0} v'(x) = +\infty$ and imposing only (4) does not qualitatively change the set of equilibria in the MIU model with no currency backing: we still get (hyperinflationary) equilibria in which money becomes worthless *at a finite date*. In the online appendix, we also relax the condition further to allow for $u'\left(y\right) > v'\left(0\right) > \left(1-\beta\right) u'\left(y\right)$; we show that this leads to equilibria in which money becomes worthless *asymptotically* (instead of at a finite date), but a currency-backing scheme still rules out these inflationary equilibria.

Compared to the setup in Obstfeld and Rogoff (1983, 2021), we have added the inequality constraints (2) and (3) to the household's optimization problem; we have also defined q_t as the inverse of their variable P_t (which represents the price level in their setup), replaced their nominal bonds with real bonds in our setup, and relaxed their Inada condition on v.

[1] Because of our endowment assumption, we do not need to impose Inada conditions on u. The Inada condition $\lim_{x\to+\infty} v'(x) = 0$, which says that demand for money is asymptotically satiated, would serve to rule out deflationary equilibria in exactly the same way as in the MIU model with no currency backing; but it does not play any role in our analysis of inflationary paths. As we discuss in the text, we relax the usual Inada condition $\lim_{x\to 0} v'(x) = +\infty$ and replace it with (4).

Let λ_t, γ_t, and δ_t denote the non-negative Lagrange multipliers on, respectively, the budget constraint (1), the non-increasing-money-stock constraint (2), and the non-negative-money-stock constraint (3). The first-order conditions are

$$u'(c_t) = \lambda_t, \tag{5}$$

$$q_t v'(q_t M_t) + \gamma_t + \left(\frac{\epsilon}{M} - q_t\right)\lambda_t = 0, \tag{6}$$

$$\gamma_t + \frac{\epsilon}{M}\lambda_t = \beta\lambda_{t+1}q_{t+1} + \delta_t, \tag{7}$$

$$\lambda_t = \beta R_t \lambda_{t+1}. \tag{8}$$

Moreover, we have

$$\lambda_t \geq 0, \;\; \gamma_t \geq 0, \;\; \delta_t \geq 0, \tag{9}$$

and the complementary slackness conditions

$$\gamma_t\left(M_t - \tilde{M}_t\right) = 0, \tag{10}$$

$$\delta_t \tilde{M}_t = 0.$$

The household optimization problem is also subject to a limit on borrowing (as in, e.g., Woodford, 2003, pages 67-68), and a standard transversality condition is also necessary for optimality. This transversality condition only serves to rule out deflationary equilibria (and rules them out with or without currency backing in exactly the same way); we abstract from this condition, as we focus on inflationary equilibria.

The government supplies the nominal money stock $M > 0$ at the start of Period 0, and thereafter redeems money upon demand. So, the money-market-clearing conditions are $M_0 = M$ and $\forall t \geq 1$, $M_t = \tilde{M}_{t-1}$; or, equivalently and more compactly (given that $\tilde{M}_{-1} \equiv M$):

$$M_t = \tilde{M}_{t-1}.$$

The bond-market-clearing condition is $b_t = 0$. Finally, redeeming money is financed with lump-sum taxation. So, the government budget constraint is $\tau_t = (\epsilon/M)(M_t - \tilde{M}_t)$, and the goods-market-clearing condition is

$$c_t = y. \tag{11}$$

In this setup, a perfect-foresight equilibrium is a collection of non-negative sequences $\{c_t, M_t, \tilde{M}_t, q_t, R_t, \tau_t\}_{t=0}^{+\infty}$ and a sequence $\{b_t\}_{t=0}^{+\infty}$ satisfying the household optimality conditions, the market-clearing conditions (for money, bonds and goods), and the government budget constraint.

3 Fundamental Equilibrium

We first consider candidate equilibria in which the nominal money stock and the price level are constant over time: $M_t = \tilde{M}_t = M$ and $q_t = q$, where q is to be determined. In these candidate equilibria, the non-negativity constraint (3) is lax, so $\delta_t = 0$. The equilibrium conditions (5), (7)-(8) and (11) then imply that all endogenous variables are constant over time:

$$c_t = y, \quad \lambda_t = u'(y), \; R_t = \frac{1}{\beta}, \text{ and } \; \gamma_t = \left(\beta q - \frac{\epsilon}{M}\right) u'(y).$$

The last equation can be interpreted as follows. The Lagrange multiplier γ_t represents the net marginal utility gain from relaxing the non-increasing-money-stock constraint (2), i.e. from allowing households to exchange goods for newly created money with the government (thus making the government facility work both ways). On the one hand, increasing one's nominal money stock by one unit would cost ϵ/M units of good, which reduces current utility by $(\epsilon/M)u'(y)$; on the other hand, it would enable one to increase future consumption by q units of good, which increases current utility by $q\beta u'(y)$.

Since $\gamma_t \geq 0$, we have

$$q \geq \frac{\epsilon}{\beta M}. \tag{12}$$

The price of money in units of goods cannot be below $\epsilon/\beta M$; otherwise, the marginal utility benefit of exchanging money for goods, $(\epsilon/M)u'(y)$, would outweigh the marginal utility cost, $q\beta u'(y)$; so, households would prefer to exchange money for goods, rather than keeping their money balances constant.

Finally, the equilibrium condition (6) implies

$$v'(qM) = (1-\beta)\, u'(y),$$

which determines q uniquely. Given the strict concavity of v, this value of q satisfies the inequality (12) if and only if

$$v'\left(\frac{\epsilon}{\beta}\right) \geq (1-\beta)\, u'(y). \tag{13}$$

From now on, we assume that ϵ is sufficiently small for the inequality (13) to be satisfied. We have just shown that this assumption is necessary for existence of the "fundamental equilibrium" (as we call the unique equilibrium in which the nominal money stock and the price level are constant over time). In the next section, we show that this assumption is also sufficient for ruling out other equilibria.

4 Equilibrium Uniqueness

The model's candidate equilibria can be classified into three groups: those with no demonetization ($\forall t \geq 0,\ M_t = M$), among which the fundamental equilibrium; those with partial demonetization ($\exists T \geq 1$, $M_T < M$, and $\forall t \geq 0$, $M_t > 0$); and those with complete demonetization ($\exists T \geq 1$, $M_T = 0$). We will first show below that the only equilibrium with no demonetization is the fundamental equilibrium, and we will then show that the model has no equilibria with (partial or complete) demonetization.

4.1 No Demonetization

In any no-demonetization equilibrium of our model, we have $M_t = \tilde{M}_t = M$. Since the non-negative-money-stock constraint (3) is lax, we also have $\delta_t = 0$. The equilibrium conditions (5), (7) and (11) then imply

$$\gamma_t = \left(\beta q_{t+1} - \frac{\epsilon}{M}\right) u'(y), \tag{14}$$

which has the same interpretation as in the previous section (in which we had $q_{t+1} = q$). So, $\gamma_t \geq 0$ implies

$$q_{t+1} \geq \frac{\epsilon}{\beta M}. \tag{15}$$

This inequality says that the price of money at date $t+1$ needs to be sufficiently high for households to be willing to keep all their money balances at date t.

Using (6) and (14), and defining equilibrium real money balances as $m_t \equiv M q_t$, we get the dynamic equation

$$\beta u'(y)\, m_{t+1} = m_t \left[u'(y) - v'(m_t)\right].$$

This is the same equation as Equation (3) in Obstfeld and Rogoff (2021). Given their Inada condition $\lim_{x\to 0} v'(x) = +\infty$, they show that absent any currency backing and provided that $\lim_{x\to 0} x v'(x) = 0$, this equation admits two types of solution: the fundamental equilibrium, and a countable infinity of "hyperinflationary equilibria" in which money becomes worthless at a finite date. This result is unchanged when their Inada condition $\lim_{x\to 0} v'(x) = +\infty$ is replaced with (4), as we show in the online appendix. However, the hyperinflationary equilibria cannot be no-demonetization equilibria of our model with a currency-backing scheme, because (15) implies that money cannot become worthless in the latter equilibria. Therefore, the only no-demonetization equilibrium of our model with a currency-backing scheme is the fundamental equilibrium.

4.2 Partial Demonetization

Consider a candidate equilibrium with partial demonetization, implying $M_t > \tilde{M}_t > 0$ for some date t. Both the non-negative-money-stock constraint (3) and the non-increasing-money-stock constraint (2) are lax, so we have $\gamma_t = \delta_t = 0$. The equilibrium conditions (5), (7) and (11) then imply

$$q_{t+1} = \frac{\epsilon}{\beta M}. \tag{16}$$

This equality says that the price of money at date $t+1$ needs to be exactly equal to $\epsilon/(\beta M)$ for households to be willing to redeem part, but not all, of their money balances at date t. If the price of money were higher, then households would want to keep all their money balances (as in the no-demonetization case above). Alternatively, if it were lower, then households would want to redeem all their money balances (as in the complete-demonetization case below).

Using (16), we can rewrite (6) at date $t+1$ as

$$v'\left(\frac{\epsilon M_{t+1}}{\beta M}\right)+\left(\frac{\beta M}{\epsilon}\right)\gamma_{t+1}=(1-\beta)\,u'(y).$$

Using $\gamma_{t+1}\geq 0$, we get

$$v'\left(\frac{\epsilon M_{t+1}}{\beta M}\right)\leq(1-\beta)\,u'(y).$$

Noting that $M_{t+1}=\tilde{M}_t<M_t\leq M$ and using the strict concavity of v, we then obtain

$$v'\left(\frac{\epsilon}{\beta}\right)<(1-\beta)\,u'(y),$$

which contradicts our assumption (13). So, there are no equilibria with partial demonetization.

4.3 Complete Demonetization

Finally, consider a candidate equilibrium with complete demonetization, implying $M_t>\tilde{M}_t=0$ for some date t. In this case, the non-increasing-money-stock constraint (2) is lax, so we have $\gamma_t=0$. Since $\delta_t\geq 0$, (7) implies

$$q_{t+1}\leq\frac{\epsilon}{\beta M}. \tag{17}$$

This inequality says that the price of money at date $t+1$ needs to be sufficiently low for households to be willing to redeem all their money balances at date t. If the price of money were higher, then households would want to keep at least part of their money balances (as in the no-demonetization and partial-demonetization cases above).

Since q_{t+1} is finite and $M_{t+1}=\tilde{M}_t=0$, we can rewrite (6) at date $t+1$ as

$$v'(0)+\left[\gamma_{t+1}+\frac{\epsilon}{M}u'(y)\right]\frac{1}{q_{t+1}}=u'(y).$$

This equation, together with $\gamma_{t+1}\geq 0$ and (17), implies

$$v'(0)\leq(1-\beta)\,u'(y).$$

Since v is strictly concave, we have $v'(\epsilon/\beta)<v'(0)$, and therefore

$$v'\left(\frac{\epsilon}{\beta}\right)<(1-\beta)\,u'(y),$$

which contradicts our assumption (13). So, there are no equilibria with complete demonetization.

5 Conclusion and Extensions

In this note, we have presented a formal proof of Obstfeld and Rogoff's (1983) claim that their fractional-currency-backing scheme eliminates the inflationary equilibria in the separable MIU model. To do so, we have fully articulated the optimization problem of consumers who have the option to redeem (part or all of) their cash for a small amount of goods at any date.

In our online appendix, we extend the result in three directions. First, we show that the result also holds under the alternative timing assumption in which money is redeemed at the *beginning* of each period (rather than at the end of each period). Second, we show that the result extends to the *non-separable* MIU model as long as money and consumption are "complements," in the sense that an increase in consumption raises the marginal utility of real money balances.[2] Finally, we consider relaxing the Inada condition on utility from real money balances further than what we assumed in (4). If we assume $u'(y) > v'(0) > (1-\beta)\, u'(y)$, then the MIU model with no currency backing has equilibria in which money becomes worthless *asymptotically* (rather than at a finite date). The equilibrium dynamics in this case are similar to those of the overlapping-generations model developed by Wallace (1981) and discussed in Obstfeld and Rogoff (2021). We show that a currency-backing scheme rules out these inflationary equilibria as well.

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[2] As Obstfeld (1984) shows, the MIU model exhibits local-equilibrium indeterminacy if money and consumption are substitutes.

Online Appendix to "Revisiting Speculative Hyperinflations in Monetary Models: A Rejoinder"

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In this online appendix, we extend our main result in three directions. First, we show that the result also holds under the alternative timing assumption in which money is redeemed at the *beginning* of each period (rather than at the end of each period). Second, we show that the result extends to the *non-separable* MIU model as long as money and consumption are "complements," in the sense that an increase in consumption raises the marginal utility of real money balances. Finally, we allow for $u'(y) > v'(0) > (1-\beta)\,u'(y)$; in this case, as we show, the MIU model with no currency backing has equilibria in which money becomes worthless *asymptotically* (rather than at a finite date), but a currency-backing scheme rules out these inflationary equilibria as well.

In what follows, instead of repeating the main text's analysis in each of the three extensions, we just highlight the changes from the main text's analysis.

1 Reverse Within-Period Timing

We consider the same model as in the main text, except that we now assume that the option to trade some money for goods can be exercised at the *beginning* of each period (before getting utility from cash balances during the period), rather than at the end of each period. We show that this alternative within-period-timing assumption leads to the same conclusion: as long as ϵ is sufficiently small for the fundamental equilibrium to exist, the backstop mechanism eliminates all inflationary equilibria.

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1.1 Model

Households now maximize

$$\sum_{t=0}^{+\infty} \beta^t \left[u\left(c_t\right) + v\left(q_t \tilde{M}_t\right)\right].$$

The only equilibrium conditions that are affected by this change are (6)-(7), which become

$$\gamma_t + \left(\frac{\epsilon}{M} - q_t\right)\lambda_t = 0, \tag{A.1}$$

$$\gamma_t + \frac{\epsilon}{M}\lambda_t = \beta\lambda_{t+1}q_{t+1} + \delta_t + q_t v'\left(q_t \tilde{M}_t\right). \tag{A.2}$$

1.2 Fundamental Equilibrium

The variables c_t, λ_t, and R_t in the fundamental equilibrium are unchanged, while we now have

$$\gamma_t = \left(q - \frac{\epsilon}{M}\right)u'(y).$$

This new expression for γ_t can be interpreted in essentially the same way as the previous expression. The Lagrange multiplier γ_t represents the net marginal utility gain from relaxing the non-increasing-money-stock constraint (2), i.e. from allowing households to exchange goods for newly created money with the government (thus making the government facility work both ways). On the one hand, increasing one's nominal money stock by one unit would cost ϵ/M units of good, which reduces current utility by $(\epsilon/M)u'(y)$, as in the main text. On the other hand, it would enable one to increase current consumption by q units of good, which increases current utility by $qu'(y)$ (while, in the main text, it increased future consumption by q units of goods, which increased current utility by $q\beta u'(y)$).

So, (12) is replaced by

$$q \geq \frac{\epsilon}{M}.$$

We get the same value of q as before, but the necessary and sufficient condition for fundamental-equilibrium existence is now

$$v'\left(\epsilon\right) \geq \left(1 - \beta\right)u'\left(y\right), \tag{A.3}$$

which replaces (13).

1.3 No Demonetization

The equilibrium conditions (14) and (15) are respectively replaced by

$$\gamma_t = \left(q_t - \frac{\epsilon}{M}\right)u'(y), \tag{A.4}$$

$$q_t \geq \frac{\epsilon}{M}.$$

Using (A.4), $\delta_t = 0$, $\tilde{M}_t = M$, and $m_t \equiv Mq_t$, we can rewrite (A.2) as the same dynamic equation as previously. So, the conclusion is unchanged: the only no-demonetization equilibrium is the fundamental equilibrium.

1.4 Partial Demonetization

Since $\gamma_t = 0$, (A.1) can be rewritten as

$$q_t = \frac{\epsilon}{M},$$

and therefore (A.2) can be rewritten as

$$v'\left(\frac{\epsilon \tilde{M}_t}{M}\right) = \left(1 - \frac{\beta M q_{t+1}}{\epsilon}\right) u'(y).$$

Using $\tilde{M}_t < M_t \leq M$, the strict concavity of v, and $q_{t+1} \geq \epsilon/M$, we get

$$v'(\epsilon) < v'\left(\frac{\epsilon \tilde{M}_t}{M}\right) = \left(1 - \frac{\beta M q_{t+1}}{\epsilon}\right) u'(y) \leq (1-\beta) u'(y),$$

which contradicts our assumption (A.3). So, the conclusion is unchanged: there are no equilibria with partial demonetization.

1.5 Complete Demonetization

Since $\gamma_t = 0$, (A.1) can be rewritten as

$$q_t = \frac{\epsilon}{M}.$$

Using $\gamma_t = 0$, $q_t = \epsilon/M$, and $\tilde{M}_t = 0$, we can then rewrite (A.2) as

$$v'(0) + \left[\delta_t + \beta q_{t+1} u'(y)\right] \frac{M}{\epsilon} = u'(y).$$

Using the strict concavity of v, $\delta_t \geq 0$, and $q_{t+1} \geq \epsilon/M$, we then get

$$v'(\epsilon) < v'(0) \leq \left(1 - \frac{\beta M q_{t+1}}{\epsilon}\right) u'(y) \leq (1-\beta) u'(y),$$

which contradicts our assumption (A.3). So, the conclusion is unchanged: there are no equilibria with complete demonetization.

2 Non-Separable MIU Model

We now consider the non-separable MIU model in which money and consumption are "complements." We show that this alternative model leads to the same conclusion: as long as ϵ is sufficiently small for the fundamental equilibrium to exist, the backstop mechanism eliminates all inflationary equilibria.

2.1 Model

Households now maximize

$$\sum_{t=0}^{+\infty} \beta^t u\left(c_t, q_t M_t\right),$$

where u is continuously differentiable, strictly increasing and strictly concave in each of its arguments, and such that $u_{cm} > 0$. We do not impose Inada conditions on u.[1] Instead, we only impose the restriction

$$u_m\left(y, 0\right) > u_c\left(y, 0\right). \tag{A.5}$$

Relaxing the usual Inada condition $\lim_{x \to 0} u_m(y, x) = +\infty$ and imposing only (A.5) does not qualitatively change the set of equilibria in the MIU model with no currency backing: we still get (hyperinflationary) equilibria in which money becomes worthless *at a finite date*, as we show below.

The only equilibrium conditions that are affected by this change are (5)-(6), which become

$$u_c\left(c_t, q_t M_t\right) = \lambda_t, \tag{A.6}$$

$$q_t u_m\left(c_t, q_t M_t\right) + \gamma_t + \left(\frac{\epsilon}{M} - q_t\right)\lambda_t = 0. \tag{A.7}$$

2.2 Fundamental Equilibrium

Most of the fundamental-equilibrium analysis in the main text remains valid if we replace $u'\left(y\right)$ and $v'\left(qM\right)$ by, respectively, $u_c\left(y, qM\right)$ and $u_m\left(y, qM\right)$. Let $z(x) \equiv u_m(y,x)/u_c(y,x)$; our assumption $u_{cm} > 0$ implies $z'(x) < 0$. So, the equilibrium condition

$$z\left(qM\right) = 1 - \beta$$

determines q uniquely. The condition for fundamental-equilibrium existence, which was formerly (13), is now

$$z\left(\frac{\epsilon}{\beta}\right) \geq 1 - \beta, \tag{A.8}$$

or equivalently

$$m^* \geq \frac{\epsilon}{\beta},$$

where $m^* \equiv z^{-1}(1-\beta)$ denotes the value of qM at the fundamental equilibrium.

[1] Because of our endowment assumption, we do not need to impose Inada conditions on u_c. The Inada condition $\lim_{x \to +\infty} u_m(y,x) = 0$, which says that demand for money is asymptotically satiated, would serve to rule out deflationary equilibria in exactly the same way as in the MIU model with no currency backing; but it does not play any role in our analysis of inflationary paths. As we discuss in the text, we relax the usual Inada condition $\lim_{x \to 0} u_m(y,x) = +\infty$ and replace it with (A.5).

2.3 Model Without Redeeming Scheme

Consider for a moment the model without redeeming scheme. Households maximize

$$\sum_{t=0}^{+\infty} \beta^t u\left(c_t, q_t M_t\right)$$

subject to

$$y + R_{t-1}b_{t-1} + q_t M_{t-1} - c_t - b_t - q_t M_t - \tau_t \geq 0.$$

Two first-order conditions are

$$u_c\left(c_t, q_t M_t\right) = \lambda_t,$$

$$q_t u_m\left(c_t, q_t M_t\right) = \lambda_t q_t - \beta \lambda_{t+1} q_{t+1}.$$

Using the goods-market-clearing condition $c_t = y$ and the money-market-clearing condition $M_t = M$, we get the dynamic equation

$$\frac{m_{t+1}u_c(y, m_{t+1})}{m_t u_c(y, m_t)} = 1 + \left(\frac{1-\beta}{\beta}\right)\left[1 - \frac{z(m_t)}{z(m^*)}\right], \tag{A.9}$$

where $m_t \equiv q_t M$. Given this dynamic equation, the only candidate dynamic equilibria are inflationary paths $m^* > m_0 > m_1 > ...$ and deflationary paths $m^* < m_0 < m_1 < ....$

The deflationary paths are not equilibria if we impose the Inada condition $\lim_{x \to +\infty} u_m(y, x) = 0$, which says that demand for money is asymptotically satiated: with this Inada condition, the deflationary paths violate the transversality condition (for any $m_0 > m^*$).

To study the inflationary paths, we first note that the right-hand side of (A.9) is strictly increasing in m_t, strictly negative for $m_t = 0$ because of our assumption (A.5), and strictly positive for $m_t = m^*$. Let $\tilde{m}$ denote the unique value of m_t that makes the right-hand side of (A.9) equal to zero. We can rewrite the dynamic equation (A.9) as

$$F\left(m_{t+1}\right) = G\left(m_t\right),$$

where $F(x) \equiv x u_c(y, x)$, $F(0) = 0$, $F(x) > 0$ and $F'(x) > 0$ for $x > 0$, $G(x) \equiv x u_c(y, x)\{1 + [(1-\beta)/\beta][1 - z(x)/z(m^*)]\}$, $G(x) < 0$ for $x \in (0, \tilde{m})$, $G(x) > 0$ and $G'(x) > 0$ for $x > \tilde{m}$. So, if $\lim_{x \to 0} G(x) < 0$, that is to say equivalently if the "super Inada condition" $\lim_{x \to 0} x u_m(y, x) > 0$ is satisfied, then any inflationary path ends up violating the dynamic equation at some date; so, there are no inflationary equilibria. Alternatively, if $\lim_{x \to 0} G(x) = 0$ or equivalently if $\lim_{x \to 0} x u_m(y, x) = 0$, then there exist "hyperinflationary" equilibria in which money becomes worthless at a finite date: for any $T \in \mathbb{N} \setminus \{0\}$, the sequence characterized by $m_t = 0$ for $t \geq T$, $m_{T-1} = \tilde{m}$, and $(m_t)_{T-2 \geq t \geq 0}$ derived sequentially from m_{T-1} with the dynamic equation, is an equilibrium. There exists a countable infinity of such equilibria, which can be indexed by $T \in \mathbb{N} \setminus \{0\}$.

2.4 No Demonetization

In any no-demonetization equilibrium of our model, using $M_t = M$, $\delta_t = 0$, (7), (11), and (A.6), we get

$$\gamma_t = \beta q_{t+1} u_c\left(y, q_{t+1}M\right) - \frac{\epsilon}{M} u_c\left(y, q_t M\right), \tag{A.10}$$

which can be interpreted in the same way as previously. Using (A.6), (A.7), and (A.10), we then get the same dynamic equation (A.9) as without redeeming scheme. So, the candidate inflationary equilibria are the hyperinflationary equilibria of the model without redeeming scheme, in which money becomes worthless at a finite date (if $\lim_{x\to 0} x u_m(y,x) = 0$). However, (A.10) and $\gamma_t \geq 0$ together imply that we cannot have q_t positive and q_{t+1} equal to zero. Therefore, the hyperinflationary paths described above cannot be no-demonetization equilibria of our model with redeeming scheme, and the only no-demonetization equilibrium of our model with redeeming scheme is the fundamental equilibrium.

2.5 Partial Demonetization

Let $m_t \equiv q_t M_t$. Under partial demonetization (meaning there exists at least one date at which the money stock strictly decreases and there exists no date at which the money stock becomes zero), we have $\delta_t = 0$ at all dates $t \geq 0$; so, (7), (11), (A.6) and (A.7) imply

$$q_t u_m\left(y, m_t\right) + \beta q_{t+1} u_c\left(y, m_{t+1}\right) = q_t u_c\left(y, m_t\right).$$

Consider a given date $t \geq 0$. If $M_t > M_{t+1}$, then $\gamma_t = 0$; so, (7), (11) and (A.6) imply

$$\frac{\epsilon}{M} u_c\left(y, m_t\right) = \beta q_{t+1} u_c\left(y, m_{t+1}\right).$$

Alternatively, if $M_t = M_{t+1}$, then $\gamma_t \geq 0$; so, (7), (11) and (A.6) imply

$$\frac{\epsilon}{M} u_c\left(y, m_t\right) \leq \beta q_{t+1} u_c\left(y, m_{t+1}\right).$$

These equilibrium conditions can be rewritten as

$$z(m_t) + \beta \frac{q_{t+1}}{q_t} \frac{u_c\left(y, m_{t+1}\right)}{u_c\left(y, m_t\right)} = 1,$$

and

$$\text{either (Case A)} \quad M_t > M_{t+1} \text{ and } \frac{u_c\left(y, m_{t+1}\right)}{u_c\left(y, m_t\right)} = \frac{q^*}{q_{t+1}},$$

$$\text{or (Case B)} \quad M_t = M_{t+1} \text{ and } \frac{u_c\left(y, m_{t+1}\right)}{u_c\left(y, m_t\right)} \geq \frac{q^*}{q_{t+1}},$$

where $q^* \equiv \epsilon/(\beta M)$. In both Cases A and B, we have

$$z(m_t) + \beta \frac{q^*}{q_t} \leq 1.$$

Using $z'(x) < 0$ and $M_t \leq M$, we get $z(m_t) = z(q_t M_t) \geq z(q_t M)$ and therefore

$$z\left(q_t M\right) + \beta \frac{q^*}{q_t} \leq 1.$$

The left-hand side is strictly decreasing in q_t and takes the value $z(\epsilon/\beta)+\beta \geq 1$ for $q_t = q^*$, where the inequality comes from the condition for fundamental-equilibrium existence (A.8). Therefore,

$$q_t \geq q^*$$

at all dates $t \geq 0$. In Case A, therefore, we have

$$\frac{u_c\left(y, m_{t+1}\right)}{u_c\left(y, m_t\right)} = \frac{q^*}{q_{t+1}} \leq 1 \quad \text{and} \quad z(m_t) = 1 - \beta \frac{q^*}{q_t} \geq 1 - \beta,$$

which implies

$$m^* \geq m_t \geq m_{t+1}.$$

In Case B, we have $M_t = M_{t+1}$ and therefore

$$z(m_t) + \beta \frac{m_{t+1} u_c\left(y, m_{t+1}\right)}{m_t u_c\left(y, m_t\right)} = 1,$$

which can be rewritten as

$$\frac{m_{t+1} u_c(y, m_{t+1})}{m_t u_c(y, m_t)} = 1 + \left(\frac{1-\beta}{\beta}\right)\left[1 - \frac{z(m_t)}{z(m^*)}\right].$$

This dynamic equation is, of course, the same as the dynamic equation (A.9) of the model without redeeming scheme; it implies that either $m_{t+1} < m_t < m^*$, or $m^* < m_t < m_{t+1}$.

So, if $m_0 > m^*$, then we are always in Case B, m_t is strictly increasing over time, and we are on a deflationary path. If we impose the Inada condition $\lim_{x \to +\infty} u_m(y, x) = 0$, this path is not an equilibrium as it violates the transversality condition; so, we do not have an equilibrium with partial demonetization and $m_0 > m^*$.

Alternatively, if $m_0 < m^*$, then we can have either Case A or Case B at each date, and m_t is strictly decreasing over time. Since the sequence $(m_t)_{t \geq 0}$ is decreasing and non-negative, it converges to a value $\underline{m} \geq 0$. We cannot have $\underline{m} > 0$, because we would then get $\lim_{t \to +\infty} u_c(y, m_t)/u_c(y, m_{t+1}) = 1$ and $\lim_{t \to +\infty} z(m_t) = z(\underline{m}) > z(m^*) = 1 - \beta$, implying

$$\lim_{t \to +\infty} \frac{q_{t+1}}{q_t} = \lim_{t \to +\infty} \left[\frac{1 - z(m_t)}{\beta}\right] \frac{u_c\left(y, m_t\right)}{u_c\left(y, m_{t+1}\right)} = \frac{1 - z(\underline{m})}{\beta} < 1,$$

which would contradict the fact that the price of money q_t is bounded below by $q^* > 0$. So, we have $\underline{m} = 0$.

Given that the dynamic equation in Case B is the same as in the model without redeeming scheme, we can have only a finite number of Case-B dates (otherwise the price of money would be zero at a finite date, which would contradict the fact that the price of money q_t is bounded below by $q^* > 0$). So, we are always in Case A from a certain date onwards. From this certain date onwards, the dynamic equation is

$$\frac{u_c(y, m_{t+1})}{u_c(y, m_t)} = 1 + \left(\frac{1-\beta}{\beta}\right)\left[1 - \frac{z(m_{t+1})}{z(m^*)}\right].$$

Because of our assumption (A.5), the right-hand side of this dynamic equation is negative as m_{t+1} approaches zero (which is its limit as $t \to +\infty$). The left-hand side, however, is always non-negative. So, m_{t+1} needs to reach its limit zero at a finite date, which requires that the price of money be zero at a finite date, which contradicts the fact that the price of money is bounded below by $q^* > 0$. So, we do not have an equilibrium with partial demonetization and $m_0 < m^*$. We conclude that there is no equilibrium with partial demonetization.

2.6 Complete Demonetization

Under complete demonetization at date t (meaning $M_t > M_{t+1} = 0$), using $M_{t+1} = 0$, (11) and (A.6), we can rewrite (A.7) at date $t+1$ as

$$z(0) + \left[\frac{\gamma_{t+1}}{u_c(y,0)} + \frac{\epsilon}{M}\right]\frac{1}{q_{t+1}} = 1,$$

which implies that $z(0) \leq 1$, which in turn contradicts our assumption (A.5). So, we do not have an equilibrium with complete demonetization.

3 Further relaxation of the Inada condition

In the main text, we relaxed the standard Inada condition $\lim_{x\to 0} v'(x) = +\infty$ and replaced it with $v'(0) > u'(y)$. We now relax the Inada condition further and allow for $(1-\beta)u'(y) < v'(0) < u'(y)$. We do not allow for $v'(0) < (1-\beta)u'(y)$ because it would eliminate the fundamental equilibrium, both in the presence and in the absence of a redeeming scheme, since no value of ϵ (not even zero) would then satisfy the condition for fundamental-equilibrium existence (13).

The results that we obtain below can be summarized as follows. If $v'(0) > u'(y)$, then the model without redeeming scheme has (a countable infinity of) equilibria in which the price of money reaches zero at a finite date. By contrast, if $(1-\beta)u'(y) < v'(0) < u'(y)$, then the model without redeeming scheme has (a non-countable infinity of) equilibria in which the price of money converges asymptotically to zero without ever reaching zero. In both cases, however, as long as ϵ is sufficiently small for the fundamental equilibrium to exist, the redeeming scheme eliminates all inflationary equilibria. So, further relaxing the Inada condition leaves our conclusion unchanged.

3.1 Model Without Redeeming Scheme

Consider the separable MIU model without redeeming scheme. Households maximize

$$\sum_{t=0}^{+\infty} \beta^t \left[u(c_t) + v(q_t M_t)\right]$$

subject to

$$y + R_{t-1}b_{t-1} + q_t M_{t-1} - c_t - b_t - q_t M_t - \tau_t \geq 0.$$

Two first-order conditions are

$$u'(c_t) = \lambda_t,$$

$$q_t v'(q_t M_t) = \lambda_t q_t - \beta \lambda_{t+1} q_{t+1}.$$

Using the goods-market-clearing condition $c_t = y$ and the money-market-clearing condition $M_t = M$, we get the dynamic equation

$$\frac{m_{t+1}}{m_t} = 1 + \left(\frac{1-\beta}{\beta}\right)\left[1 - \frac{v'(m_t)}{v'(m^*)}\right], \tag{A.11}$$

where $m_t \equiv q_t M$ and where m^* is implicitly and uniquely defined by $v'(m^*) = (1-\beta)u'(y)$. Given this dynamic equation, the only candidate dynamic equilibria are inflationary paths $m^* > m_0 > m_1 > ...$ and deflationary paths $m^* < m_0 < m_1 < ....$

The deflationary paths are not equilibria if we impose the Inada condition $\lim_{x \to +\infty} v'(x) = 0$, which says that demand for money is asymptotically satiated: with this Inada condition, the deflationary paths violate the transversality condition (for any $m_0 > m^*$).

We study the inflationary paths as follows. First, since the sequence $(m_t)_{t \geq 0}$ is decreasing and non-negative, it converges to a value $\underline{m} \geq 0$. We cannot have $\underline{m} > 0$, because the left-hand side of (A.11) would then converge to 1 as $t \to +\infty$, while the right-hand side of (A.11) would converge to a value lower than 1. So, we have $\underline{m} = 0$. We then rewrite (A.11) as

$$\frac{m_{t+1}}{m_t} = \frac{1}{\beta}\left[1 - \frac{v'(m_t)}{u'(y)}\right], \tag{A.12}$$

and we distinguish between two cases, depending on whether $v'(0) > u'(y)$ or $(1-\beta)u'(y) < v'(0) < u'(y)$.

If $v'(0) > u'(y)$, then the right-hand side of (A.12) is strictly negative for $m_t = 0$. It is also strictly increasing in m_t and strictly positive for $m_t = m^*$. Let $\tilde{m}$ denote the unique value of m_t that makes this right-hand side equal to zero. We can rewrite the dynamic equation (A.12) as

$$m_{t+1} = G(m_t),$$

where $G(x) \equiv (x/\beta)[1 - v'(x)/u'(y)]$, $G(x) < 0$ for $x \in (0, \tilde{m})$, and $G(x) > 0$ and $G'(x) > 0$ for $x > \tilde{m}$. So, if $\lim_{x \to 0} G(x) < 0$, that is to say equivalently if the "super Inada condition" $\lim_{x \to 0} x v'(x) > 0$ is satisfied, then any inflationary path ends up violating the dynamic equation at some date; so, there are no inflationary equilibria. Alternatively, if $\lim_{x\to 0} G(x) = 0$ or equivalently if $\lim_{x\to 0} xv'(x) = 0$, then there exist "hyperinflationary" equilibria in which money becomes worthless at a finite date: for any $T \in \mathbb{N} \setminus \{0\}$, the sequence characterized by $m_t = 0$ for $t \geq T$, $m_{T-1} = \tilde{m}$, and $(m_t)_{T-2\geq t\geq 0}$ derived sequentially from m_{T-1} with the dynamic equation, is an equilibrium. There exists a countable infinity of such equilibria, which can be indexed by $T \in \mathbb{N} \setminus \{0\}$.

Alternatively, if $(1-\beta)u'(y) < v'(0) < u'(y)$, then the right-hand side of (A.12) is strictly positive for $m_t = 0$. It is also strictly increasing in m_t and, therefore, strictly positive for any $m_t \geq 0$. In this case, for any $m_0 < m^*$, the sequence $(m_t)_{t\geq 0}$ derived sequentially from m_0 with the dynamic equation is an equilibrium path that converges asymptotically to zero without ever reaching zero. There exists a non-countable infinity of such equilibria, which can be indexed by $m_0 \in (0, m^*)$.

3.2 Model With Redeeming Scheme

We have shown in the main text that if $v'(0) > u'(y)$, then the redeeming scheme eliminates all inflationary equilibria. More specifically, we have shown that: (i) the only no-demonetization equilibrium is the fundamental equilibrium; (ii) there are no equilibria with partial demonetization; and (iii) there are no equilibria with complete demonetization.

Our proofs for (ii)-(iii) in the main text do not use the restriction $v'(0) > u'(y)$; these proofs work equally well in the case $(1-\beta)u'(y) < v'(0) < u'(y)$.

Our proof for (i) in the main text uses the fact that the candidate no-demonetization equilibria in our model with redeeming scheme are the equilibria of the model without redeeming scheme. This proof uses the restriction $v'(0) > u'(y)$ *only* to get that the price of money converges to zero in the latter equilibria; but the proof works equally well independently of whether the price of money reaches zero at a finite date and remains at zero thereafter (as in the case $v'(0) > u'(y)$), or converges asymptotically to zero without ever reaching zero (as in the case $(1-\beta)u'(y) < v'(0) < u'(y)$).

So, the redeeming scheme eliminates all inflationary equilibria also in the case $(1-\beta)u'(y) < v'(0) < u'(y)$.