Pegging the Interest Rate on Bank Reserves: A Resolution of New Keynesian Puzzles and Paradoxes

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Abstract

We consider various ways of introducing money, interpreted or modeled as bank reserves, into the basic New Keynesian (NK) model. In the resulting models, the central bank can set independently the interest rate on bank reserves and the nominal stock of bank reserves. As long as there is a monetary friction, the models deliver local-equilibrium determinacy under permanently exogenous monetary-policy instruments. As a result, they do not share the puzzling and paradoxical implications of the basic NK model under a temporary interest-rate peg (e.g., in the context of a liquidity trap). More specifically, they offer a resolution of the "forward-guidance puzzle," a related puzzle about fiscal multipliers, and the "paradox of flexibility," even for an *arbitrarily* small monetary friction. As the monetary friction becomes *vanishingly* small, the models converge to the basic NK model and serve to select a particular equilibrium of that model – still (fully or partially) solving the above puzzles and paradox, and also offering a resolution of the "paradox of toil."

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1 Introduction

The Great Recession led central banks to peg their policy rates near zero and provide forward guidance about future policy rates. The recession also rekindled interest in the use of discretionary fiscal policy as a stabilization tool, and, in Europe, sparked debate about implementing structural reforms. The mainstream New Keynesian (NK) literature is of limited help for economists and policymakers in this context, because standard NK models have puzzling and paradoxical implications about the consequences of forward guidance, fiscal policy, and structural reforms under a temporary interest-rate peg – e.g., in the context of a liquidity trap during which the interest rate is pegged at its zero lower bound (ZLB).

In this paper, we show that a simple departure from the basic NK model (Woodford, 2003, Galí, 2008) offers a resolution of these puzzles and paradoxes. This departure adds a (possibly small) monetary friction to the basic NK model – using, in turn, an ad-hoc money-demand function, or a familiar money-in-utility (MIU) setup, or the model with a more explicit role for bank reserves presented in Diba and Loisel (2020). Adding money – either interpreted or modeled as bank reserves – to the NK model allows us to capture more precisely what central banks actually did during the Great Recession: they pegged the interest rate on bank reserves (the IOR rate) and switched to balance-sheet policies that essentially set the nominal stock of bank reserves.

In standard NK models, the effects of a marginal increase in the duration of an interest-rate peg on initial inflation and output become unboundedly large as the duration of the peg goes to infinity (the so-called "forward-guidance puzzle") or as prices become perfectly flexible (the so-called "paradox of flexibility"). Moreover, the effects of a fiscal expansion at the end of the peg on initial inflation and output also grow explosively as the duration of the peg goes to infinity (what we henceforth call the "fiscal-multiplier puzzle"). The forward-guidance and fiscal-multiplier puzzles make policy interventions in the vanishingly distant future have unboundedly large effects, instead of vanishingly small effects, on current outcomes; while the paradox of flexibility makes the effects of policy interventions grow explosively as prices become more and more flexible, instead of converging towards their finite flexible-price effects. In addition to these three "limit puzzles," the basic NK model has another perplexing implication known as the "paradox of toil:" positive supply shocks – such as downward shifts in the labor-disutility function, labor-tax cuts, technology improvements, and reductions in market power – are not expansionary, but contractionary, under a temporary interest-rate peg.

The literature, cited below, has linked some of these NK puzzles and paradoxes to local-equilibrium indeterminacy (i.e., multiplicity of equilibria in the neighborhood of the steady state) under a permanently exogenous policy rate. The basic NK model has a stable eigenvalue under exogenous policy rates, but no predetermined variable. Under a permanently exogenous policy rate – e.g., under a permanent peg – this loose stable eigenvalue leads to indeterminacy. Under a temporarily exogenous policy rate – e.g., under a temporary peg – determinacy is ensured by future policy normalization; but as the model is iterated backward in time, the loose stable eigenvalue is inverted and magnifies the effects of future conditions (close to or at the end of the peg) on initial outcomes (at the start of the peg). These effects grow explosively as the duration of the peg goes to infinity, thus giving rise to the forward-guidance and fiscal-multiplier puzzles. Moreover, the indeterminacy property of the basic NK model is also behind the paradox of flexibility, as we highlight in the text: the stable eigenvalue converges towards zero as prices become more and more flexible; this leads to explosive initial outcomes even for a peg of given (short) duration.

Indeterminacy under a permanently exogenous policy rate, in turn, is a familiar feature of standard monetary models that, following Sargent and Wallace (1975), assume the central bank pegs the interest rate on a bond, committing to buy or sell this bond at the implied price. This makes the money supply endogenous, and it makes any arbitrary price level (with the associated nominal money stock) consistent with an equilibrium.

During the Great Recession, however, central banks did not peg the (market-determined) interest rate on a bond; the lower bound on nominal interest rates forced them to peg the IOR rate, which is the interest rate that they directly control (a policy instrument). Since bank reserves essentially serve as the medium of exchange and as the unit of account, pegging the IOR rate did not prevent central banks from setting also the nominal stock of reserves (or, more precisely, the size of their balance sheet). Indeed, they did announce and implement balance-sheet policies that effectively set targets for the nominal stock of bank reserves.

In the three monetary models that we consider, setting exogenously these two monetary-policy instruments, the IOR rate and the nominal stock of bank reserves, always delivers determinacy.[1] This determinacy result is familiar in our simple framework that adds an ad-hoc money-demand nexus to the basic NK model (which is isomorphic to the MIU model with separable utility). In this framework, as pointed out by Woodford (2003, Chapter 4), setting exogenously the IOR rate and the money stock amounts to following (what we call) a "shadow Wicksellian rule" for the interest rate on bonds: it is as if the central bank directly controlled this interest rate and set it as an increasing function of the price level and output. This is because an increase in the price level or output would raise demand for money; and the interest rate on bonds would have to rise to restore equilibrium, given the prevailing nominal money stock and IOR rate set by policy. In the MIU model with non-separable utility, and in our model in which banks hold reserves to reduce banking costs (Diba and Loisel, 2020), setting the two monetary-policy instruments exogenously also amounts to following a shadow Wicksellian rule. We show that this specific shadow Wicksellian rule, given its implied coefficients, also ensures determinacy in these models for all functional forms of the utility and production functions and all values of the structural

[1] In a recent independent contribution, Piazzesi et al. (2019) also show that setting exogenously these two monetary-policy instruments delivers determinacy in the MIU model and in a banking model of theirs. They do not formally study the implications for the NK puzzles and paradoxes.

and (steady-state) policy parameters.

Since they deliver determinacy under exogenous monetary-policy instruments, all these models solve the three limit puzzles: policy interventions in the vanishingly distant future have vanishingly small effects, instead of unboundedly large effects, on current outcomes; and, as prices become more and more flexible, the effects of policy interventions converge towards their finite flexible-price effects, instead of growing explosively.

The ad-hoc model and the MIU model enable us to make our points in simple and familiar frameworks. They have, however, an awkward implication: namely, they allow for the possibility of complex eigenvalues, opening the door to a "reversal puzzle" (e.g., recurrent sign reversals in the effect of forward guidance on current inflation and output as we change the guidance horizon). Our more structured model with banks has the advantage of always delivering positive real eigenvalues and thus always avoiding this reversal puzzle.

The models that we consider solve the three limit puzzles even for *arbitrarily* small monetary frictions. In this sense, our resolution of these puzzles does not require a big departure from the basic NK model. In the limit, as we make the monetary frictions *vanishingly* small, the models converge to the basic NK model and serve to select uniquely a particular equilibrium of that model under a permanently exogenous policy rate. The selected equilibrium does not depend on how we add monetary frictions to the basic NK model, and we argue that this robustness feature of our selected equilibrium would extend to other models that deliver determinacy under exogenous policy instruments and converge to the basic NK model as we shrink some friction.[2] We show that our selected equilibrium does not exhibit the fiscal-multiplier puzzle, nor the paradox of flexibility, nor the paradox of toil; and that it exhibits a distinctively weaker form of forward-guidance puzzle. In the text, we compare this equilibrium to the equilibria considered in Cochrane (2017) and Bilbiie (2018). One notable difference is that our selected equilibrium never exhibits Neo-Fisherian effects: current and future interest-rate hikes are always deflationary in our equilibrium.

Our resolution of the NK puzzles and paradoxes rests on the assumption that demand for bank reserves is not fully satiated. Indeed, if it were fully satiated, then our monetary models would exactly coincide with the basic NK model, and all the NK puzzles and paradoxes would re-emerge. Our non-satiation assumption stands in contrast to views often expressed about the US economy in recent years (e.g., Cochrane, 2014, 2018; Reis, 2016). We defend this assumption in the text, drawing on more formal arguments presented in a companion paper (Diba and Loisel, 2020). We also show that we still solve the NK puzzles and paradoxes when demand for bank reserves is *arbitrarily* close to satiation, and even when it is *asymptotically* satiated.

[2]There are, of course, monetary models that are not amenable to our equilibrium-selection exercise, for instance because their reduced form does not converge to the reduced form of the basic NK model as we shrink the monetary friction. The standard cash-in-advance model with cash and credit goods is a case in point, as we explain in the text.

In the latter case, indeed, the unique equilibrium of our monetary models converges to our selected equilibrium of the basic NK model (in which the fiscal-multiplier puzzle, the paradox of flexibility, and the paradox of toil are fully solved, and the forward-guidance puzzle is partially solved). The reason is that going to satiation asymptotically alleviates the monetary friction and has the same effect as asymptotically removing the monetary friction from the model (as in our equilibrium-selection exercise).

Some brief remarks may serve to put our contribution in the context of the recent literature on NK puzzles and paradoxes. The phrases "forward-guidance puzzle," "paradox of flexibility," and "paradox of toil" were coined respectively by Del Negro et al. (2015), Eggertsson and Krugman (2012), and Eggertsson (2010), while Farhi and Werning (2016) were the first to expose the fiscal-multiplier puzzle. Other early contributions related to at least one of the NK puzzles and paradoxes include Christiano et al. (2011), Eggertsson (2011, 2012), Eggertsson et al. (2014), Werning (2012), and Woodford (2011). Wieland (2019) presents empirical evidence against the paradox of toil. Carlstrom et al. (2015) and Cochrane (2017) clearly show the link between the forward-guidance and fiscal-multiplier puzzles and indeterminacy under a permanently exogenous policy rate.

A fast-growing number of contributions have proposed departures from the basic NK model that solve or attenuate at least one of the NK puzzles and paradoxes ("attenuate" in the sense of reducing the quantitative effects of policy interventions under an interest-rate peg for a *given* duration of the peg and a *given* degree of price stickiness). These departures may involve non-rational expectations (e.g. Farhi and Werning, 2019, Gabaix, 2019, García-Schmidt and Woodford, 2019); information frictions (e.g. Angeletos and Lian, 2018, Kiley, 2016, Wiederholt, 2015); incomplete markets (e.g. Bilbiie, 2019, Hagedorn et al., 2019, McKay et al., 2016); overlapping generations (Del Negro et al., 2015); non-Ricardian fiscal policy (Cochrane, 2017, 2018); and government bonds with a convenience yield (e.g. Bredemeier et al., 2018, Hagedorn, 2018a, 2018b, Michaillat and Saez, 2019).

Against this literature background, our work has three distinctive features. First, we are solving (not just attenuating) *all* the NK puzzles and paradoxes. Second, our resolution of the puzzles and paradoxes is purely monetary (not fiscal), and it is based on what central banks actually did during the Great Recession. And third, we stay close to the NK paradigm in the sense that we solve the puzzles and paradoxes even for arbitrarily small monetary frictions, and we still − fully or partially − solve them for vanishingly small frictions (leading to a new equilibrium-selection approach).

Several contributions to the literature propose models that can solve the forward-guidance puzzle by "discounting" the IS equation or the Phillips curve of the basic NK model, i.e. by scaling down the coefficients of their expectational terms. We generalize Cochrane's (2016) comments on Gabaix (2019) to highlight three differences between these models and ours: (i) discounting models do not solve the paradox of flexibility; (ii) they require a discrete (sufficiently large) departure from the basic NK model to solve the forward-guidance puzzle; and (iii) their resolution of the forward-guidance puzzle comes at the expense of non-standard implications for equilibrium determinacy in normal times and of overturning the standard Fisher effect (i.e. replacing the *one-to-one* long-term relationship between the inflation rate and the nominal interest rate with a *negative* relationship).

The rest of the paper is organized as follows. Section 2 briefly exposes the puzzles and paradoxes in the basic NK model. Section 3 presents our main points in a simple setting that adds a money-demand nexus to the basic NK model. Section 4 confirms and strengthens our points in the MIU model and a model with banks. Section 5 clarifies and defends our non-satiation assumption. Section 6 compares our monetary models with a class of discounting models. We then conclude and provide a technical appendix that contains proofs and some numerical illustrations.

2 Puzzles and Paradoxes in the Basic NK Model

We start with a brief exposition of the puzzles and paradoxes in the basic NK model. The log-linearized IS equation and Phillips curve of this model are

$$y_t = \mathbb{E}_t\{y_{t+1}\} - \frac{1}{\sigma}(i_t - \mathbb{E}_t\{\pi_{t+1}\} - r_t) + g_t - \mathbb{E}_t\{g_{t+1}\} \quad (1)$$

$$\pi_t = \beta\mathbb{E}_t\{\pi_{t+1}\} + \kappa(y_t - \delta_g g_t - \delta_\varphi \varphi_t), \quad (2)$$

where y_t denotes the output level, π_t the inflation rate, i_t the nominal interest rate on a bond (in zero net supply) that households can trade with each other, r_t a preference shock (stemming from variations in the discount factor), g_t a government-purchases shock, and φ_t a supply shock (stemming from shifts in the labor-disutility function, labor-tax modifications, technology changes, or variations in the elasticity of substitution between intermediate goods). All variables and shocks are expressed in log-deviation from their steady-state value, except g_t (which is proportional to the log-deviation of government purchases from steady state). The parameters satisfy $\beta \in (0,1)$, $\sigma > 0$, $\delta_g \in (0,1)$, $\delta_\varphi > 0$, $\kappa > 0$; and κ increases without bound as the degree of price stickiness goes to zero. Finally, $\mathbb{E}_t\{.\}$ denotes the rational-expectations operator at date t.

The monetary-policy instrument, in this model, is the interest rate i_t. Under a permanently exogenous policy rate, the IS equation (1) and the Phillips curve (2) lead to the following dynamic equation relating π_t to $\mathbb{E}_t\{\pi_{t+2}\}$, $\mathbb{E}_t\{\pi_{t+1}\}$, and exogenous terms:

$$\mathbb{E}_t\left\{\mathcal{P}_b\left(L^{-1}\right)\pi_t\right\} = Z_t^b, \quad (3)$$

where L denotes the lag operator, $\mathcal{P}_b(X) \equiv X^2 - [1 + 1/\beta + \kappa/(\beta\sigma)]X + 1/\beta$, and

$$Z_t^b \equiv \frac{-\kappa}{\beta\sigma}(i_t - r_t) + \frac{(1-\delta_g)\kappa}{\beta}(g_t - \mathbb{E}_t\{g_{t+1}\}) - \frac{\delta_\varphi\kappa}{\beta}(\varphi_t - \mathbb{E}_t\{\varphi_{t+1}\})$$

(the subscript "b" and superscript "b" stand for "basic"). Since $\mathcal{P}_b(0) = 1/\beta > 0$, $\mathcal{P}_b(1) = -\kappa/(\beta\sigma) < 0$, and $\lim_{X\to+\infty} \mathcal{P}_b(X) = +\infty > 0$, the roots of $\mathcal{P}_b(X)$ are two real numbers ρ_b and ω_b such that $0 < \rho_b < 1 < \omega_b$. We can thus rewrite the dynamic equation (3) as

$$\mathbb{E}_t \left\{ \left(L^{-1} - \omega_b\right) \left(L^{-1} - \rho_b\right) \pi_t \right\} = Z_t^b. \tag{4}$$

Because $\rho_b \in (0,1)$, however, we cannot iterate this equation forward to $+\infty$ and get π_t as a bounded function of current and expected future exogenous shocks. Nor can we iterate the equation backward to pin down a unique solution for π_t, given the absence of a π_{t-1} term in this equation. The dynamic system has one stable eigenvalue (ρ_b) and no predetermined variable (no π_{t-1} term), Blanchard and Kahn's (1980) conditions are not satisfied, and local-equilibrium indeterminacy arises under a permanently exogenous policy rate.

Under a *temporarily* exogenous policy rate, however, local-equilibrium determinacy can be obtained by assuming that policy will switch in the future to a rule that sets a nominal anchor (e.g., a Taylor rule like $i_t = \phi\pi_t$ with $\phi > 1$). The literature about the NK puzzles and paradoxes typically assumes that after a finite date T, there are no expected shocks and the economy is expected to be back to its steady state: $\mathbb{E}_t\{r_{T+k}\} = \mathbb{E}_t\{g_{T+k}\} = \mathbb{E}_t\{\varphi_{T+k}\} = \mathbb{E}_t\{\pi_{T+k}\} = \mathbb{E}_t\{y_{T+k}\} = \mathbb{E}_t\{i_{T+k}\} = 0$ for all $t \leq T$ and $k \geq 1$. Under this assumption, we can use the method of partial fractions to iterate the dynamic equation (4) forward until date T; we get[3]

$$\begin{aligned}
\pi_t &= \mathbb{E}_t \left\{ \frac{Z_t^b}{\left(L^{-1} - \omega_b\right)\left(L^{-1} - \rho_b\right)} \right\} \\
&= \frac{\mathbb{E}_t}{\omega_b - \rho_b} \left\{ \frac{\rho_b^{-1} Z_t^b}{1 - (\rho_b L)^{-1}} - \frac{\omega_b^{-1} Z_t^b}{1 - (\omega_b L)^{-1}} \right\} \\
&= \frac{\mathbb{E}_t}{\omega_b - \rho_b} \left\{ \sum_{k=0}^{T-t} \left(\rho_b^{-k-1} - \omega_b^{-k-1} \right) Z_{t+k}^b \right\}.
\end{aligned}$$

Using the definition of Z_t^b, we then get

$$\begin{aligned}
\pi_t = \; & \frac{\kappa \mathbb{E}_t}{\beta\left(\omega_b - \rho_b\right)} \left\{ \frac{-1}{\sigma} \sum_{k=0}^{T-t} \left(\rho_b^{-k-1} - \omega_b^{-k-1} \right) \left(i_{t+k} - r_{t+k}\right) \right. \\
& \left. + \sum_{k=0}^{T-t} \left[\left(1 - \rho_b\right) \rho_b^{-k-1} + \left(\omega_b - 1\right) \omega_b^{-k-1} \right] \left[\left(1 - \delta_g\right) g_{t+k} - \delta_\varphi \varphi_{t+k} \right] \right\}.
\end{aligned} \tag{5}$$

Finally, using (5) to replace π_t and π_{t+1} in the Phillips curve (2), and using also $\beta\rho_b\omega_b = 1$ and $\mathcal{P}_b(\rho_b) = \mathcal{P}_b(\omega_b) = 0$, we get

$$\begin{aligned}
y_t = \; & \frac{\kappa \mathbb{E}_t}{\beta\sigma\left(\omega_b - \rho_b\right)} \left\{ \frac{-1}{\sigma} \sum_{k=0}^{T-t} \left(\frac{\rho_b^{-k}}{1 - \rho_b} + \frac{\omega_b^{-k}}{\omega_b - 1} \right) \left(i_{t+k} - r_{t+k}\right) \right. \\
& \left. + \sum_{k=1}^{T-t} \left(\rho_b^{-k} - \omega_b^{-k} \right) \left[\left(1 - \delta_g\right) g_{t+k} - \delta_\varphi \varphi_{t+k} \right] \right\} + g_t.
\end{aligned} \tag{6}$$

[3] Our presentation in this section is a discrete-time version of the presentations in Werning (2012), Farhi and Werning (2016), and Cochrane (2017).

Equations (5)-(6) form what Cochrane (2017) calls the "standard equilibrium" of the basic NK model. Because the stable eigenvalue ρ_b has been inverted to obtain these equations, the effects of future shocks on current inflation and output grow at rate ρ_b^{-k} in the horizon k of the shocks. In particular, a persistently negative preference shock ($r_{t+k} < 0$ for all $k \in \{0, ..., T-t\}$) exerts deflationary and contractionary pressures that grow exponentially in the duration of the shock ($\lim_{T \to +\infty} \partial \pi_t / \partial r_T = \lim_{T \to +\infty} \partial y_t / \partial r_T = +\infty$). Setting the current policy rate i_t to its ZLB may not be enough to offset these pressures. But the central bank can easily offset or even overturn them by promising a low policy rate in the sufficiently distant future (i.e. a small i_{t+T} for a sufficiently large T), because the power of forward guidance grows exponentially in the guidance horizon ($\lim_{T \to +\infty} \partial \pi_t / \partial i_T = \lim_{T \to +\infty} \partial y_t / \partial i_T = -\infty$). Similarly, the government can easily offset or overturn these pressures by promising a fiscal expansion in the sufficiently distant future (i.e. $g_{t+T} > 0$ for a sufficiently large T), because the effectiveness of future fiscal expansions grows exponentially in their implementation horizon ($\lim_{T \to +\infty} \partial \pi_t / \partial g_T = \lim_{T \to +\infty} \partial y_t / \partial g_T = +\infty$). The equilibrium, thus, exhibits the forward-guidance and fiscal-multiplier puzzles.

The inversion of the stable eigenvalue ρ_b, leading to the terms ρ_b^{-k} in Equations (5)-(6), is also responsible for the paradox of flexibility. Indeed, as the degree of price stickiness θ (i.e. the probability that a firm cannot change its price at a given date) goes to zero, we have $\kappa \to +\infty$ and therefore $\omega_b = [1 + \beta + \kappa/\sigma + \sqrt{(1 + \beta + \kappa/\sigma)^2 - 4\beta}]/(2\beta) \to +\infty$ and $\rho_b = \mathcal{P}_b(0)/\omega_b = 1/(\beta\omega_b) \to 0$. As a consequence, the terms ρ_b^{-k} in Equations (5)-(6) make current inflation and output explode in response to future shocks as prices are made more and more flexible, for any given horizon of the shocks ($\lim_{\theta \to 0} \partial \pi_t / \partial i_{t+k} = \lim_{\theta \to 0} \partial y_t / \partial i_{t+k} = -\infty$ and $\lim_{\theta \to 0} \partial \pi_t / \partial g_{t+k} = \lim_{\theta \to 0} \partial y_t / \partial g_{t+k} = +\infty$ for any $k \in \{t+1, ..., T\}$).

Finally, in addition to these three limit puzzles and for a related reason, the standard equilibrium also exhibits the paradox of toil: current and future positive supply shocks are not expansionary; instead, current ones are neutral ($\partial y_t / \partial \varphi_t = 0$) and future ones are contractionary ($\partial y_t / \partial \varphi_{t+k} < 0$ for any $k \in \{1, ..., T-t\}$, given the presence of $\rho_b^{-k} > \omega_b^{-k}$ in the second line of (6)).

3 Resolution of the Puzzles and Paradoxes in a Simple Model

We now add a money-demand nexus to the basic NK model, and show how setting exogenously the interest rate on money and the stock of money solves the NK puzzles and paradoxes. The model is simple, but not micro-founded; we will consider two micro-founded models in the next section.

3.1 Resolution of the Three Limit Puzzles

We consider a standard log-linear money-demand equation, of the kind estimated in the empirical literature, linking demand for real money balances m_t positively to output y_t and negatively to the opportunity cost of holding money $i_t - i_t^m$:

$$m_t = \chi_y y_t - \chi_i \left(i_t - i_t^m\right), \tag{7}$$

where i_t^m denotes the interest rate on money, $\chi_y > 0$, and $\chi_i > 0$. Real money balances m_t are also linked to nominal money balances M_t and the price level p_t through the identity $m_t = M_t - p_t$ (where all variables are expressed in log-deviation from some constant value). The monetary-policy instruments are, now, i_t^m and M_t. When these instruments are permanently exogenous, the IS equation (1), the Phillips curve (2), the money-demand equation (7), and the identities $m_t = M_t - p_t$ and $\pi_t = p_t - p_{t-1}$ lead to the following dynamic equation relating p_t to $\mathbb{E}_t\{p_{t+2}\}$, $\mathbb{E}_t\{p_{t+1}\}$, p_{t-1}, and exogenous terms:

$$\mathbb{E}_t\left\{L\mathcal{P}\left(L^{-1}\right)p_t\right\} = Z_t,$$

where

$$\begin{aligned}
\mathcal{P}(X) \equiv\ & X^3 - \left(2 + \frac{1}{\beta} + \frac{\kappa}{\beta\sigma} + \frac{\chi_y}{\sigma\chi_i}\right)X^2 \\
& + \left[1 + \frac{2}{\beta} + \left(1 + \frac{1}{\chi_i}\right)\frac{\kappa}{\beta\sigma} + \left(1 + \frac{1}{\beta}\right)\frac{\chi_y}{\sigma\chi_i}\right]X - \left(\frac{1}{\beta} + \frac{\chi_y}{\beta\sigma\chi_i}\right),
\end{aligned}$$

$$\begin{aligned}
Z_t \equiv\ & \frac{-\kappa}{\beta\sigma}\left(i_t^m - r_t\right) + \frac{\kappa}{\beta\sigma\chi_i}M_t + \left[1 - \left(1 + \frac{\chi_y}{\sigma\chi_i}\right)\delta_g\right]\frac{\kappa}{\beta}g_t \\
& - \frac{\left(1 - \delta_g\right)\kappa}{\beta}\mathbb{E}_t\left\{g_{t+1}\right\} - \left(1 + \frac{\chi_y}{\sigma\chi_i}\right)\frac{\delta_\varphi\kappa}{\beta}\varphi_t + \frac{\delta_\varphi\kappa}{\beta}\mathbb{E}_t\left\{\varphi_{t+1}\right\}.
\end{aligned}$$

We show in Appendix A.1 that the characteristic polynomial $\mathcal{P}(X)$ has one root inside the unit circle ($\rho \in (0,1)$) and two roots outside the unit circle (ω_1 and ω_2 with $|\omega_1| \leq |\omega_2|$); and that the latter roots are either positive real numbers or complex conjugates. We assume that they are distinct real numbers (so that $\omega_2 > \omega_1 > 1$), and postpone the discussion of this assumption to the next section. Using the roots of $\mathcal{P}(X)$, we can rewrite the dynamic equation as

$$\mathbb{E}_t\left\{\left(L^{-1} - \omega_1\right)\left(L^{-1} - \omega_2\right)(1 - \rho L)p_t\right\} = Z_t,$$

and we can use the method of partial fractions to solve this equation forward and get the unique bounded solution for $p_t - \rho p_{t-1}$:[4]

$$\begin{aligned}
p_t - \rho p_{t-1} &= \mathbb{E}_t\left\{\frac{Z_t}{\left(L^{-1} - \omega_1\right)\left(L^{-1} - \omega_2\right)}\right\} = \frac{\mathbb{E}_t}{\omega_2 - \omega_1}\left\{\frac{\omega_1^{-1}Z_t}{1 - \left(\omega_1 L\right)^{-1}} - \frac{\omega_2^{-1}Z_t}{1 - \left(\omega_2 L\right)^{-1}}\right\} \\
&= \frac{\mathbb{E}_t}{\omega_2 - \omega_1}\left\{\sum_{k=0}^{+\infty}\left(\omega_1^{-k-1} - \omega_2^{-k-1}\right)Z_{t+k}\right\}.
\end{aligned} \tag{8}$$

[4] Confining our analysis to bounded solutions amounts to following the common practice to set aside the global indeterminacy inherent in models with fiat money.

At any date t, p_{t-1} is known, and (8) pins down p_t uniquely. If time starts at $-\infty$, we can iterate (8) backward and get p_t as a unique bounded function of the exogenous forcing variables $\mathbb{E}_{t-j}\{Z_{t-j+k}\}$ for all $(j,k) \in \mathbb{N}^2$. Thus, the model delivers local-equilibrium determinacy under permanently exogenous monetary-policy instruments. The dynamic system still has one stable eigenvalue (ρ), but this eigenvalue is now matched by a predetermined variable (p_{t-1}), so that Blanchard and Kahn's (1980) conditions are satisfied.

This determinacy result can be interpreted as follows. Under exogenous monetary-policy instruments i_t^m and M_t, the money-demand equation (7) makes the interest rate on bonds i_t a strictly increasing function of output and, crucially, the price level: $i_t = (\chi_y/\chi_i)y_t + (1/\chi_i)p_t + [i_t^m - (1/\chi_i)M_t]$. If the price level (or output) rises, demand for nominal money balances increases; and, given the exogenous policy instruments, the interest rate on bonds must increase to clear the money market. Thus, our model with exogenous monetary-policy instruments is isomorphic to the basic NK model with a "Wicksellian rule" for the interest rate i_t (which is the policy rate in the latter model); and Wicksellian rules are well known to ensure determinacy in the basic NK model (as shown in Woodford, 2003, Chapter 4). At the ZLB, the central bank has to peg the IOR rate i_t^m; but it is as if it could set the interest rate on bonds i_t according to this (shadow) Wicksellian rule ensuring determinacy.

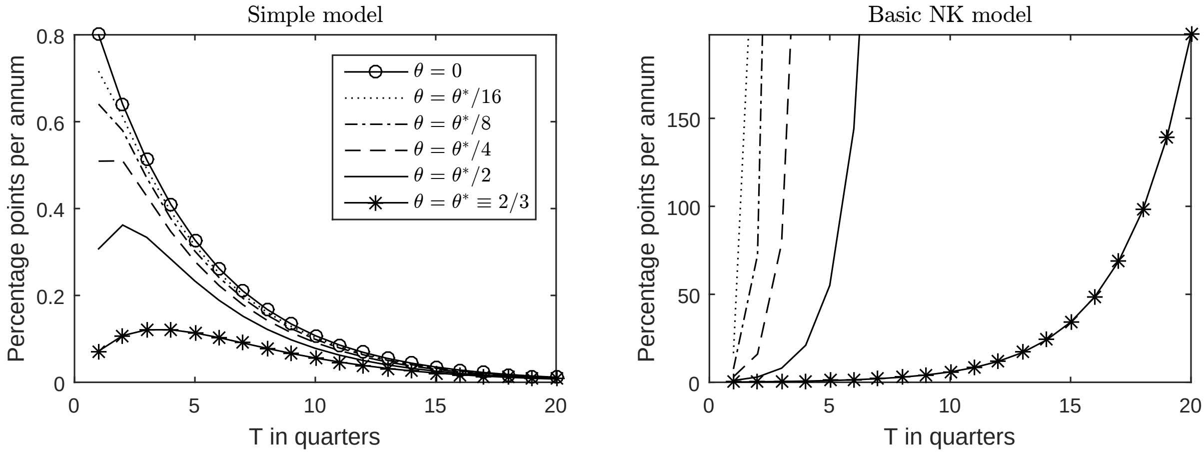
Using the price-level solution (8), the identity $\pi_t = p_t - p_{t-1}$, and the Phillips curve (2), we get

$$\pi_t = -(1-\rho)\,p_{t-1} + \frac{\mathbb{E}_t}{\omega_2 - \omega_1}\left\{\sum_{k=0}^{+\infty}\left(\omega_1^{-k-1} - \omega_2^{-k-1}\right) Z_{t+k}\right\}, \tag{9}$$

$$y_t = -\vartheta p_{t-1} + \delta_g g_t + \delta_\varphi \varphi_t - \frac{\mathbb{E}_t}{(\omega_2 - \omega_1)\,\kappa}\left\{\sum_{k=0}^{+\infty}\left(\xi_1\omega_1^{-k-1} - \xi_2\omega_2^{-k-1}\right) Z_{t+k}\right\}, \tag{10}$$

where $\vartheta \equiv (1-\rho)(1-\beta\rho)/\kappa$ and $\xi_j \equiv \beta(\omega_j + \rho - 1) - 1$ for $j \in \{1,2\}$. One key difference between our simple model's equilibrium (9)-(10) and the basic NK model's standard equilibrium (5)-(6) is that the former involves only ω_1^{-k} and ω_2^{-k} terms with $\omega_1 > 1$ and $\omega_2 > 1$. The NK model's standard equilibrium involves ρ_b^{-k} terms with $\rho_b \in (0,1)$, which are responsible for the limit puzzles discussed in the previous section. As a consequence, the implications of our simple model for the response of inflation and output to anticipated future shocks are in sharp contrast to the corresponding implications of the basic NK model: the later shocks are expected to occur, the bigger their current effects in the basic NK model, but the smaller their current effects in our model, regardless of which type of shock (preference, monetary, fiscal, or supply) we consider. More specifically, in our model, shocks occurring at date $t+k$ and announced at date t do not affect p_{t-1}; their effects on inflation and output at date t decay at an exponential rate, converging to zero essentially like ω_1^{-k}. So, neither the forward-guidance puzzle nor the fiscal-multiplier puzzle can arise in our model. The central bank in our model can provide forward guidance not only about low future policy rates (i_{t+k}^m), but also about large future balance sheets (M_{t+k}), in order to offset the deflationary pressures exerted by the negative preference shock r_t; but the effectiveness of both types of forward guidance decreases in the guidance horizon k.

Figure 1 – Effect of a policy-rate cut at date T on inflation at date 1



Note: The figure displays the effect on π_1 of announcing at date 1 a one-percentage-point-per-annum cut in i_T^m (for the simple model) or i_T (for the basic NK model), as a function of $T \in \{1, ..., 20\}$. Benchmark parameter values are set as in Galí (2008, Chapter 3): $\beta = 0.99$, $\sigma = 1$, $\chi_y = 1$, $\chi_i = 4$, and $\kappa = \lambda[(1-\theta)(1-\beta\theta)/\theta] = 0.13$, where $\lambda = 3/4$ and $\theta = \theta^* \equiv 2/3$. As θ takes the values $\theta^*/2$, $\theta^*/4$, $\theta^*/8$, and $\theta^*/16$, κ takes respectively the values 1.00, 3.13, 7.57, and 16.54.

Moreover, because we have $0 < \rho < 1 < |\omega_1| \leq |\omega_2|$ for any degree of price stickiness $\theta \in (0, 1)$ and in particular as $\theta \to 0$, the paradox of flexibility does not arise in our simple model. In Appendix A.2, we show that the limits of π_t and y_t as $\theta \to 0$ take finite values, unlike their counterparts in the basic NK model, and that these values coincide with the values that π_t and y_t take under perfectly flexible prices (in particular, $\lim_{\theta \to 0} y_t = \delta_g g_t + \delta_\varphi \varphi_t$). So, the model involves no discontinuity at the $\theta = 0$ point, in contrast to the basic NK model.

To illustrate graphically our resolution of the forward-guidance puzzle and the paradox of flexibility, Figure 1 shows the effects of cutting the policy rate by 25 basis points (one percentage point per annum) in Quarter T on the annualized inflation rate in Quarter 1 (when the rate cut is announced). We start from a benchmark calibration borrowed from Galí (2008, Chapter 3), which sets the degree of price stickiness θ to 2/3 (corresponding to "3-quarter price rigidity"), and we then cut θ in half step by step to make prices more flexible.[5] The right panel in Figure 1 replicates the implausible implications of the basic NK model: cutting the policy rate in a later quarter leads to an exponentially larger effect on current inflation; and making prices more flexible accelerates these explosive effects. The left panel shows the results for our simple model: with our benchmark value of $\theta = 2/3$, the inflationary effects of the policy-rate cut are modest (about 10 basis points for a cut in one of the first five quarters) and die off relatively quickly with the horizon of the cut; as we make prices more flexible, these inflationary effects smoothly converge to the effects under perfect price flexibility ($\theta = 0$). Figure 1 is, of course, only illustrative, as our simple model is not really suitable for a quantitative assessment of the

[5] The roots ω_1 and ω_2 are positive real numbers under all these calibrations, consistently with our assumption above.

effects of forward guidance or other policies.[6] Our contribution here is to show that this model offers a qualitative resolution of the three limit puzzles under any calibration.

3.2 Equilibrium Selection and Resolution of the Paradox of Toil

In our simple model, we can asymptotically remove the monetary friction by making χ_i go to infinity. We then get a sequence of money-demand equations that converge to $i_t = i_t^m$, and hence a sequence of models that converge to the basic NK model. The corresponding sequence of unique bounded solutions converges to a particular equilibrium (out of an infinity of equilibria) of the basic NK model under a permanently exogenous policy rate. To characterize the equilibrium that we select uniquely in this way, we first note that as $\chi_i \to +\infty$, we have $\mathcal{P}(X) \to (X-1)\mathcal{P}_b(X)$ for any $X \in \mathbb{R}$. Therefore, we get $\lim_{\chi_i \to +\infty}(\rho, \omega_1, \omega_2) = (\rho_b, 1, \omega_b)$. Using this result, $\beta\rho_b\omega_b = 1$, $\rho_b + \omega_b = 1 + 1/\beta + \kappa/(\beta\sigma)$, and $\mathcal{P}_b(\rho_b) = 0$, we then easily obtain that the limits of (9) and (10) as $\chi_i \to +\infty$ are

$$\begin{aligned} \pi_t &= -(1-\rho_b)\,p_{t-1} + \mathbb{E}_t\left\{-(1-\rho_b)\sum_{k=0}^{+\infty}\left(1-\omega_b^{-k-1}\right)(i_{t+k}-r_{t+k}) \right. \\ &\quad \left. +\kappa\rho_b\sum_{k=0}^{+\infty}\omega_b^{-k}\left[(1-\delta_g)\,g_{t+k} - \delta_\varphi\varphi_{t+k}\right]\right\}, \qquad (11) \\ y_t &= -\frac{\rho_b}{\sigma}p_{t-1} - \frac{\rho_b}{\sigma}\mathbb{E}_t\left\{\sum_{k=0}^{+\infty}\left[1+\beta(1-\rho_b)\,\omega_b^{-k}\right](i_{t+k}-r_{t+k}) \right. \\ &\quad \left. +\kappa\sum_{k=0}^{+\infty}\omega_b^{-k}\left[(1-\delta_g)\,g_{t+k} - \delta_\varphi\varphi_{t+k}\right]\right\} + g_t. \qquad (12) \end{aligned}$$

We can highlight three main differences between our selected equilibrium (11)-(12) and the standard equilibrium (5)-(6) of the basic NK model. First, there are no ρ_b^{-k} terms in (11)-(12), and as a consequence our selected equilibrium does not exhibit any of the three limit puzzles – at least in the same form as the standard equilibrium. More specifically, the effects of announcing at date t a fiscal expansion at date $t+k$ on inflation and output at date t decrease at rate ω_b^{-k} in the horizon k of the fiscal expansion, and asymptotically $\lim_{k\to+\infty}\partial\pi_t/\partial g_{t+k} = \lim_{k\to+\infty}\partial y_t/\partial g_{t+k} = 0$ (no fiscal-multiplier puzzle). Moreover, given that $\lim_{\theta\to 0}(\rho_b, \omega_b, \kappa/\omega_b) = (0, +\infty, \beta\sigma)$ and $\rho_b\omega_b = 1/\beta$, (11) and (12) imply that $\lim_{\theta\to 0}\pi_t$ and $\lim_{\theta\to 0} y_t$ are finite, and in particular that $\lim_{\theta\to 0} y_t$ takes the same value (i.e., $\delta_g g_t + \delta_\varphi \varphi_t$) as y_t under perfectly flexible prices (no paradox of flexibility).[7] Finally, in our selected equilibrium (11)-(12), the effects of announcing at date t a policy-rate cut at date $t+k$ on inflation and output at date t do not explode as the horizon k of the policy-rate cut goes to infinity (unlike in the standard equilibrium). However,

[6] We provide a sensitivity analysis in Appendix F.2 – with a particular focus on the sensitivity to the value of the interest-rate semi-elasticity of money demand χ_i, given the wide range of estimates for χ_i in the empirical literature (Galí's value standing in the middle of the range). We also provide additional numerical illustrations under our benchmark calibration in Appendix F.1 (for the effects of forward guidance on output, and the effects of anticipated changes in fiscal policy on inflation and output).

[7] We cannot compare $\lim_{\theta\to 0}\pi_t$ with π_t at $\theta = 0$, since π_t at $\theta = 0$ is indeterminate in the basic NK model.

they do not converge to zero either (unlike in our simple model outside its basic-NK-model limit). Instead, they converge to non-zero finite values: $\lim_{k\to+\infty} \partial\pi_t/\partial i_{t+k} = -(1-\rho_b)$ and $\lim_{k\to+\infty} \partial y_t/\partial i_{t+k} = -\rho_b/\sigma$. In this sense, the forward-guidance puzzle is not fully, but only partially solved in our selected equilibrium, unlike the fiscal-multiplier puzzle and the paradox of flexibility.

We relate this partial resolution of the forward-guidance puzzle in our selected equilibrium to price-level stationarity. More generally, we show that the forward-guidance puzzle cannot be fully solved in any equilibrium of the basic NK model under exogenous policy rates in which the price level is stationary in response to temporary policy-rate shocks (which is the case, in particular, of our selected equilibrium). To see this, note that our simple model implies price-level stationarity in response to a one-off IOR-rate change ($p_\infty = p_0$ when $i_T^m = i^*$ and $i_t^m = 0$ for $t \geq 1$ and $t \neq T$). Therefore, in our selected equilibrium of the basic NK model, the price level is also stationary in response to a one-off interest-rate change ($p_\infty = p_0$ when $i_T = i^*$ and $i_t = 0$ for $t \geq 1$ and $t \neq T$). Now, iterating the IS equation (1) forward to $+\infty$ under this interest-rate change at date T, using price-level stationarity, and using the terminal condition $y_\infty = 0$, leads to

$$y_1 = \frac{-i^*}{\sigma} - \frac{\pi_1}{\sigma}.$$

This relationship is consistent with y_1 and π_1 converging towards non-zero finite values as $T \to +\infty$ (partial resolution of the puzzle), but inconsistent with y_1 and π_1 both converging towards zero (full resolution of the puzzle).

The second difference between our selected equilibrium and the standard equilibrium is that they have *opposite* implications about the sign of the effects of future government-purchases on current output. In the standard equilibrium (6), anticipated fiscal expansions are expansionary ($\partial y_t/\partial g_T > 0$ for $t < T$) – suggesting that fiscal policy is a potent stabilization tool at the ZLB. This implication of the standard equilibrium arises from a feedback loop, first described in Farhi and Werning (2016), that works back in time via the IS equation and the Phillips curve: given that $\pi_{T+1} = y_{T+1} = 0$, a fiscal expansion at date T raises inflation at date T, which lowers the real interest rate at date $T-1$, which raises output and inflation at date $T-1$, etc. This feedback loop is also present in our selected equilibrium, but it is counteracted by the presence of the state variable p_{t-1} in (11)-(12): the starting point of the loop, π_{T+1} and y_{T+1}, now reacts endogenously to prior developments. More specifically, (11) and the identity $\pi_t = p_t - p_{t-1}$ imply that a pre-announced fiscal expansion at date T ($g_T > 0$) raises the price level at all dates starting from the announcement date, and in particular at date T ($p_T > 0$); in turn, (11) and (12) then imply that $\pi_{T+1} = -(1-\rho_b)p_T < 0$ and $y_{T+1} = -(\rho_b/\sigma)p_T < 0$. As a result, expected future fiscal expansions are now contractionary, reflecting the familiar wealth effect present in standard Real-Business-Cycle models (through a reduction in permanent income).

The third difference between our selected equilibrium and the standard equilibrium is about the effects of current and expected future supply shocks on current output. In the standard equilibrium (6), anticipated positive supply shocks are contractionary ($\partial y_t/\partial \varphi_T < 0$ for $t < T$). In other words, the standard equilibrium exhibits the paradox of toil, and suggests that structural reforms of labor and product markets may be best put on hold during a ZLB episode. This implication of the standard equilibrium arises from essentially the same feedback loop as the one at work for government-purchases shocks (described above). In our selected equilibrium (12), this feedback loop is, again, counteracted by the presence of the state variable p_{t-1}. As a result, anticipated positive supply shocks are now expansionary ($\partial y_t/\partial \varphi_T > 0$ for $t < T$). Thus, our selected equilibrium does not exhibit the paradox of toil, and suggests that structural reforms are not counter-productive during a ZLB episode. Similarly, current positive supply shocks are expansionary in our selected equilibrium ($\partial y_t/\partial \varphi_t > 0$). By contrast, they are neutral in the standard equilibrium ($\partial y_t/\partial \varphi_t = 0$). The mechanical reason for this neutrality is that the standard equilibrium is obtained by iterating the IS equation and the Phillips curve backward in time (from $\pi_{T+1} = y_{T+1} = 0$); so, the date-$t+1$ variables π_{t+1} and y_{t+1} do not depend on the date-t shock φ_t; and the date-t IS equation then implies that y_t does not depend on φ_t either.

3.3 Comparison With Other Equilibria in the Literature

We now briefly compare our selected equilibrium with some other equilibria studied in the literature. Cochrane (2017) characterizes the set of all equilibrium paths, in the basic NK model, with $(i_t, r_t) = (i^*, r^*)$ for $1 \leq t \leq T$ and $(i_t, r_t) = (0, 0)$ for $t \geq T+1$. He shows that any of these paths can be obtained as the unique local equilibrium under a temporary interest-rate peg followed by a suitably designed interest-rate rule. He considers a "local-to-frictionless" equilibrium-selection criterion, which requires that equilibrium outcomes converge towards flexible-price equilibrium outcomes as prices become more and more flexible. This criterion does not select a unique equilibrium, but rules out some equilibria (including the standard NK equilibrium). Our selected equilibrium satisfies this criterion, since it does not exhibit the paradox of flexibility.

Among the equilibria satisfying the local-to-frictionless criterion, Cochrane (2017) describes more specifically two particular equilibria that do not exhibit any of the puzzles and paradoxes: (i) the "backward-stable" equilibrium, in which inflation goes to zero backward in time ($\lim_{t \to -\infty} \pi_t = 0$) when the interest-rate peg between 1 and T is announced at date $-\infty$; and (ii) the "no-inflation-jump" equilibrium, in which inflation is zero at the start of the peg ($\pi_1 = 0$). Two differences between these equilibria and our selected equilibrium are worth emphasizing. First, the forward-guidance puzzle is fully solved in these equilibria, but only partially solved in our equilibrium. Second, at the start of a liquidity trap caused by $r^* < 0$, inflation is negative in our equilibrium ($\partial \pi_1/\partial r^* > 0$), while it is positive in the backward-stable equilibrium ($\partial \pi_1/\partial r^* < 0$) and, by construction, zero in the no-inflation-jump equilibrium ($\partial \pi_1/\partial r^* = 0$).[8]

[8]The result $\partial \pi_1/\partial r^* < 0$ in the backward-stable equilibrium is straightforwardly obtained from Cochrane's

The latter difference can be equivalently restated as follows. Unlike the backward-stable equilibrium (and, to a lesser extent, the no-inflation-jump equilibrium), our selected equilibrium does not imply any *Neo-Fisherian* effects: announcing a current or future interest-rate hike, in our equilibrium, always reduces current inflation ($\partial\pi_1/\partial i^* < 0$). In the context of a monetary-policy normalization process, in particular, our equilibrium thus highlights the deflationary pressures that arise from expected future interest-rate hikes. This property also distinguishes our equilibrium from the one considered by Bilbiie (2018). Bilbiie (2018) uses McCallum's (1999) Minimal-State-Variable (MSV) criterion to select an equilibrium of the basic NK model under an exogenous AR(1) stochastic process for the interest rate. He finds that this MSV equilibrium may imply Neo-Fisherian effects – like Cochrane's (2017) equilibria, and unlike ours.

Another interesting parallel is between our selected equilibrium of the basic NK model and the equilibrium of Mankiw and Reis's (2002) sticky-information model. Carlstrom et al. (2015) and Kiley (2016) show that Mankiw and Reis's (2002) model fully solves the fiscal-multiplier puzzle, the paradox of flexibility, and the paradox of toil, and partially solves the forward-guidance puzzle – exactly like the basic NK model with our selected equilibrium.[9] Thus, our selected equilibrium brings the canonical sticky-price model at par with its sticky-information cousin in terms of their ability to fully or partially solve all four NK puzzles and paradoxes. Kiley (2016) also points out that Mankiw and Reis's (2002) model implies price-level stationarity when the central bank follows a non-inertial interest-rate rule after the temporary interest-rate peg; in this case, we can attribute the inability of that model to fully solve the forward-guidance puzzle to price-level stationarity, for the same reason as in the basic NK model with our selected equilibrium.

4 Resolution of the Puzzles and Paradoxes in Other Models

We have so far presented our main points in the context of a simple but ad hoc setup. It remains to show that these points can be made in more structured models generating a demand for money. In this section, we consider in turn a standard money-in-utility (MIU) model, with money interpreted as interest-bearing reserve balances at the central bank, and a model with a more explicit role for banks and bank reserves that we develop in a companion paper (Diba and Loisel, 2020).

Throughout the section, for convenience, we keep the same notations as in the previous section for the reduced-form parameters (σ, κ, δ_g, δ_φ, χ_y, χ_i), the characteristic polynomial ($\mathcal{P}(X)$), the roots of this polynomial (ρ, ω_1, ω_2), and the exogenous driving term in the dynamic equation (Z_t), although all of them are in fact model-specific.

(2017) Equation (34) by setting $C = 0$ and $t = T_l$; the result $\partial\pi_1/\partial r^* > 0$ in our equilibrium is straightforwardly obtained from (11).

[9] Of course, the paradox of flexibility solved by Mankiw and Reis's (2002) model is about the effects of information flexibility, not price flexibility.

4.1 MIU Model

We consider essentially the same MIU model as in Woodford (2003, Chapter 4).[10] In Appendix B, for completeness, we present this model, derive the necessary and sufficient condition for steady-state existence and uniqueness under exogenous monetary-policy instruments, and log-linearize the equilibrium conditions around the unique steady state. We obtain the following IS equation, Phillips curve, and money-demand equation:

$$y_t = \mathbb{E}_t\{y_{t+1}\} - \frac{1}{\sigma}(i_t - \mathbb{E}_t\{\pi_{t+1}\} - r_t) + \eta(m_t - \mathbb{E}_t\{m_{t+1}\}) + g_t - \mathbb{E}_t\{g_{t+1}\}, \quad (13)$$

$$\pi_t = \beta\mathbb{E}_t\{\pi_{t+1}\} + \kappa(y_t - \delta_m m_t - \delta_g g_t - \delta_\varphi \varphi_t), \quad (14)$$

$$m_t = \chi_y(y_t - g_t) - \chi_i(i_t - i_t^m), \quad (15)$$

where $\beta \in (0,1)$, $\eta \geq 0$, $\delta_m \geq 0$, $\delta_g \in (0,1)$, and all the other parameters are positive.

In the case in which the utility function is not separable in consumption and money, we have $\eta > 0$ and $\delta_m > 0$. In this case, the IS equation (13) involves real-money terms (in m_t and $\mathbb{E}_t\{m_{t+1}\}$) because the marginal utility of consumption in the consumption Euler equation depends on real money. Similarly, the Phillips curve (14) involves a real-money term (in m_t) because real money increases the marginal utility of consumption, which in turn decreases the real wage and hence the marginal cost of production of firms. In the alternative case in which the utility function is separable in consumption and money, we have $\eta = \delta_m = 0$, and these two equations become identical to the IS equation (1) and the Phillips curve (2) of the two models considered so far (the basic NK model in Section 2 and our simple model in Section 3). The money-demand equation (15) is isomorphic to its counterpart (7) in our simple model, except for the presence of the government-purchases shock g_t. This shock appears in (15) because money demand now depends on consumption, which we have eliminated using the goods-market-clearing condition.

Our MIU model implies, in particular, the following two restrictions on the reduced-form parameters:

$$\eta = \frac{\delta_m}{\delta_g}, \quad (16)$$

$$\delta_m \chi_y < 1, \quad (17)$$

as we show in Appendix B.6. These restrictions will play an important role in our determinacy result below (as we will see). The equality (16) says that the weight of m_t relative to g_t (and $\mathbb{E}_t\{m_{t+1}\}$ relative to $\mathbb{E}_t\{g_{t+1}\}$) in the IS equation, η, is identical to the weight of m_t relative to g_t in the Phillips curve, δ_m/δ_g. The reason is that m_t and g_t come exclusively from the marginal utility of consumption in both equations. The marginal utility of consumption depends negatively on consumption, and therefore positively on g_t for a given y_t (through the

[10] The main difference is that he considers differentiated types of labor, while we consider, for simplicity, a single type of labor.

goods-market-clearing condition); and it depends non-negatively on m_t, with a weight of m_t relative to g_t equal to $\eta = \delta_m/\delta_g$. The inequality (17) reflects how holding money mitigates changes in firms' marginal cost of production (through the real wage). For a given spread $i_t - i_t^m$, a rise in output y_t has two opposite effects on firms' marginal cost of production (i.e., on the term in factor of κ in the Phillips curve): a standard positive direct effect (with elasticity 1), and a negative indirect effect via the implied rise in money m_t (with elasticity $\delta_m \chi_y$). The inequality states that the direct effect dominates the indirect one (i.e., $\delta_m \chi_y < 1$).

Under permanently exogenous monetary-policy instruments i_t^m and M_t, the IS equation (13), the Phillips curve (14), the money-demand equation (15), and the identities $m_t = M_t - p_t$ and $\pi_t = p_t - p_{t-1}$ lead to the following dynamic equation relating p_t to $\mathbb{E}_t\{p_{t+2}\}$, $\mathbb{E}_t\{p_{t+1}\}$, p_{t-1}, and exogenous terms:

$$\mathbb{E}_t \left\{ L\mathcal{P}\left(L^{-1}\right) p_t \right\} = Z_t$$

with

$$\begin{aligned}
\mathcal{P}(X) \quad \equiv \quad & X^3 - \left[2 + \frac{1}{\beta} + \frac{\kappa}{\beta\sigma} + \frac{(1-\delta_g)\,\delta_m \kappa}{\beta\delta_g} + \frac{\chi_y}{\sigma\chi_i}\right] X^2 + \left[1 + \frac{2}{\beta} + \frac{\kappa}{\beta\sigma} \right. \\
& \left. + \frac{(1-\delta_g)\,\delta_m \kappa}{\beta\delta_g} + \frac{(1+\beta)\,\chi_y}{\beta\sigma\chi_i} + \frac{(1-\delta_m\chi_y)\,\kappa}{\beta\sigma\chi_i}\right] X - \left(\frac{1}{\beta} + \frac{\chi_y}{\beta\sigma\chi_i}\right),
\end{aligned}$$

$$\begin{aligned}
Z_t \quad \equiv \quad & \frac{-\kappa}{\beta\sigma}\left(i_t^m - r_t\right) + \left[\frac{(1-\delta_g)\,\delta_m}{\delta_g} + \frac{1-\delta_m\chi_y}{\sigma\chi_i}\right]\frac{\kappa}{\beta} M_t - \frac{(1-\delta_g)\,\delta_m\kappa}{\beta\delta_g}\mathbb{E}_t\left\{M_{t+1}\right\} \\
& + \left(1 + \frac{\chi_y}{\sigma\chi_i}\right)\frac{(1-\delta_g)\,\kappa}{\beta} g_t - \frac{(1-\delta_g)\,\kappa}{\beta}\mathbb{E}_t\left\{g_{t+1}\right\} - \left(1 + \frac{\chi_y}{\sigma\chi_i}\right)\frac{\delta_\varphi\kappa}{\beta}\varphi_t + \frac{\delta_\varphi\kappa}{\beta}\mathbb{E}_t\left\{\varphi_{t+1}\right\},
\end{aligned}$$

where we have used the equality (16) to replace η by δ_m/δ_g. Using the inequality (17), we show in Appendix C.1 that, as in our simple model in the previous section, the characteristic polynomial $\mathcal{P}(X)$ has one root inside the unit circle ($\rho \in (0,1)$), and two roots outside the unit circle (ω_1 and ω_2 with $|\omega_1| \leq |\omega_2|$), either positive real numbers or complex conjugates. With one eigenvalue inside the unit circle (ρ) for one predetermined variable (p_{t-1}), thus, our MIU model satisfies Blanchard and Kahn's (1980) conditions and has a unique bounded solution under permanently exogenous monetary-policy instruments.

In the MIU model, as in the simple model of the previous section, setting exogenously i_t^m and M_t amounts to following a "shadow Wicksellian rule" for i_t. Indeed, if the price level rises (making real money fall, given that nominal money is fixed), or if output rises, then the marginal utility of real money increases. Since the IOR rate is fixed, the interest rate on bonds has then to increase for private agents to remain indifferent between holding money and holding bonds. What is different from the previous section, however, is that existing results for Wicksellian rules in the basic NK model (e.g., Woodford, 2003, Chapter 4) do not apply to the MIU model with non-separable utility (i.e. with $\eta = \delta_m/\delta_g > 0$). In fact, not all Wicksellian rules would ensure determinacy in this model. What our determinacy result says, thus, is that the specific shadow Wicksellian rule that arises under permanently exogenous monetary-policy instruments, given the restriction (17) that the model imposes on its coefficients, always delivers determinacy.

Since it delivers determinacy under permanently exogenous monetary-policy instruments, the MIU model solves the three limit puzzles in the same way as the previous section's simple setup. We assume again that the roots ω_1 and ω_2 are distinct real numbers, so that $\omega_2 > \omega_1 > 1$, and postpone the discussion of this assumption to the end of the subsection. We solve the dynamic equation forward in the same way as in the previous section, and obtain that inflation in the unique bounded solution is again characterized by (9) – keeping in mind, though, that the roots ρ, ω_1, ω_2, and the exogenous driving term Z_t have changed. Using the solution for inflation (9), the Phillips curve (14), and the identities $m_t = M_t - p_t$ and $\pi_t = p_t - p_{t-1}$, we then get the solution for output:

$$y_t = -\vartheta p_{t-1} + \delta_m M_t + \delta_g g_t + \delta_\varphi \varphi_t - \frac{\mathbb{E}_t}{(\omega_2 - \omega_1)\,\kappa} \left\{ \sum_{k=0}^{+\infty} \left(\xi_1 \omega_1^{-k-1} - \xi_2 \omega_2^{-k-1} \right) Z_{t+k} \right\}, \quad (18)$$

where now $\vartheta \equiv (1-\rho)(1-\beta\rho)/\kappa + \delta_m \rho$ and $\xi_j \equiv \beta(\omega_j + \rho - 1) + \kappa\delta_m - 1$ for $j \in \{1, 2\}$. Like our simple model's equilibrium (9)-(10), and unlike the basic NK model's standard equilibrium (5)-(6), the MIU model's equilibrium (9) and (18) involves only ω_1^{-k} and ω_2^{-k} terms with $\omega_1 > 1$ and $\omega_2 > 1$. Therefore, the longer the horizon k, the smaller the effects of shocks occurring at date $t + k$ on inflation and output at date t in the MIU model, regardless of which type of shock (preference, monetary, fiscal, or supply) we consider. In particular, neither the forward-guidance puzzle nor the fiscal-multiplier puzzle can arise in the MIU model. Moreover, because determinacy obtains for any degree of price stickiness $\theta \in (0, 1)$ and in particular as $\theta \to 0$, the paradox of flexibility does not arise either. In Appendix C.2, we show that the limits of π_t and y_t as $\theta \to 0$ take finite values, and that these values coincide with the values that π_t and y_t take under perfectly flexible prices (in particular, $\lim_{\theta \to 0} y_t = \delta_m M_t + \delta_g g_t + \delta_\varphi \varphi_t$).

As with the simple model of the previous section, we can asymptotically remove the monetary friction from (some specifications of) our MIU model for equilibrium-selection purposes. In the separable-utility case, for instance, we can make the scale parameter of the money-utility function go to zero. In the case of utility over a constant-elasticity-of-substitution (CES) aggregator of money and consumption, we can make the quasi-share parameter on money go to zero. Making this scale or quasi-share parameter γ go to zero removes asymptotically the marginal benefit of holding money. To prevent real money balances from shrinking to zero, we need to concomitantly remove the opportunity cost of holding money, i.e. to make the steady-state IOR rate go to the steady-state interest rate on bonds β^{-1}. In Appendix C.3, we show that as I^m goes to β^{-1} at the same speed as γ goes to zero, the steady state and reduced form of our MIU model with separable utility or with utility over a CES aggregator converge to the steady state and reduced form of the basic NK model, with real money balances bounded away from zero and infinity along the way.

Thus, as we asymptotically remove the monetary friction from our MIU model, the reduced-form parameters σ, κ, δ_g, and δ_φ converge to their counterparts in the basic NK model, while η, δ_m, χ_i^{-1}, and $\chi_y\chi_i^{-1}$ converge to zero. As a result, the characteristic polynomial $\mathcal{P}(X)$ goes to $(X-1)\mathcal{P}_b(X)$; its roots ρ, ω_1, and ω_2 go respectively to ρ_b, 1, and ω_b; and the exogenous driving term Z_t goes to Z_t^b. Using these limit results, we get that the unique local equilibrium of our MIU model (9) and (18) converges to (11)-(12). Thus, our MIU model serves to select the same equilibrium of the basic NK model under a permanently exogenous policy rate as our simple model in the previous section. This selected equilibrium, in particular, does not exhibit the paradox of toil.

We have so far assumed that the roots ω_1 and ω_2 are real numbers − both in this subsection in the context of the MIU model, and in the previous section in the context of the simple model. These roots, however, can be complex (non-real) numbers in either model. In the case of complex roots, all the equations that we have derived are still valid, and all the results that we have obtained still hold. In particular, these two models still solve the three limit puzzles, and still serve to select the equilibrium (11)-(12) of the basic NK model, which solves the paradox of toil.

However, the case of complex roots is awkward for our purposes because it implies recurrent sign reversals in the effects of future shocks on current inflation and output, as we change the horizon of the shocks. We illustrate this "reversal puzzle" (as we call it) with the effect of forward-guidance policy on inflation. When ω_1 and ω_2 are complex, we can write them as $\omega_1 = re^{i\phi}$ and $\omega_2 = re^{-i\phi}$, where $r > 0$ and $\phi \in (0, 2\pi)$. Therefore, the effect of a future IOR-rate change $i_{t+k}^m = i^* \neq 0$ on current inflation π_t in (9) can be written as

$$\pi_t = \left(\frac{\omega_2^{-k-1} - \omega_1^{-k-1}}{\omega_2 - \omega_1}\right)\frac{\kappa i^*}{\beta\sigma} = \frac{-\kappa r^{-k-2}\sin\left[(k+1)\,\phi\right] i^*}{\beta\sigma\sin(\phi)}.$$

The sign of this effect changes an infinity of times as the horizon k of the IOR-rate change moves to $+\infty$.

4.2 Model with Banks

We now turn to a model in which money is explicitly made of bank reserves. In this model, which we present in detail in Diba and Loisel (2020), firms must borrow the wage bill (or some fraction of it) from banks. Banks incur costs making loans, and holding reserves mitigates these costs. The model makes weak standard assumptions about utility and production functions, like monotonicity and concavity, without specifying any functional form. The key log-linearized equilibrium conditions are, again, an IS equation, a Phillips curve, and a money-demand equation. The IS equation is the same as the IS equation (1) of the basic NK model, while the

Phillips curve and the money-demand equation are

$$\pi_t = \beta \mathbb{E}_t \{\pi_{t+1}\} + \kappa (y_t - \delta_m m_t - \delta_g g_t - \delta_\varphi \varphi_t), \tag{19}$$

$$m_t = \chi_y y_t - \chi_i (i_t - i_t^m) - \chi_g g_t - \chi_\varphi \varphi_t, \tag{20}$$

where $\beta \in (0,1)$, $\delta_g \in (0,1)$, $\chi_\varphi \geq 0$, and all the other parameters are positive. The Phillips curve (19) is isomorphic to its counterpart (14) in the MIU model $-$ but not identical to it, since the reduced-form parameters κ, δ_m, δ_g, and δ_φ have changed (even though we keep, for convenience, the same notations). The reason why real reserves m_t appear in the Phillips curve (19) is that they reduce banking costs, which in turn lowers the borrowing costs of firms and hence their marginal cost of production. Like its counterpart (15) in the MIU model, the money-demand equation (20) involves the government-purchases shock g_t because money demand depends on consumption, which we have eliminated using the goods-market-clearing condition.[11] Unlike (15), however, it also involves the supply shock φ_t, because the demand for reserves now depends on the volume of loans, which in turn depends on firms' wage bill, which in turn depends on the supply shock for a given output level.[12]

Our model with banks implies, in particular, the following restriction on the reduced-form parameters:

$$\sigma < \chi_y < \frac{1}{\delta_m}, \tag{21}$$

as we show in Diba and Loisel (2020). This double inequality will play a key role in our results below (as we will see). The first inequality in (21) arises from the fact that bank loans serve to finance the wage bill (or some fraction of it). If output y_t increases by 1% for given government purchases g_t, the marginal utility of consumption decreases by σ%; so, the wage, the wage bill, and loans all increase by more than σ%; and, in turn, so does the demand for reserves m_t for a given spread $i_t - i_t^m$ (i.e., $\chi_y > \sigma$). The second inequality in (21) is similar to the inequality (17) in the MIU model. Here, it reflects how holding reserves mitigates changes in banking costs. For a given spread $i_t - i_t^m$, a rise in output y_t has two opposite effects on firms' marginal cost of production (i.e., on the term in factor of κ in the Phillips curve): a standard positive direct effect (with elasticity 1), and a negative indirect effect via the implied rise in reserves m_t (with elasticity $\delta_m \chi_y$). The inequality states that the direct effect dominates the indirect one (i.e., $\delta_m \chi_y < 1$).

Under permanently exogenous monetary-policy instruments i_t^m and M_t, the IS equation (1), the Phillips curve (19), the money-demand equation (20), and the identities $m_t = M_t - p_t$ and $\pi_t = p_t - p_{t-1}$ lead to the following dynamic equation relating p_t to $\mathbb{E}_t\{p_{t+2}\}$, $\mathbb{E}_t\{p_{t+1}\}$, p_{t-1}, and exogenous terms:

$$\mathbb{E}_t \left\{ L\mathcal{P}\left(L^{-1}\right) p_t \right\} = Z_t,$$

[11] We no longer have $\chi_g = \chi_y$, though, because money demand now depends on y_t not only through consumption (via the goods-market-clearing condition), but also through loans (which are proportional to the wage bill).

[12] The only exception is when the supply shock is a markup shock $-$ in which case $\chi_\varphi = 0$.

where

$$\mathcal{P}(X) \equiv X^3 - \left[2 + \frac{1}{\beta} + \frac{\chi_y}{\sigma\chi_i} + \left(\frac{1}{\sigma} - \delta_m\right)\frac{\kappa}{\beta}\right] X^2 + \left[1 + \frac{2}{\beta} + \left(1 + \frac{1}{\beta}\right)\frac{\chi_y}{\sigma\chi_i} + \left(\frac{1}{\sigma} - \delta_m\right)\frac{\kappa}{\beta} + (1 - \delta_m\chi_y)\frac{\kappa}{\beta\sigma\chi_i}\right] X - \left(\frac{1}{\beta} + \frac{\chi_y}{\beta\sigma\chi_i}\right),$$

$$Z_t \equiv \frac{-\kappa}{\beta\sigma}\left(i_t^m - r_t\right) + \left[\frac{1}{\sigma\chi_i} - \left(1 + \frac{\chi_y}{\sigma\chi_i}\right)\delta_m\right]\frac{\kappa}{\beta}M_t + \frac{\delta_m\kappa}{\beta}\mathbb{E}_t\left\{M_{t+1}\right\} + \left[\left(1 + \frac{\chi_g}{\sigma\chi_i}\right) - \left(1 + \frac{\chi_y}{\sigma\chi_i}\right)\delta_g\right]\frac{\kappa}{\beta}g_t - \frac{(1-\delta_g)\kappa}{\beta}\mathbb{E}_t\left\{g_{t+1}\right\} + \left[\frac{\chi_\varphi}{\sigma\chi_i} - \left(1 + \frac{\chi_y}{\sigma\chi_i}\right)\delta_\varphi\right]\frac{\kappa}{\beta}\varphi_t + \frac{\delta_\varphi\kappa}{\beta}\mathbb{E}_t\left\{\varphi_{t+1}\right\}.$$

Using the double inequality (21), we show in Appendix D.1 that the roots of the characteristic polynomial $\mathcal{P}(X)$ are three real numbers ρ, ω_1, and ω_2 such that $0 < \rho < 1 < \omega_1 < \omega_2$. With one eigenvalue inside the unit circle (ρ) for one predetermined variable (p_{t-1}), thus, our model with banks satisfies Blanchard and Kahn's (1980) conditions and has a unique bounded solution under permanently exogenous monetary-policy instruments.

In our model with banks, setting exogenously i_t^m and M_t also amounts to following a "shadow Wicksellian rule" for i_t, as in the previous two models (the simple model of Section 3 and the MIU model of Subsection 4.1). Existing results for Wicksellian rules in the basic NK model do not apply to our model with banks, and not all Wicksellian rules would ensure determinacy in this model. What our determinacy result says is that the specific shadow Wicksellian rule that arises under permanently exogenous monetary-policy instruments, given the restriction (21) that the model imposes on its coefficients, always delivers determinacy.

Our model with banks, thus, solves the three limit puzzles in the same way as our previous two models. We determine the unique local equilibrium of this model in the same way as in the previous subsection, and obtain that inflation and output in this equilibrium are again characterized by (9) and (18) − keeping in mind, though, that the roots ρ, ω_1, ω_2, the reduced-form parameters κ, δ_m, δ_g, δ_φ, and the exogenous driving term Z_t have changed. Since (9) and (18) involve only ω_1^{-k} and ω_2^{-k} terms with $\omega_1 > 1$ and $\omega_2 > 1$, neither the forward-guidance puzzle nor the fiscal-multiplier puzzle can arise in our model with banks. Moreover, because determinacy obtains for any degree of price stickiness $\theta \in (0, 1)$ and in particular as $\theta \to 0$, the paradox of flexibility does not arise either in this model, as we formally show in Appendix D.2.

Unlike our previous two models, however, our model with banks solves the three limit puzzles without *ever* giving rise to the reversal puzzle discussed in the previous subsection. In other words, in our model with banks, we do not have to *assume* that ω_1 and ω_2 are positive real numbers in order to avoid the reversal puzzle. Because its coefficients are restricted by (21), the specific shadow Wicksellian rule that arises under permanently exogenous monetary-policy instruments *always* delivers positive real eigenvalues ω_1 and ω_2 in this model.

As in our previous two models, we can asymptotically remove the monetary friction from our model with banks for equilibrium-selection purposes. In a previous version of this paper (Diba and Loisel, 2019), we show that as the scale parameter of banking costs and the steady-state interest-rate spread are shrunk to zero (at suitable rates, to keep a positive and finite level of steady-state real reserve balances in the limit), the steady state and reduced form of our model with banks converge to the steady state and reduced form of the basic NK model. Therefore, as previously, the characteristic polynomial $\mathcal{P}(X)$ goes to $(X-1)\mathcal{P}_b(X)$; its roots ρ, ω_1, and ω_2 go respectively to ρ_b, 1, and ω_b; and the exogenous driving term Z_t goes to Z_t^b. Using these limit results, we get again that the unique local equilibrium of our model with banks (9) and (18) converges to (11)-(12). Thus, our model with banks serves to select the same equilibrium of the basic NK model under a permanently exogenous policy rate as our previous two models − an equilibrium that, in particular, does not exhibit the paradox of toil.

4.3 Robustness of the Selected Equilibrium

The basic NK model has an infinity of (local) equilibria under a permanently exogenous policy rate. The equilibrium-selection mechanism that we have proposed consists in adding a monetary friction to the basic NK model, getting a unique equilibrium under exogenous monetary-policy instruments in the resulting monetary model, and considering the limit of this equilibrium as the monetary friction is shrunk to zero. Despite their differences, all the three monetary models that we have considered (the simple model, the MIU model, and the model with banks) have served to select the same equilibrium of the basic NK model.

This robustness feature of our selected equilibrium, we argue, extends to other models that deliver determinacy under exogenous policy instruments and converge to the basic NK model as we shrink some friction. At least, any model that has these two properties and whose dynamic equation under exogenous policy instruments relates p_t to $\mathbb{E}_t\{p_{t+2}\}$, $\mathbb{E}_t\{p_{t+1}\}$, p_{t-1}, and exogenous terms (like our three monetary models) will lead to our selected equilibrium. Indeed, any model of this kind will have inflation in the unique equilibrium given by (9), although the roots ρ, ω_1, ω_2, and the exogenous driving term Z_t in (9) will be model-specific. As the friction is shrunk, the characteristic polynomial will go to $(X-1)\mathcal{P}_b(X)$, so its roots ρ, ω_1, and ω_2 will converge to ρ_b, 1, and ω_b. Similarly, the exogenous driving term Z_t will converge to Z_t^b. Therefore, inflation in the unique equilibrium of the model, (9), will converge to inflation in our selected equilibrium of the basic NK model, (11). Our conjecture is that models with the same properties but dynamic systems of higher orders will also lead to our selected equilibrium.

There are, of course, monetary models that are not amenable to our equilibrium-selection exercise, because they do not (always) deliver determinacy under exogenous monetary-policy instruments, or because they do not converge to the basic NK model as we shrink the monetary friction, or simply because we cannot (smoothly) shrink the monetary friction in these models.

One example is the sticky-price and separable-utility version of the standard cash-in-advance (CIA) model with cash and credit goods (Lucas and Stokey, 1983). As the scale parameter of the cash-goods-consumption utility function is shrunk to zero, the reduced form of this model converges to a reduced form whose IS equation, unlike the IS equation (1) of the basic NK model, involves the interest rate at date $t+1$, not t. Another example is the CIA model that we consider in a previous version of this paper (Diba and Loisel, 2019), in which leisure implicitly serves as credit good. We show in this previous version that this model may generate indeterminacy under exogenous monetary-policy instruments, and that the monetary friction cannot be (smoothly) shrunk to zero in this model. We have no reason to think, however, that a model that does deliver determinacy under exogenous policy instruments and does converge to the basic NK model as some friction is shrunk to zero will not lead to our selected equilibrium.

5 Discussion of the Non-Satiation Assumption

Our resolution of the NK puzzles and paradoxes rests on the assumption that demand for reserves is not fully satiated. In our monetary models, demand for reserves would be fully satiated if the marginal benefit of holding reserves were exactly zero − implying that $\eta = 0$ in the IS equation (13) and $\delta_m = 0$ in the Phillips curves (14) and (19). In that case, the opportunity cost of holding reserves would also have to be exactly zero − implying that the money-demand equations (7), (15), and (20) would be replaced by $i_t = i_t^m$. Thus, our monetary models would then exactly coincide with the basic NK model, and all the NK puzzles and paradoxes would re-emerge.

Our non-satiation assumption stands in contrast to views often expressed about the US economy in recent years (e.g., Cochrane, 2014, 2018; Reis, 2016). In this section, we clarify and defend our non-satiation assumption in two steps. First, we show that our resolution of the NK puzzles and paradoxes survives a thought experiment that involves arbitrarily large increases in the real stock of reserves, and even asymptotic satiation of the demand for reserves. As this result makes clear, thus, the assumption we need to make is *only* that demand for reserves not be *fully* satiated. Second, we defend this assumption, drawing on more formal arguments presented in a companion paper (Diba and Loisel, 2020).

5.1 Resolution of the Puzzles and Paradoxes Under Asymptotic Satiation

Even though we no longer solve the NK puzzles and paradoxes when demand for reserves is *fully* satiated, we still solve them when it is *arbitrarily* close to satiation, and even when it is *asymptotically* satiated. To establish this result, we make the steady-state IOR rate go to the steady-state interest rate on bonds β^{-1} in the MIU model and the model with banks.[13] This

[13] We cannot conduct this policy experiment in the simple model, because there is no steady-state IOR rate in this ad-hoc model.

policy experiment asymptotically removes the opportunity cost of holding reserves. This time, unlike previously, we do not concomitantly remove the marginal benefit of holding reserves (by shrinking a scale or quasi-share parameter). Therefore, the steady-state real stock of reserves goes to infinity, and demand for reserves is asymptotically satiated.

As we show in Appendices C.3 and D.3, the steady states and reduced forms of our MIU model with separable utility and of our model with banks then converge to the steady state and reduced form of the basic NK model – under a condition that is met, in particular, for isoelastic money-utility and banking-cost functions. Therefore, as previously, the characteristic polynomial $\mathcal{P}(X)$ goes to $(X-1)\mathcal{P}_b(X)$; its roots ρ, ω_1, and ω_2 go respectively to ρ_b, 1, and ω_b; the exogenous driving term Z_t goes to Z_t^b; and, again, the unique local equilibrium (9) and (18) converges to our selected equilibrium (11)-(12) of the basic NK model. Thus, at the limit, when demand for reserves is asymptotically satiated, we fully solve the fiscal-multiplier puzzle, the paradox of flexibility, and the paradox of toil; and we partially solve the forward-guidance puzzle.

This policy experiment involves a sequence of different policies in a given model, and the corresponding sequence of unique equilibrium outcomes converging to satiation. By contrast, our previous equilibrium-selection exercises involved a sequence of different policies in different models (with smaller and smaller monetary frictions), and the corresponding sequence of unique equilibrium outcomes staying away from satiation. Despite this difference, however, the two sequences of unique equilibrium outcomes converge to the same limit equilibrium outcome (namely, our selected equilibrium of the basic NK model). The reason is that going to satiation asymptotically alleviates the monetary friction and has the same effect as asymptotically removing the monetary friction from the model.

5.2 Defense of the Non-Satiation Assumption

In a companion paper (Diba and Loisel, 2020), we present in detail our model with banks (used in Subsection 4.2) and show in particular that this model can account, in qualitative terms, for three key features of US inflation during the Great Recession: no significant deflation, little inflation volatility, and no significant inflation following quantitative-easing (QE) policies. These results, like our resolution of NK puzzles and paradoxes in the present paper, rest on the assumption that demand for bank reserves was not fully satiated in the US. For this reason, in our companion paper, we address in detail two types of arguments that go against our non-satiation view.

First, some observers may make a case for satiation noting that the federal-funds rate and Treasury-bill (T-bill) returns were below the IOR rate for several years in the aftermath of the crisis. We do not think this contradicts our claim that reserves still had a positive marginal convenience yield during this period. Most of the trading activity in the federal-funds market over this period involved banks borrowing funds from entities that do not have direct access to the IOR rate (particularly from Federal Home Loan Banks). Given the presence of such eager lenders, the federal-funds rate had to be below the IOR rate to incentivize the borrowers (banks with direct access to the IOR rate). As to T-bill returns, the low rates could reflect strong demand by non-bank entities – using T-bills as, e.g., collateral or international reserve asset. We formalize this counter-argument in Diba and Loisel (2020) by introducing government bonds providing liquidity services into our model with banks, and showing that the resulting model reconciles the observed negative spread between T-bill and IOR rates with our non-satiation assumption.

The second argument making a case for satiation of demand for reserves is the fact that large increases in reserve balances during the second and third rounds of quantitative easing (QE2 and QE3) had no apparent effect on expected inflation, as Reis (2016) points out. Our counter-argument is that this evidence may also be consistent with demand for reserves being close to satiation, rather than fully satiated. More specifically, we show in Diba and Loisel (2020) that in our model with banks, large increases in the money supply (say, doubling the stock of reserves) can have very small inflationary effects (around twenty basis points) if the demand for reserves is close to satiation and the monetary expansion is perceived as temporary (say, balance-sheet normalization is expected to occur in about five years). Distinguishing between the two possibilities (arbitrarily close to satiation versus fully satiated demand) may be difficult in practice. In fact, in contrast to Reis's (2016) evidence about expected inflation, Krishnamurthy and Lustig (2019) find statistically significant effects of monetary policy, during QE2 and QE3, on the convenience yield of US Treasury bills and the foreign-exchange value of the dollar.

6 Comparison With Discounting Models

In this section, we generalize Cochrane's (2016) comments on Gabaix (2019) to highlight four properties of the "discounting models" proposed in the literature to solve the forward-guidance puzzle, and we show how our monetary models are different.[14] First, discounting models do not solve the paradox of flexibility. Second, they require a discrete departure from the basic NK model to solve the forward-guidance puzzle, and cannot serve to select a unique equilibrium of the basic NK model under a permanently exogenous policy rate. Third, they cannot solve the forward-guidance puzzle without generating a *negative* long-term relationship between the inflation rate and the interest rate on bonds. By contrast, our monetary models generate the standard Fisher effect, i.e. a *one-to-one* long-term relationship between these two variables. And fourth, discounting models cannot solve the forward-guidance puzzle without having non-standard implications for equilibrium determinacy in "normal times," i.e. away from the ZLB on nominal interest rates. By contrast, our monetary models do not necessarily question the implications of the basic NK model about how monetary policy works during normal times; for

[14]None of these four properties is related to the fiscal-multiplier puzzle or the paradox of toil. Exploring the implications of discounting models for this puzzle and this paradox would be interesting, but is beyond the scope of this paper.

example, under a corridor system, our models become identical or isomorphic to the basic NK model and inherit its familiar conditions for equilibrium determinacy.

6.1 A Class of Discounting Models

In this section, by "discounting model," we mean more specifically any model whose reduced form, in the absence of shocks other than interest-rate shocks, is made of an IS equation and a Phillips curve of type

$$y_t = \xi_1 \mathbb{E}_t \{y_{t+1}\} - \frac{\xi_2}{\sigma} \mathbb{E}_t \{i_t - \pi_{t+1}\}, \tag{22}$$

$$\pi_t = \beta \xi_3(\theta) \mathbb{E}_t \{\pi_{t+1}\} + \kappa(\theta) [y_t - \xi_4(\theta) \mathbb{E}_t \{y_{t+1}\}], \tag{23}$$

where $\beta \in (0,1)$, $\sigma > 0$, $\xi_1 > 0$, $\xi_2 > 0$, and, for all $\theta \in (0,1)$, $\xi_3(\theta) \geq 0$, $\xi_4(\theta) \in [0,1)$, and $\kappa(\theta) > 0$, with $\lim_{\theta \to 0} \xi_3(\theta) < +\infty$ and $\lim_{\theta \to 0} \kappa(\theta) = +\infty$.[15] This class of reduced forms nests the reduced form of the basic NK model as a special case in which $\xi_1 = \xi_2 = \xi_3(\theta) = 1$ and $\xi_4(\theta) = 0$. More generally, this class allows the coefficients of $\mathbb{E}_t\{y_{t+1}\}$ and $\mathbb{E}_t\{\pi_{t+1}\}$ to be smaller ("positive discounting") or larger ("negative discounting") than in the basic NK model, and also allows for a $\mathbb{E}_t\{y_{t+1}\}$ term in the Phillips curve. In particular, this class encompasses the reduced forms of three models that have been shown to be able to solve the forward-guidance puzzle: (i) Gabaix's (2019) benchmark model, in which $(\xi_1, \xi_3(\theta)) \in (0,1)^2$ and $\xi_4(\theta) = 0$; (ii) Angeletos and Lian's (2018) model, in which $(\xi_1, \xi_2, \xi_3(\theta), \xi_4(\theta)) \in (0,1)^4$; and (iii) Bilbiie's (2019) model with external price-adjustment costs, in which $(\xi_1, \xi_2) \in (0,1)^2$ and $\xi_3(\theta) = \xi_4(\theta) = 0$. In addition, it also encompasses the reduced forms of: (iv) Bilbiie's (2019) model with internal price-adjustment costs, in which $(\xi_1, \xi_2) \in (0,1)^2$, $\xi_3(\theta) = 1$, and $\xi_4(\theta) = 0$; (v) McKay et al.'s (2017) model, in which also $(\xi_1, \xi_2) \in (0,1)^2$, $\xi_3(\theta) = 1$, and $\xi_4(\theta) = 0$; (vi) Ravn and Sterk's (2018) model with risk-neutral equity investors, in which $\xi_3(\theta) = 1$ and $\xi_4(\theta) \in (0,1)$; and (vii) Woodford's (2019) model with exponentially distributed planning horizons and no learning, in which $\xi_1 = \xi_2 = \xi_3(\theta) \in (0,1)$ and $\xi_4(\theta) = 0$.[16]

6.2 Paradox of Flexibility

Like the basic NK model, and unlike our monetary models, discounting models do not solve the paradox of flexibility.[17] They make inflation and output explode in response to future shocks as the degree of price stickiness θ goes to zero. We establish this result formally in Appendix E.1 in the same way as with the basic NK model in Section 2. More specifically, we assume that

[15] We focus on discrete-time discounting models for the sake of comparability with our monetary models, but we have no reason to expect that continuous-time discounting models behave differently. Indeed, Michaillat and Saez (2019) show that their continuous-time discounting model has the same four properties as the ones listed above.

[16] However, it does not encompass the reduced forms of McKay et al.'s (2016) and Del Negro et al.'s (2015) models, which involve some discounting too but are more complex.

[17] They may *attenuate* this paradox, though, as Angeletos and Lian (2018) show in the context of their discounting model.

the interest rate is set exogenously from date 1 to some date $T \geq 2$, and that the economy is at its steady state at date $T+1$. We show that the responses of $|\pi_1|$ and $|y_1|$ to an interest-rate change at date T go to infinity as $\theta \to 0$.

This result is the consequence of two properties of discounting models: (i) these models generate indeterminacy under a permanently exogenous policy rate when prices are sufficiently flexible, as their dynamic system then has one stable eigenvalue not matched by any predetermined variable, and (ii) this stable eigenvalue goes to zero as prices are made more and more flexible. As in the basic NK model in Section 2, this stable eigenvalue magnifies the effects of future conditions (at date T) on initial outcomes (at date 1), and these effects grow explosively as this eigenvalue goes to zero – thus giving rise to the paradox of flexibility.

In discounting models, indeterminacy under sufficiently flexible prices follows, by continuity, from indeterminacy under perfectly flexible prices. Under perfectly flexible prices, the Phillips curve (23) collapses to the dynamic equation $y_t = [\lim_{\theta \to 0} \xi_4(\theta)]\mathbb{E}_t\{y_{t+1}\}$, which pins down y_t uniquely if $\lim_{\theta \to 0} \xi_4(\theta) \neq 1$. Under an exogenous policy rate i_t, the IS equation (22) then pins down expected future inflation $\mathbb{E}_t\{\pi_{t+1}\}$, but not current inflation π_t. Thus, discounting models may deliver determinacy under a permanently exogenous policy rate for some degrees of price stickiness, but cannot do it for sufficiently small degrees of price stickiness.

In our monetary models, by contrast, the interest rate pegged at the ZLB is the IOR rate i_t^m, not the interest rate on bonds i_t. Setting exogenously the IOR rate and the nominal stock of reserves – two monetary-policy instruments under the direct control of central banks – makes the (market-determined) interest rate on bonds evolves according to a shadow Wicksellian rule, as we have explained. This shadow Wicksellian rule ensures determinacy whatever the degree of price stickiness, and in particular for perfectly flexible prices – thus solving the paradox of flexibility.

6.3 Distance From the Basic NK Model

Unlike our monetary models, discounting models require a *discrete* departure from the basic NK model to deliver determinacy under a permanently exogenous policy rate and, therefore, to solve the forward-guidance puzzle. By "discrete departure," we mean that the reduced-form parameters in the IS equation (22) and Phillips curve (23) of discounting models should be sufficiently distant from their counterparts in the IS equation (1) and Phillips curve (2) of the basic NK model, i.e. that $(\xi_1, \xi_2, \xi_3(\theta), \xi_4(\theta))$ should be sufficiently distant from $(1, 1, 1, 0)$. Otherwise, as we show in Appendix E.2, indeterminacy obtains by continuity with the basic NK model.

The basic reason for this result is that the reduced form (22)-(23) of discounting models involves inflation and output, but not the price level, like the reduced form (1)-(2) of the basic NK model. As a result, under a permanently exogenous policy rate, the characteristic polynomial $\mathcal{P}(X)$ of their dynamic system is of degree two, like the characteristic polynomial $\mathcal{P}_b(X)$ of the dynamic equation of the basic NK model. As $(\xi_1, \xi_2, \xi_3(\theta), \xi_4(\theta))$ goes to $(1,1,1,0)$, the coefficients of $\mathcal{P}(X)$ converge to the coefficients of $\mathcal{P}_b(X)$, and therefore the two roots of $\mathcal{P}(X)$ converge to the two roots $\rho_b \in (0,1)$ and $\omega_b > 1$ of $\mathcal{P}_b(X)$. For $(\xi_1, \xi_2, \xi_3(\theta), \xi_4(\theta))$ sufficiently close to $(1,1,1,0)$, thus, one root of $\mathcal{P}(X)$ lies inside the unit circle, and the other lies outside. With only one eigenvalue outside the unit circle for two non-predetermined variables ($\mathbb{E}_t\{y_{t+1}\}$ and $\mathbb{E}_t\{\pi_{t+1}\}$), discounting models then generate indeterminacy.

The reduced forms of our monetary models, by contrast, involve not only inflation and output, but also the price level p_t through the real stock of reserves $m_t = M_t - p_t$ in the money-demand equations (7), (15), and (20). As a result, under permanently exogenous monetary-policy instruments, their dynamic equation cannot be written in terms of inflation – unlike the dynamic system of discounting models and the dynamic equation (3) of the basic NK model (which involve $\mathbb{E}_t\{\pi_{t+2}\}$, $\mathbb{E}_t\{\pi_{t+1}\}$, and π_t). Instead, it has to be written in terms of the price level, and to involve $\mathbb{E}_t\{p_{t+2}\}$, $\mathbb{E}_t\{p_{t+1}\}$, p_t, and p_{t-1}. As a consequence, the characteristic polynomial $\mathcal{P}(X)$ of our monetary models' dynamic equation is of degree three. As the distance from the basic NK model is shrunk to zero, the roots ρ, ω_1, and ω_2 of $\mathcal{P}(X)$ converge respectively to $\rho_b \in (0,1)$, 1, and $\omega_b > 1$, where the limit value 1 of ω_1 simply reflects the identity $\pi_t = p_t - p_{t-1}$. As long as the distance from the basic NK model remains positive, however small this distance is, ω_1 remains outside the unit circle. With two eigenvalues outside the unit circle (ω_1 and ω_2) for two non-predetermined variables ($\mathbb{E}_t\{p_{t+2}\}$ and $\mathbb{E}_t\{p_{t+1}\}$), therefore, the monetary models ensure determinacy even for an arbitrarily small departure from the basic NK model.

Put differently, the price-level term in our shadow Wicksellian rules – which directly comes from the price-level term in the money-demand equations – acts as an *error-correction* term that makes the price level stationary and hence determinate in our monetary models. As the distance from the basic NK model is shrunk to zero, the coefficient of the price level in the shadow Wicksellian rules also goes to zero. But as long as this distance remains positive, that coefficient remains positive too, and both stationarity and determinacy of the price level ensue.

This difference between discounting and monetary models has two implications. First, the degree of friction in discounting models – e.g., the degree of bounded rationality in Gabaix (2019), information frictions in Angeletos and Lian (2018), market incompleteness in Bilbiie (2019) – needs to be sufficiently large to solve the forward-guidance puzzle. By contrast, an arbitrarily small degree of monetary friction in our models is still enough to solve the NK puzzles and paradoxes. Second, shrinking the degree of monetary friction to zero in our models serves to select uniquely an equilibrium of the basic NK model under a permanently exogenous policy rate, as we have shown. By contrast, discounting models cannot be used for that purpose, since they generate indeterminacy as they approach the basic NK model.

6.4 Fisher Effect

Our monetary models generate the standard Fisher effect, i.e. a *one-to-one* long-term relationship between the inflation rate and the interest rate on bonds. Indeed, the money-demand equations (7), (15), and (20) imply that a permanent change in nominal-money growth $M_t - M_{t-1} = \delta^*$ leads to the same permanent change in inflation $\pi_t = \delta^*$. In turn, the IS equations (1) and (13) imply that $M_t - M_{t-1} = \pi_t = \delta^*$ leads to the same permanent change in the interest rate on bonds $i_t = \delta^*$.[18]

By contrast, discounting models cannot deliver determinacy under a permanently exogenous policy rate without making the inflation rate and the interest rate *negatively* related to each other in the long term, as we show in Appendix E.3. Therefore, they cannot both solve the forward-guidance puzzle and imply a long-term relationship consistent in sign (let alone in size) with the standard Fisher effect.[19]

The proof for this result in Appendix E.3 is simple, but mechanical. In what follows, we offer an interpretation of this result that involves a shadow interest-rate rule and the Taylor principle. The question (negatively) answered by this result is whether the system made of the modified IS equation (22), the modified Phillips curve (23), and the permanent peg $i_t = i^*$ can have a unique stationary solution and make inflation, in this unique stationary solution, depend positively on i^*. This question will receive exactly the same answer if that system is replaced by the system made of the standard IS equation (1), the modified Phillips curve (23), and the shadow interest-rate rule

$$i_t = \xi_2 i^* + \sigma (1 - \xi_1) \mathbb{E}_t \{y_{t+1}\} + (1 - \xi_2) \mathbb{E}_t \{\pi_{t+1}\}. \tag{24}$$

Indeed, the two systems have exactly the same implications for local-equilibrium determinacy and the dynamics of inflation and output (they differ only in terms of the implied dynamics for i_t). So consider the latter system. The Taylor principle (as defined by Woodford, 2003, Chapter 4) states that a necessary condition for local-equilibrium determinacy is that the modified Phillips curve (23) and the shadow interest-rate rule (24) should make the interest rate react more than one-to-one to the inflation rate in the long term, that is to say

$$\zeta \equiv \frac{\sigma (1 - \xi_1) [1 - \beta \xi_3 (\theta)]}{\kappa (\theta) [1 - \xi_4 (\theta)]} + (1 - \xi_2) > 1. \tag{25}$$

In the unique local equilibrium, the (constant) interest rate i and the (constant) inflation rate π are therefore linked to each other by the relationship $i = \xi_2 i^* + \zeta \pi$, where $\zeta > 1$. Now, the standard IS equation (1) implies that they should be equal to each other: $i = \pi$. As a

[18] Under the assumption that non-optimized prices are indexed to steady-state inflation, the Phillips curves (2), (14), and (19) remain valid and residually determine the permanent change in output.

[19] Gabaix (2019), however, adds price indexation and "inflation guidance" to his benchmark discounting model and shows that the resulting model (which does not belong to the class of discounting models we consider) can both solve the forward-guidance puzzle and make inflation respond positively to the nominal interest rate in the long term.

consequence, we get

$$\pi = \frac{-\xi_2 i^*}{\zeta - 1}.$$

Thus, the necessary condition for local-equilibrium determinacy (25) imposed by the Taylor principle requires that π be *negatively* related to i^*.

This conflict between the Taylor principle and the Fisher effect does not arise in our monetary models. The non-discounted nature of their IS equations guarantees the Fisher effect. And the interest rate pegged at the ZLB in these models is the (directly controlled) IOR rate i_t^m, not the (market-determined) interest rate on bonds i_t. Under exogenous monetary-policy instruments, the interest rate on bonds evolves according to a shadow Wicksellian rule that always ensures determinacy.

6.5 Normal Times

Discounting models have only one interest rate, the interest rate on bonds i_t, which is the policy instrument. To solve the forward-guidance puzzle, they need to deliver determinacy when this interest rate is pegged at the ZLB. The parameter restrictions that deliver determinacy under a peg at the value zero, however, also deliver determinacy under a peg at any positive value. By continuity, therefore, policy rules away from the ZLB that make the interest rate react sufficiently *weakly* to inflation will also deliver determinacy. Thus, discounting models do not support the conventional wisdom that interest-rate rules have to be active during normal times.

By contrast, our monetary models have two interest rates: the interest rate on bonds i_t, and the IOR rate i_t^m. At the ZLB, the central bank has no other possibility than pegging the IOR rate, which is the interest rate it directly controls. Away from the ZLB, however, it has various possibilities. One of them is to operate under a "corridor system" that (i) maintains a fixed spread between the IOR rate and the interest rate on bonds, and (ii) sets a target for the interest rate on bonds that depends on the state of the economy. Under this system, the (log-deviation of the) spread $i_t - i_t^m$ is zero in the money-demand equations (7), (15), and (20), and a rule for i_t is added to the reduced forms of our monetary models.

Under such a corridor system, the reduced forms of our simple model and our MIU model with separable utility have a simple and familiar block-recursive structure. The first block, which is made of the IS equation (1), the Phillips curve (2), and the interest-rate rule, is exactly identical to the reduced form of the basic NK model. It determines inflation, output, and the interest rate on bonds. The second block is made of the money-demand equation (7) or (15) with $i_t - i_t^m = 0$, and residually determines real money balances. Therefore, these two models, under that system, have exactly the same implications for equilibrium determinacy and dynamics as the basic NK model.

The reduced forms of our MIU model with non-separable utility and our model with banks, under the corridor system, can also be rewritten in a block-recursive structure, with a first block that is not identical but *isomorphic* to the reduced form of the basic NK model. More specifically, this first block is made of an IS equation of type (1), a Phillips curve of type (2), and the interest-rate rule. The reduced-form parameters σ, κ, δ_g, and δ_φ in the IS equation and the Phillips curve still satisfy $\sigma > 0$, $\kappa > 0$, $\delta_g \in (0, 1)$, and $\delta_\varphi > 0$, but take different values than in the basic NK model. We establish this isomorphism result in Appendix C.4 for the MIU model with non-separable utility, and in a previous version of our paper (Diba and Loisel, 2019) for the model with banks. As a consequence, the determinacy conditions in these two models under the corridor system are isomorphic to the determinacy conditions in the basic NK model.

In particular, in all our monetary models under such a corridor system, a Taylor rule $i_t = \phi\pi_t$ with $\phi \geq 0$ thus needs to make the interest rate react sufficiently *strongly* to inflation ($\phi > 1$) to deliver determinacy − like in the basic NK model, and unlike in discounting models.

7 Conclusion

In this paper, we have proposed a resolution of the puzzles and paradoxes that arise in the basic NK model under a temporary interest-rate peg (e.g., in the context of a liquidity trap). In reality, it is the IOR rate, not the interest rate on bonds, that the Great Recession forced central banks to peg near zero. Central banks also conducted balance-sheet policies. To capture what they actually did, we analyze models of monetary policy in which the central bank sets exogenously the IOR rate and the nominal stock of reserves. The three models we consider − a model with an ad-hoc money-demand function, a familiar MIU setup, and a model with a more explicit role for bank reserves − lead essentially to the same conclusions. By delivering local-equilibrium determinacy under permanently exogenous monetary-policy instruments, whatever the degree of price stickiness, they offer a full resolution of the three limit puzzles − the forward-guidance puzzle, the fiscal-multiplier puzzle, and the paradox of flexibility. The more structured model with banks, however, has the advantage over the other two models of never giving rise to what we call the "reversal puzzle."

Moreover, these three models solve the limit puzzles even for an arbitrarily small departure from the basic NK model, in the sense of an arbitrarily small monetary friction. In fact, they still (fully or partially) solve the limit puzzles for a vanishingly small monetary friction, and also solve the paradox of toil in that case. We use this limit result to propose a new equilibrium-selection device in the basic NK model under a permanently exogenous interest rate. The equilibrium that we select uniquely in this way does not depend on the model we start from (neither one of the three monetary models we consider, nor, as we argue, any other suitable model). This equilibrium does not imply any Neo-Fisherian effect, unlike other equilibria of the basic NK model considered in the literature.

The resolution of NK puzzles and paradoxes has important policy implications at the ZLB: fiscal policy and monetary forward guidance may not be such powerful stabilization tools, and the pursuit of structural reforms − say, reducing monopoly markups or promoting price flexibility − may not be counterproductive. In this paper, we have made these points analytically in tractable models. We think it would be useful to explore these points quantitatively by adding a money-demand nexus to larger-scale models of monetary policy.

Appendix A: Simple Model (Analytical Results)

A.1 Root Analysis

We first show that $0 < \rho < 1 < |\omega_1| \leq |\omega_2|$. Since $\mathcal{P}(0) = -1/\beta - \chi_y/(\beta\sigma\chi_i) < 0$ and $\mathcal{P}(1) = \kappa/(\beta\sigma\chi_i) > 0$, $\mathcal{P}(X)$ has either one or three real roots inside $(0, 1)$. Moreover, since $\mathcal{P}(X) < 0$ for all $X < 0$, $\mathcal{P}(X)$ has no negative real roots. Therefore, $\mathcal{P}(X)$ has at least one real root inside $(0, 1)$, which we denote by ρ, and its other two roots, which we denote by ω_1 and ω_2 with $|\omega_1| \leq |\omega_2|$, must be (i) both real and inside $(0, 1)$, or (ii) both real and larger than 1, or (iii) both complex and conjugates of each other. Now, given that $\mathcal{P}(X)$ is of type $X^3 - a_2X^2 + a_1X - a_0$, we have $\rho + \omega_1 + \omega_2 = a_2 \equiv 2 + 1/\beta + \kappa/(\beta\sigma) + \chi_y/(\sigma\chi_i) > 3$. Therefore, Case (i) is impossible, and in Case (iii) the common real part of ω_1 and ω_2 is larger than 1. As a consequence, in the remaining two possible cases, namely Cases (ii) and (iii), ω_1 and ω_2 lie outside the unit circle.

We now show that ω_1 and ω_2 can be real numbers, and that they can also be complex (non-real) numbers. Suppose, for instance, that χ_y and χ_i go to 0, with χ_y/χ_i constant. Then, $a_1 \equiv 1 + 2/\beta + (1 + 1/\chi_i)\kappa/(\beta\sigma) + (1 + 1/\beta)\chi_y/(\sigma\chi_i)$ goes to $+\infty$, while $a_2 \equiv 2 + 1/\beta + \kappa/(\beta\sigma) + \chi_y/(\sigma\chi_i)$ and $a_0 \equiv 1/\beta + \chi_y/(\beta\sigma\chi_i)$ remain constant. Therefore, for sufficiently small values of χ_y and χ_i, $\mathcal{P}(X) = X^3 - a_2X^2 + a_1X - a_0$ is positive for all $X \geq 1$, so that Case (ii) is impossible and ω_1 and ω_2 are complex numbers. By contrast, suppose now that χ_y and χ_i go to $+\infty$, with χ_y/χ_i constant. Then, $\mathcal{P}[1 + \chi_y/(\sigma\chi_i)]$ goes to $-[1 + \chi_y/(\sigma\chi_i)]\kappa\chi_y/(\beta\sigma^2\chi_i)$, which is negative. Therefore, for sufficiently large values of χ_y and χ_i, we have $\mathcal{P}[1 + \chi_y/(\sigma\chi_i)] < 0$, which, together with $\mathcal{P}(1) > 0$, implies that ω_1 and ω_2 are positive real numbers.

A.2 Resolution of the Paradox of Flexibility

Using the definition of Z_t, and after some simple algebra, we can rewrite (9) and (10) as

$$\begin{aligned} \pi_t \quad = \quad & -(1-\rho)\,p_{t-1} + \frac{\kappa}{\beta\,(\omega_2 - \omega_1)}\mathbb{E}_t \left\{ -\frac{1}{\sigma} \sum_{k=0}^{+\infty} \left(\omega_1^{-k-1} - \omega_2^{-k-1} \right) \left(i_{t+k}^m - r_{t+k} - \frac{M_{t+k}}{\chi_i} \right) \right. \\ & \left. - \sum_{k=0}^{+\infty} \left(\xi_1^g \omega_1^{-k-1} - \xi_2^g \omega_2^{-k-1} \right) g_{t+k} + \sum_{k=0}^{+\infty} \left(\xi_1^\varphi \omega_1^{-k-1} - \xi_2^\varphi \omega_2^{-k-1} \right) \delta_\varphi \varphi_{t+k} \right\}, \qquad \text{(A.1)} \end{aligned}$$

$$
\begin{aligned}
y_t &= -\vartheta p_{t-1} + g_t + \frac{\mathbb{E}_t}{\beta(\omega_2-\omega_1)}\left\{\frac{1}{\sigma}\sum_{k=0}^{+\infty}\left(\xi_1\omega_1^{-k-1} - \xi_2\omega_2^{-k-1}\right)\left(i_{t+k}^m - r_{t+k} - \frac{M_{t+k}}{\chi_i}\right)\right.\\
&\left. + \sum_{k=0}^{+\infty}\left(\xi_1\xi_1^g\omega_1^{-k-1} - \xi_2\xi_2^g\omega_2^{-k-1}\right)g_{t+k} - \sum_{k=0}^{+\infty}\left(\xi_1\xi_1^\varphi\omega_1^{-k-1} - \xi_2\xi_2^\varphi\omega_2^{-k-1}\right)\varphi_{t+k}\right\}, \quad \text{(A.2)}
\end{aligned}
$$

where $\vartheta \equiv (1-\rho)(1-\beta\rho)/\kappa$ and

$$
\begin{aligned}
\xi_j &\equiv \beta(\omega_j + \rho - 1) - 1,\\
\xi_j^g &\equiv (1-\delta_g)(\omega_j - 1) + \frac{\delta_g \chi_y}{\sigma\chi_i},\\
\xi_j^\varphi &\equiv \delta_\varphi(\omega_j - 1) - \frac{\delta_\varphi \chi_y}{\sigma\chi_i}
\end{aligned}
$$

for $j \in \{1,2\}$.

The only parameter that depends on the degree of price stickiness θ in the structural equations (1), (2), and (7) is the slope κ of the Phillips curve (2). We have $\lim_{\theta\to 0}\kappa = +\infty$ and therefore

$$
-\beta\sigma \lim_{\theta\to 0}\left[\frac{\mathcal{P}(X)}{\kappa}\right] = X(X - \omega_1^n)
$$

for any $X \in \mathbb{R}$, where $\omega_1^n \equiv (1+\chi_i)/\chi_i > 1$, which implies in turn that

$$
\lim_{\theta\to 0}\rho = 0, \quad \lim_{\theta\to 0}\omega_1 = \omega_1^n, \quad \text{and} \quad \lim_{\theta\to 0}\omega_2 = +\infty. \tag{A.3}
$$

Using (A.3) and

$$
(1-\rho)(\omega_1 - 1)(\omega_2 - 1) = \mathcal{P}(1) = \frac{\kappa}{\beta\sigma\chi_i},
$$

we also get that

$$
\lim_{\theta\to 0}\frac{\kappa}{\omega_2} = \beta\sigma. \tag{A.4}
$$

Using (A.3) and (A.4), we can easily determine the limits of (A.1) and (A.2) as $\theta \to 0$:

$$
\begin{aligned}
\lim_{\theta\to 0}\pi_t &= -p_{t-1} - \mathbb{E}_t\left\{\sum_{k=0}^{+\infty}(\omega_1^n)^{-k-1}\left\{i_{t+k}^m - r_{t+k} - \frac{M_{t+k}}{\chi_i} + \left[\frac{\sigma(1-\delta_g) + \chi_y\delta_g}{\chi_i}\right]g_{t+k}\right.\right.\\
&\left.\left. + \left[\frac{(\chi_y - \sigma)\delta_\varphi}{\chi_i}\right]\varphi_{t+k}\right\}\right\} + \sigma(1-\delta_g)g_t - \sigma\delta_\varphi\varphi_t, \quad \text{(A.5)}\\
\lim_{\theta\to 0} y_t &= \delta_g g_t + \delta_\varphi\varphi_t. \quad \text{(A.6)}
\end{aligned}
$$

These limits are finite, unlike their counterparts in the basic NK model.

We now show that the right-hand sides of (A.5) and (A.6) coincide with the values taken by π_t and y_t when prices are perfectly flexible ($\theta = 0$). The flexible-price value of y_t is straightforwardly obtained by setting to zero the last term in the Phillips curve (2), which is proportional to (the log-deviation of) firms' marginal cost of production:

$$
y_t = \delta_g g_t + \delta_\varphi\varphi_t. \tag{A.7}
$$

This value is identical to the right-hand side of (A.6). Using the IS equation (1), the money-demand equation (7), the identity $m_t = M_t - p_t$, and the solution for flexible-price output (A.7), we get the following dynamic equation under flexible prices:

$$
\begin{aligned}
p_t \;=\; & (\omega_1^n)^{-1} \mathbb{E}_t \{p_{t+1}\} - (\omega_1^n)^{-1} \left\{ i_t^m - r_t - \frac{M_t}{\chi_i} - \left[\sigma (1-\delta_g) - \frac{\chi_y \delta_g}{\chi_i} \right] g_t \right. \\
& \left. + \sigma (1-\delta_g) \mathbb{E}_t \{g_{t+1}\} + \left(\sigma + \frac{\chi_y}{\chi_i} \right) \delta_\varphi \varphi_t - \sigma \delta_\varphi \mathbb{E}_t \{\varphi_{t+1}\} \right\}.
\end{aligned}
$$

Iterating this equation forward to $+\infty$ leads to the following value for the price level p_t in our simple model under flexible prices:

$$
\begin{aligned}
p_t \;=\; & -\mathbb{E}_t \left\{ \sum_{k=0}^{+\infty} (\omega_1^n)^{-k-1} \left\{ i_{t+k}^m - r_{t+k} - \frac{M_{t+k}}{\chi_i} + \left[\frac{\sigma (1-\delta_g) + \chi_y \delta_g}{\chi_i} \right] g_{t+k} \right. \right. \\
& \left. \left. + \left[\frac{(\chi_y - \sigma) \delta_\varphi}{\chi_i} \right] \varphi_{t+k} \right\} \right\} + \sigma (1-\delta_g) g_t - \sigma \delta_\varphi \varphi_t,
\end{aligned}
$$

which implies in turn that the value of $\pi_t \equiv p_t - p_{t-1}$ in our simple model under flexible prices coincides with the right-hand side of (A.5). Thus, our simple model solves the paradox of flexibility: the limits of π_t and y_t as $\theta \to 0$ are finite and coincide with the values of π_t and y_t when $\theta = 0$.

Appendix B: MIU Model (Presentation and Log-Linearization)

In this appendix, to lighten up the notation, we sometimes omit function arguments when no ambiguity results.

B.1 Households

Households get utility from consumption (c_t) and real money (m_t), and disutility from labor (h_t). Their intertemporal utility is

$$
\mathcal{U}_t = \mathbb{E}_t \left\{ \sum_{k=0}^{\infty} \beta^k \zeta_{t+k} \left[u(c_{t+k}, m_{t+k}) - \frac{v(h_{t+k})}{\varphi_{1,t+k}} \right] \right\},
$$

where $\beta \in (0,1)$. The utility function u, defined over the set of pairs of positive real numbers $\mathbb{R}^2_{>0}$, is twice differentiable, strictly increasing ($u_c > 0$, $u_m > 0$), strictly concave ($u_{cc} < 0$, $u_{mm} < 0$, $u_{cc}u_{mm} - (u_{cm})^2 > 0$), with $u_{cm} \geq 0$, and it satisfies the standard Inada conditions

$$
\begin{aligned}
\lim_{c_t \to 0} u_c(c_t, m_t) &= +\infty, & \lim_{c_t \to +\infty} u_c(c_t, m_t) &= 0, \\
\lim_{m_t \to 0} u_m(c_t, m_t) &= +\infty, & \lim_{m_t \to +\infty} u_m(c_t, m_t) &= 0.
\end{aligned}
$$

The labor-disutility function v, defined over the set of non-negative real numbers $\mathbb{R}_{\geq 0}$, is twice differentiable, strictly increasing ($v' > 0$), and weakly convex ($v'' \geq 0$). The intertemporal utility $\mathcal{U}_t$ is affected by two stochastic exogenous shocks of mean one: the discount-factor shock ζ_t, and the labor-disutility shock $\varphi_{1,t}$. The latter shock is the first of the four alternative supply shocks that we consider.

Households choose c_t, h_t, m_t, and real bonds b_t to maximize their utility function subject to their budget constraint

$$c_t + b_t + m_t \leq \frac{I_{t-1}}{\Pi_t} b_{t-1} + \frac{I_{t-1}^m}{\Pi_t} m_{t-1} + w_t h_t + \tau_t, \tag{B.1}$$

where I_t denotes the gross nominal interest rate on bonds, I_t^m the gross nominal interest rate on money, $\Pi_t \equiv P_t/P_{t-1}$ the gross inflation rate (with P_t the price level), w_t the real wage, and τ_t captures firm profits and the government's lump-sum taxes or transfers. Let λ_t denote the Lagrange multiplier on the period-t budget constraint. The first-order conditions of this maximization problem are

$$\lambda_t = \zeta_t u_c\left(c_t, m_t\right), \tag{B.2}$$

$$\frac{1}{I_t} = \beta \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\lambda_t \Pi_{t+1}} \right\}, \tag{B.3}$$

$$\lambda_t w_t = \frac{\zeta_t v'\left(h_t\right)}{\varphi_{1,t}}, \tag{B.4}$$

$$\zeta_t u_m\left(c_t, m_t\right) = \lambda_t - \beta I_t^m \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\Pi_{t+1}} \right\}.$$

Using (B.2) and (B.3), we can rewrite the last condition as

$$\frac{I_t^m}{I_t} = 1 - \frac{u_m\left(c_t, m_t\right)}{u_c\left(c_t, m_t\right)}. \tag{B.5}$$

B.2 Firms

There is a continuum of monopolistically competitive firms owned by households and indexed by $i \in [0,1]$. Each firm i uses $h_t(i)$ units of labor to produce

$$y_t\left(i\right) = \varphi_{2,t} f\left[h_t\left(i\right)\right] \tag{B.6}$$

units of output. The production function f, defined over $\mathbb{R}_{\geq 0}$, is twice differentiable, strictly increasing ($f' > 0$), weakly concave ($f'' \leq 0$), and such that $f(0) = 0$. The stochastic exogenous technology shock $\varphi_{2,t}$, of mean one, is the second of the four alternative supply shocks that we consider. The third supply shock that we consider, $\varphi_{3,t}$, also of mean one, captures a labor subsidy received by firms (when $\varphi_{3,t} > 1$) or labor tax paid by firms (when $\varphi_{3,t} < 1$): if W_t denotes the pre-subsidy or pre-tax nominal wage, then the after-subsidy or after-tax nominal wage paid by firms is $W_t/\varphi_{3,t}$.

Following Calvo (1983), we assume that at any date, each firm, whatever its history, has the probability $\theta \in [0,1)$ not to be allowed to reset its price. If allowed to reset its price at date t, firm i chooses its new price $P_t^*(i)$ to maximize the present value of the profits that this price will generate:

$$\mathbb{E}_t\left\{\sum_{k=0}^{+\infty}(\beta\theta)^k\frac{\lambda_{t+k}}{\lambda_t\Pi_{t,t+k}}\left[P_t^*(i)\,y_{t+k}(i)-\frac{W_{t+k}h_{t+k}(i)}{\varphi_{3,t+k}}\right]\right\},$$

subject to the production function (B.6) and the demand schedule

$$y_{t+k}(i)=\left[\frac{P_t^*(i)}{P_{t+k}}\right]^{-\varepsilon\varphi_{4,t+k}}y_{t+k}, \tag{B.7}$$

where $\Pi_{t,t+k}\equiv P_{t+k}/P_t$ for any $k\in\mathbb{N}$, $\varepsilon>0$ denotes the steady-state elasticity of substitution between differentiated goods, and $y_t\equiv[\int_0^1 y_t(i)^{(\varepsilon\varphi_{4,t}-1)/(\varepsilon\varphi_{4,t})}di]^{\varepsilon\varphi_{4,t}/(\varepsilon\varphi_{4,t}-1)}$. The stochastic exogenous shock $\varphi_{4,t}$, of mean one, affecting the elasticity of substitution between differentiated goods, is the last of the four alternative supply shocks that we consider.

Using (B.6), we can rewrite the present value of the profits generated by $P_t^*(i)$ as

$$\mathbb{E}_t\left\{\sum_{k=0}^{+\infty}(\beta\theta)^k\frac{\lambda_{t+k}}{\lambda_t\Pi_{t,t+k}}\left[P_t^*(i)\,y_{t+k}(i)-\frac{W_{t+k}}{\varphi_{3,t+k}}f^{-1}\left[\frac{y_{t+k}(i)}{\varphi_{2,t+k}}\right]\right]\right\}.$$

Choosing $P_t^*(i)$ to maximize this present value subject to (B.7) leads to the first-order condition

$$\mathbb{E}_t\left\{\sum_{k=0}^{+\infty}(\beta\theta)^k\frac{\lambda_{t+k}(\varepsilon\varphi_{4,t+k}-1)}{\lambda_t\Pi_{t,t+k}}\left[P_t^*(i)-\left(\frac{\varepsilon\varphi_{4,t+k}}{\varepsilon\varphi_{4,t+k}-1}\right)\frac{W_{t+k}}{\varphi_{2,t+k}\varphi_{3,t+k}f'[h_{t+k}(i)]}\right]y_{t+k}(i)\right\}=0. \tag{B.8}$$

In the limit case of perfectly flexible prices ($\theta=0$), and in a symmetric equilibrium ($P_t^*(i)=P_t$ and $h_t(i)=h_t$), this first-order condition becomes

$$P_t=\left(\frac{\varepsilon\varphi_{4,t}}{\varepsilon\varphi_{4,t}-1}\right)\frac{W_t}{\varphi_{2,t}\varphi_{3,t}f'(h_t)}. \tag{B.9}$$

B.3 Government

The government consists of a fiscal authority and a monetary authority. The fiscal authority consumes an exogenous quantity $g_t\geq 0$ of goods, does not issue bonds, and sets lump-sum taxes on households so as to balance its budget (making fiscal policy Ricardian). We assume for simplicity that government purchases g_t are wasted, but the results would be unchanged if they entered households' utility function in a separable way.

The monetary authority – i.e., the central bank – has two independent instruments: the nominal stock of money $M_t>0$, or equivalently its (gross) growth rate $\mu_t\equiv M_t/M_{t-1}>0$, and the (gross) nominal interest rate on money $I_t^m\geq 0$. We assume that the central bank injects reserves via lump-sum transfers.[20] The consolidated budget constraint of the government is thus

$$M_t=I_{t-1}^m M_{t-1}+P_t g_t-T_t, \tag{B.10}$$

[20] It would be straightforward to modify our model and allow changes in money balances to be matched by changes in the monetary authority's holdings of bonds issued by households or the fiscal authority; such features, however, would not play a role in our analysis.

where T_t denotes the net lump-sum tax imposed by the government (the fiscal authority's tax minus the monetary authority's transfer).

To capture a lower bound on I_t^m in a simple and stark way, we assume that cash (with no interest payments) is a perfect substitute for deposits at the central bank in terms of providing liquidity services to households. This introduces a zero lower bound (ZLB) for the net nominal IOR rate $I_t^m - 1$ in our model. In an equilibrium with $I_t^m > 1$, households will hold no cash. In an equilibrium with $I_t^m = 1$, the decomposition of money into reserves and cash will be indeterminate, but also inconsequential.

B.4 Market-Clearing Conditions

The bond-market-clearing condition is

$$b_t = 0,$$

the money-market-clearing condition is

$$m_t = \frac{M_t}{P_t}, \tag{B.11}$$

and the goods-market-clearing condition is

$$c_t + g_t = y_t. \tag{B.12}$$

B.5 Steady-State Existence and Uniqueness

We consider steady-state values of policy-instruments such that $I^m \geq 1$, $\mu = 1$, and $g \geq 0$. Since $\mu = 1$, the set of steady states is the same under sticky prices ($\theta > 0$) as under flexible prices ($\theta = 0$), so that we can use the first-order condition of firms' optimization problem under flexible prices (B.9) to characterize this set. We first use (B.2), (B.4), (B.6), (B.9), and (B.12) to get

$$u_c \left[f(h) - g, m \right] = \left(\frac{\varepsilon}{\varepsilon - 1} \right) \frac{v'(h)}{f'(h)}. \tag{B.13}$$

We then consider two alternative cases in turn, separable utility ($u_{cm} = 0$) and non-separable utility ($u_{cm} > 0$). We show that in both cases, the necessary and sufficient condition for steady-state existence and uniqueness is $I^m < \beta^{-1}$.

In the separable-utility case, the left-hand side of (B.13) does not depend on m and decreases from $+\infty$ to 0 as h increases from $\underline{h} \equiv f^{-1}(g)$ to $+\infty$. The right-hand side of (B.13) increases as h increases from $\underline{h}$ to $+\infty$. Therefore, there is a unique value of h in $(\underline{h}, +\infty)$ that satisfies (B.13). Moreover, (B.3), (B.5), and (B.13) imply

$$\left(\frac{\varepsilon - 1}{\varepsilon} \right) \frac{f'(h)}{v'(h)} u_m \left[f(h) - g, m \right] = 1 - \beta I^m. \tag{B.14}$$

The left-hand side of (B.14) decreases from $+\infty$ to 0 as m increases from 0 to $+\infty$. Therefore, there is a unique value of m that satisfies (B.14) if and only if the right-hand side of (B.14) is positive. In other words, there exists a unique steady state if and only if $I^m < \beta^{-1}$.

In the non-separable-utility case, (B.13) implicitly and uniquely defines a function $\mathcal{M}$ such that

$$m = \mathcal{M}(h). \tag{B.15}$$

This function is defined over $(\underline{h}, +\infty)$, and it is strictly increasing ($\mathcal{M}' > 0$). We then use (B.3), (B.5), (B.13), and (B.15) to get

$$\left(\frac{\varepsilon - 1}{\varepsilon}\right) \frac{f'(h)}{v'(h)} u_m \left[f(h) - g, \mathcal{M}(h)\right] = 1 - \beta I^m. \tag{B.16}$$

The function $z(h) \equiv u_m [f(h) - g, \mathcal{M}(h)]$ is strictly decreasing in h. The reason is that (B.13) implies that $u_c [f(h) - g, \mathcal{M}(h)]$ is strictly increasing in h, i.e. that

$$u_{cc} [f(h) - g, \mathcal{M}(h)] f'(h) + u_{cm} [f(h) - g, \mathcal{M}(h)] \mathcal{M}'(h) > 0,$$

which implies in turn that

$$z'(h) = u_{cm} f'(h) + u_{mm} \mathcal{M}'(h) < \frac{-f'(h)}{u_{cm}} \left(u_{cc} u_{mm} - u_{cm}^2\right) \leq 0,$$

where the functions u_{cc}, u_{mm}, and u_{cm} are evaluated at $[f(h) - g, \mathcal{M}(h)]$. Since $z'(h) < 0$, the left-hand side of (B.16) decreases from $+\infty$ to 0 as h increases from $\underline{h}$ to $+\infty$. Therefore, there is a unique value of h that satisfies this equation if and only if its right-hand side is positive. In other words, there exists a unique steady state if and only if $I^m < \beta^{-1}$.

B.6 Log-Linearization

We assume that $I^m < \beta^{-1}$ and log-linearize the equilibrium conditions of the model around its unique steady state. To derive the Phillips curve (14), we log-linearize firms' first-order condition (B.8), and use the definition of the real wage $w_t \equiv W_t/P_t$, to get

$$\widehat{P}_t^* = (1 - \beta\theta) \mathbb{E}_t \left\{ \sum_{k=0}^{+\infty} (\beta\theta)^k \left[\widehat{w}_{t+k} + \widehat{P}_{t+k} - \widehat{mp}_{t+k|t} - \widehat{\varphi}_{3,t+k} - \frac{\widehat{\varphi}_{4,t+k}}{\varepsilon - 1} \right] \right\}, \tag{B.17}$$

where variables with hats denote log-deviations from steady-state values, $i_t \equiv \widehat{I}_t$, and $mp_{t+k|t}$ denotes the marginal productivity in period $t + k$ for a firm whose price was last set in period t. Log-linearizing the production function (B.6) gives

$$\widehat{h}_t = \frac{f}{f'h} \left(\widehat{y}_t - \widehat{\varphi}_{2,t}\right), \tag{B.18}$$

so that we can rewrite $\widehat{mp}_{t+k|t}$ as

$$\begin{aligned} \widehat{mp}_{t+k|t} &= \widehat{\varphi}_{2,t} + \frac{f''h}{f'} \widehat{h}_{t+k|t} = \widehat{mp}_{t+k} + \frac{f''h}{f'} \left(\widehat{h}_{t+k|t} - \widehat{h}_{t+k}\right) \\ &= \widehat{mp}_{t+k} + \frac{ff''}{(f')^2} \left(\widehat{y}_{t+k|t} - \widehat{y}_{t+k}\right) = \widehat{mp}_{t+k} - \frac{\varepsilon f f''}{(f')^2} \left(\widehat{P}_t^* - \widehat{P}_{t+k}\right), \end{aligned} \tag{B.19}$$

where mp_{t+k} denotes the average marginal productivity in period $t+k$. Using this result and

$$\pi_t \equiv \log(\Pi_t) = (1-\theta)\left(\widehat{P}_t^* - \widehat{P}_{t-1}\right),$$

and following the same steps as in, e.g., Galí (2008, Chapter 3), we can rewrite (B.17) as

$$\pi_t = \beta \mathbb{E}_t\{\pi_{t+1}\} + \frac{(1-\theta)(1-\beta\theta)}{\theta\left[1-\frac{\varepsilon f f''}{(f')^2}\right]}\left(\widehat{w}_t - \widehat{mp}_t - \widehat{\varphi}_{3,t} - \frac{\widehat{\varphi}_{4,t}}{\varepsilon-1}\right). \tag{B.20}$$

Now, log-linearizing the goods-market-clearing condition (B.12) gives

$$\widetilde{c}_t + \widetilde{g}_t = \widehat{y}_t, \tag{B.21}$$

where $\widetilde{c}_t \equiv (c/y)\widehat{c}_t$ and $\widetilde{g}_t \equiv (g/y)\widehat{g}_t$. Log-linearizing the first-order condition (B.4), and using (B.18) and (B.21), gives

$$\widehat{w}_t = \left(-\frac{u_{cc}y}{u_c} + \frac{v''h}{v'}\frac{f}{f'h}\right)\widehat{y}_t - \frac{u_{cm}m}{u_c}\widehat{m}_t + \frac{u_{cc}y}{u_c}\widetilde{g}_t - \widehat{\varphi}_{1,t} - \frac{v''h}{v'}\frac{f}{f'h}\widehat{\varphi}_{2,t}. \tag{B.22}$$

Moreover, we have

$$\widehat{mp}_t = \widehat{\varphi}_{2,t} + \frac{ff''}{(f')^2}\left(\widehat{y}_t - \widehat{\varphi}_{2,t}\right). \tag{B.23}$$

Using (B.22) and (B.23), we can then rewrite (B.20) as the Phillips curve

$$\pi_t = \beta \mathbb{E}_t\{\pi_{t+1}\} + \kappa\left(\widehat{y}_t - \delta_m \widehat{m}_t - \delta_g \widetilde{g}_t - \delta_\varphi \widehat{\varphi}_t\right) \tag{B.24}$$

with

$$\begin{aligned}
\kappa &\equiv \frac{(1-\theta)(1-\beta\theta)}{\theta\left[1-\frac{\varepsilon f f''}{(f')^2}\right]}\psi > 0,\\
\delta_m &\equiv \left(\frac{u_{cm}m}{u_c}\right)\psi^{-1} \geq 0,\\
\delta_g &\equiv \left(\frac{-u_{cc}y}{u_c}\right)\psi^{-1} \in (0,1),\\
\delta_\varphi &\equiv \left\{\mathbb{1}_{\varphi_t=\varphi_{1,t}} + \left[1 + \frac{v''h}{v'}\frac{f}{f'h} - \frac{ff''}{(f')^2}\right]\mathbb{1}_{\varphi_t=\varphi_{2,t}} + \mathbb{1}_{\varphi_t=\varphi_{3,t}} + \left(\frac{1}{\varepsilon-1}\right)\mathbb{1}_{\varphi_t=\varphi_{4,t}}\right\}\psi^{-1} > 0,
\end{aligned}$$

where

$$\psi \equiv \frac{-u_{cc}y}{u_c} + \frac{v''h}{v'}\frac{f}{f'h} - \frac{ff''}{(f')^2} > 0.$$

Note that, to write this Phillips curve in a compact way, we have considered a single supply shock $\varphi_t \in \{\varphi_{1,t}, \varphi_{2,t}, \varphi_{3,t}, \varphi_{4,t}\}$ and used indicator functions in the definition of δ_φ: for any $k \in \{1,2,3,4\}$, $\mathbb{1}_{\varphi_t=\varphi_{k,t}}$ takes the value one if $\varphi_t = \varphi_{k,t}$ and the value zero otherwise.

Log-linearizing the first-order condition (B.5) and using (B.21) gives the money-demand equation

$$\widehat{m}_t = \chi_y\left(\widehat{y}_t - \widetilde{g}_t\right) - \chi_i\left(i_t - i_t^m\right), \tag{B.25}$$

where $i_t^m \equiv \widehat{I}_t^m$ and

$$\chi_y \equiv \left(\frac{u_{cm}m}{u_c} - \frac{u_{mm}m}{u_m}\right)^{-1}\left(\frac{u_{cm}y}{u_m} - \frac{u_{cc}y}{u_c}\right) > 0,$$
$$\chi_i \equiv \left(\frac{u_{cm}m}{u_c} - \frac{u_{mm}m}{u_m}\right)^{-1}\left(\frac{\beta I^m}{1-\beta I^m}\right) > 0.$$

Finally, log-linearizing the first-order condition (B.3) and using (B.21) gives the IS equation

$$\widehat{y}_t = \mathbb{E}_t\left\{\widehat{y}_{t+1}\right\} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t\left\{\pi_{t+1}\right\} - r_t\right) - \eta\mathbb{E}_t\left\{\Delta\widehat{m}_{t+1}\right\} - \mathbb{E}_t\left\{\Delta\widetilde{g}_{t+1}\right\}, \tag{B.26}$$

where $\Delta \equiv 1 - L$ denotes the first-difference operator, $r_t \equiv -\mathbb{E}_t\{\Delta\widehat{\zeta}_{t+1}\}$, and

$$\sigma \equiv \frac{-u_{cc}y}{u_c} > 0,$$
$$\eta \equiv \left(\frac{-u_{cc}y}{u_c}\right)^{-1}\frac{u_{cm}m}{u_c} \geq 0.$$

The IS equation (13), the Phillips curve (14), and the money-demand equation (15) in the main text correspond respectively to equations (B.26), (B.24), and (B.25) in which, for simplicity, the notations $\widehat{y}_t$, $\widehat{m}_t$, $\widetilde{g}_t$, and $\widehat{\varphi}_t$ are replaced by the notations y_t, m_t, g_t, and φ_t (as everywhere in the main text). The reduced-form parameters η, δ_m, and δ_g are straightforwardly linked to each other through the equality (16), while the reduced-form parameters δ_m and χ_y satisfy the inequality (17) because

$$\begin{aligned}
1-\delta_m\chi_y &= 1 - \left[\frac{-u_{cc}y}{u_c} + \frac{v''h}{v'}\frac{f}{f'h} - \frac{ff''}{(f')^2}\right]^{-1}\left(\frac{u_{cm}m}{u_c} - \frac{u_{mm}m}{u_m}\right)^{-1}\left(\frac{u_{cm}c}{u_m} - \frac{u_{cc}y}{u_c}\right)\frac{u_{cm}m}{u_c} \\
&= \left[\frac{-u_{cc}y}{u_c} + \frac{v''h}{v'}\frac{f}{f'h} - \frac{ff''}{(f')^2}\right]^{-1}\left(\frac{u_{cm}m}{u_c} - \frac{u_{mm}m}{u_m}\right)^{-1}\left\{\left[\frac{v''h}{v'}\frac{f}{f'h} - \frac{ff''}{(f')^2}\right]\right. \\
&\quad \left.\left(\frac{u_{cm}m}{u_c} - \frac{u_{mm}m}{u_m}\right) + \frac{(y-c)\,mu_{cc}u_{mm}}{u_cu_m} + \frac{cm}{u_cu_m}\left(u_{cc}u_{mm} - u_{cm}^2\right)\right\} \\
&> 0.
\end{aligned}$$

Appendix C: MIU Model (Log-Linearized Version)

In this appendix, to lighten up the notation, we sometimes omit function arguments when no ambiguity results.

C.1 Root Analysis

We first show that $0 < \rho < 1 < |\omega_1| \leq |\omega_2|$. To that aim, we write the polynomial $\mathcal{P}(X)$ as

$$\mathcal{P}(X) = X^3 - a_2X^2 + a_1X - a_0$$

with

$$
\begin{aligned}
a_2 &\equiv 2+\frac{1}{\beta}+\frac{\kappa}{\beta\sigma}+\frac{(1-\delta_g)\,\delta_m\kappa}{\beta\delta_g}+\frac{\chi_y}{\sigma\chi_i}>3,\\
a_1 &\equiv 1+\frac{2}{\beta}+\frac{\kappa}{\beta\sigma}+\frac{(1-\delta_g)\,\delta_m\kappa}{\beta\delta_g}+\frac{(1+\beta)\,\chi_y}{\beta\sigma\chi_i}+\frac{(1-\delta_m\chi_y)\,\kappa}{\beta\sigma\chi_i}>0,\\
a_0 &\equiv \frac{1}{\beta}+\frac{\chi_y}{\beta\sigma\chi_i}>0,
\end{aligned}
$$

where the inequality $a_2 > 3$ comes from $\beta \in (0,1)$ and $\delta_g \in (0,1)$, and the inequality $a_1 > 0$ from $\delta_g \in (0,1)$ and (17).

We have $\mathcal{P}(0) = -a_0 < 0$ and $\mathcal{P}(1) = (1-\delta_m\chi_y)\kappa/(\beta\sigma\chi_i) > 0$, where the last inequality comes from (17). Therefore, $\mathcal{P}(X)$ has either one or three real roots inside $(0,1)$. Moreover, the inequalities $a_2 > 0$, $a_1 > 0$, and $a_0 > 0$ imply that $\mathcal{P}(X) < 0$ for all $X < 0$, so that $\mathcal{P}(X)$ has no negative real roots. Therefore, $\mathcal{P}(X)$ has at least one real root inside $(0,1)$, which we denote by ρ, and its other two roots, which we denote by ω_1 and ω_2 with $|\omega_1| \leq |\omega_2|$, must be (i) both real and inside $(0,1)$, or (ii) both real and larger than 1, or (iii) both complex and conjugates of each other. Now, we have $\rho+\omega_1+\omega_2 = a_2 > 3$. Therefore, Case (i) is impossible, and in Case (iii) the common real part of ω_1 and ω_2 is larger than 1. As a consequence, in the remaining two possible cases, namely Cases (ii) and (iii), ω_1 and ω_2 lie outside the unit circle.

We now show that ω_1 and ω_2 can be real numbers, and that they can also be complex (non-real) numbers. Consider, for example, the separable and iso-elastic specification

$$
u(c_t, m_t) = \frac{c_t^{1-\sigma_c}-1}{1-\sigma_c}+\frac{m_t^{1-\sigma_m}-1}{1-\sigma_m},
$$

where $\sigma_c > 0$ and $\sigma_m > 0$. Under this specification, σ, κ, δ_m, δ_g, and χ_y/χ_i do not depend on σ_m, but χ_i and χ_y do. Therefore, a_2 and a_0 do not depend on σ_m, but a_1 does. Since $\lim_{\sigma_m\to+\infty}\chi_i = 0$, we have $\lim_{\sigma_m\to+\infty} a_1 = +\infty$. As a consequence, for sufficiently large values of σ_m, $\mathcal{P}(X) = X^3 - a_2X^2 + a_1X - a_0$ is positive for all $X \geq 1$, so that Case (ii) is impossible and ω_1 and ω_2 are complex numbers. Moreover, since $\lim_{\sigma_m\to 0}\chi_i = +\infty$, we have

$$
\lim_{\sigma_m\to 0}\mathcal{P}\left(1+\frac{\chi_y}{\sigma\chi_i}\right) = -\left(1+\frac{\chi_y}{\sigma\chi_i}\right)\frac{\chi_y\kappa}{\beta\sigma^2\chi_i} < 0.
$$

Therefore, for sufficiently small values of σ_m, we have $\mathcal{P}[1+\chi_y/(\sigma\chi_i)] < 0$, which, together with $\mathcal{P}(1) > 0$, implies that ω_1 and ω_2 are positive real numbers.

By continuity, there also exist non-separable specifications of u that can make ω_1 and ω_2 real or complex depending on the calibration. Consider, for instance, the iso-elastic specification

$$
u(c_t, m_t) = \frac{c_t^{1-\sigma_c}-1}{1-\sigma_c}+\frac{m_t^{1-\sigma_m}-1}{1-\sigma_m}+\epsilon c_t^{\nu} m_t^{1-\nu},
$$

where $\nu \in (0,1)$ and $\epsilon > 0$. If ϵ is sufficiently small, then, as above, ω_1 and ω_2 will be real for sufficiently small values of σ_m and complex for sufficiently large values of σ_m.

C.2 Resolution of the Paradox of Flexibility

Using the definition of Z_t, and after some simple algebra, we can rewrite (9) and (18) as

$$\begin{aligned} \pi_t &= -(1-\rho)\,p_{t-1} + \frac{\kappa}{\beta(\omega_2-\omega_1)}\mathbb{E}_t\left\{-\frac{1}{\sigma}\sum_{k=0}^{+\infty}\left(\omega_1^{-k-1}-\omega_2^{-k-1}\right)\left(i_{t+k}^m - r_{t+k}\right)\right. \\ &\quad + \sum_{k=0}^{+\infty}\left(\xi_1^M\omega_1^{-k-1}-\xi_2^M\omega_2^{-k-1}\right)M_{t+k} - \sum_{k=0}^{+\infty}\left(\xi_1^g\omega_1^{-k-1}-\xi_2^g\omega_2^{-k-1}\right)g_{t+k} \\ &\quad \left. + \sum_{k=0}^{+\infty}\left(\xi_1^\varphi\omega_1^{-k-1}-\xi_2^\varphi\omega_2^{-k-1}\right)\varphi_{t+k}\right\}, \end{aligned} \tag{C.1}$$

$$\begin{aligned} y_t &= -\vartheta p_{t-1} + \frac{\delta_m}{\delta_g}M_t + g_t + \frac{\mathbb{E}_t}{\beta(\omega_2-\omega_1)}\left\{\frac{1}{\sigma}\sum_{k=0}^{+\infty}\left(\xi_1\omega_1^{-k-1}-\xi_2\omega_2^{-k-1}\right)\left(i_{t+k}^m - r_{t+k}\right)\right. \\ &\quad - \sum_{k=0}^{+\infty}\left(\xi_1\xi_1^M\omega_1^{-k-1}-\xi_2\xi_2^M\omega_2^{-k-1}\right)M_{t+k} + \sum_{k=0}^{+\infty}\left(\xi_1\xi_1^g\omega_1^{-k-1}-\xi_2\xi_2^g\omega_2^{-k-1}\right)g_{t+k} \\ &\quad \left. - \sum_{k=0}^{+\infty}\left(\xi_1\xi_1^\varphi\omega_1^{-k-1}-\xi_2\xi_2^\varphi\omega_2^{-k-1}\right)\varphi_{t+k}\right\}, \end{aligned} \tag{C.2}$$

where $\vartheta \equiv (1-\rho)(1-\beta\rho)/\kappa + \delta_m\rho$ and

$$\begin{aligned} \xi_j &\equiv \beta(\omega_j+\rho-1)+\kappa\delta_m - 1, \\ \xi_j^M &\equiv \frac{1-\delta_m\chi_y}{\sigma\chi_i} - \frac{(1-\delta_g)(\omega_j-1)\delta_m}{\delta_g}, \\ \xi_j^g &\equiv \left(\omega_j - 1 - \frac{\chi_y}{\sigma\chi_i}\right)(1-\delta_g), \\ \xi_j^\varphi &\equiv \left(\omega_j - 1 - \frac{\chi_y}{\sigma\chi_i}\right)\delta_\varphi \end{aligned}$$

for $j \in \{1,2\}$.

The only parameter that depends on the degree of price stickiness θ in the structural equations (13), (14), and (15) is the slope κ of the Phillips curve (14). We have $\lim_{\theta\to 0}\kappa = +\infty$ and hence

$$\left[\frac{-\beta\sigma\delta_g}{\delta_g+(1-\delta_g)\sigma\delta_m}\right]\lim_{\theta\to 0}\left[\frac{\mathcal{P}(X)}{\kappa}\right] = X\left(X-\omega_1^n\right)$$

for any $X \in \mathbb{R}$, where

$$\omega_1^n \equiv 1 + \left[\frac{1-\delta_m\chi_y}{\delta_g+(1-\delta_g)\sigma\delta_m}\right]\frac{\delta_g}{\chi_i} > 1,$$

where in turn the inequality follows from $\delta_g \in (0,1)$ and (17). Therefore, we get

$$\lim_{\theta\to 0}\rho = 0, \quad \lim_{\theta\to 0}\omega_1 = \omega_1^n, \quad \text{and} \quad \lim_{\theta\to 0}\omega_2 = +\infty. \tag{C.3}$$

Using (C.3) and

$$(1-\rho)(\omega_1-1)(\omega_2-1) = \mathcal{P}(1) = \frac{(1-\delta_m\chi_y)\kappa}{\beta\sigma\chi_i},$$

we also get that

$$\lim_{\theta\to 0}\frac{\kappa}{\omega_2}=\frac{\beta\sigma\delta_g}{\delta_g+(1-\delta_g)\,\sigma\delta_m}. \tag{C.4}$$

Using (C.3) and (C.4), we can easily determine the limits of (C.1) and (C.2) as $\theta \to 0$:

$$\begin{aligned}
\lim_{\theta\to 0}\pi_t &= -p_{t-1}+\frac{\delta_g}{\delta_g+(1-\delta_g)\,\sigma\delta_m}\mathbb{E}_t\left\{\sum_{k=0}^{+\infty}(\omega_1^n)^{-k-1}\left\{-\left(i_{t+k}^m-r_{t+k}\right)\right.\right.\\
&\quad\left.\left.+\left(\omega_1^n-1\right)M_{t+k}+\left[\frac{\chi_y}{\chi_i}-\sigma\left(\omega_1^n-1\right)\right]\left[(1-\delta_g)\,g_{t+k}-\delta_\varphi\varphi_{t+k}\right]\right\}\right\}\\
&\quad+\frac{\sigma\delta_g}{\delta_g+(1-\delta_g)\,\sigma\delta_m}\left[\frac{(1-\delta_g)\,\delta_m}{\delta_g}M_t+(1-\delta_g)\,g_t-\delta_\varphi\varphi_t\right], \qquad (C.5)\\
\lim_{\theta\to 0}y_t &= \frac{\delta_g\delta_m}{\delta_g+(1-\delta_g)\,\sigma\delta_m}\mathbb{E}_t\left\{\sum_{k=0}^{+\infty}(\omega_1^n)^{-k-1}\left\{i_{t+k}^m-r_{t+k}-\left(\omega_1^n-1\right)M_{t+k}\right.\right.\\
&\quad\left.\left.+\left[\sigma\left(\omega_1^n-1\right)-\frac{\chi_y}{\chi_i}\right]\left[(1-\delta_g)\,g_{t+k}-\delta_\varphi\varphi_{t+k}\right]\right\}\right\}\\
&\quad+\frac{\delta_g}{\delta_g+(1-\delta_g)\,\sigma\delta_m}\left[\delta_m M_t+\delta_g g_t+\left(1+\frac{\sigma\delta_m}{\delta_g}\right)\delta_\varphi\varphi_t\right]. \qquad (C.6)
\end{aligned}$$

These limits are finite, unlike their counterparts in the basic NK model.

We now show that the right-hand sides of (C.5) and (C.6) coincide with the values taken by π_t and y_t when prices are perfectly flexible ($\theta = 0$). To determine these values, we first log-linearize the first-order condition of firms' optimization problem under flexible prices (B.9), and use (B.18), to get

$$\widehat{w}_t=\frac{ff''}{(f')^2}\widehat{y}_t+\left[1-\frac{ff''}{(f')^2}\right]\widehat{\varphi}_{2,t}+\widehat{\varphi}_{3,t}+\frac{\widehat{\varphi}_{4,t}}{\varepsilon-1}. \tag{C.7}$$

Using (B.22) and (C.7), considering a single supply shock $\varphi_t \in \{\varphi_{1,t}, \varphi_{2,t}, \varphi_{3,t}, \varphi_{4,t}\}$, and replacing the notations $\widehat{y}_t$, $\widehat{m}_t$, $\widetilde{g}_t$, and $\widehat{\varphi}_t$ by the notations y_t, m_t, g_t, and φ_t (for simplicity and consistency with the main text), we then get

$$y_t=\delta_m m_t+\delta_g g_t+\delta_\varphi\varphi_t. \tag{C.8}$$

Finally, using the IS equation (13), the money-demand equation (15), the identity $m_t = M_t - p_t$, and the solution for flexible-price output (C.8), we get the following dynamic equation under flexible prices:

$$\begin{aligned}
p_t &= (\omega_1^n)^{-1}\,\mathbb{E}_t\{p_{t+1}\}+\frac{\delta_g\,(\omega_1^n)^{-1}}{\delta_g+(1-\delta_g)\,\sigma\delta_m}\left\{-\left(i_t^m-r_t\right)+\left[\frac{1-\delta_m\chi_y}{\chi_i}+\frac{(1-\delta_g)\,\sigma\delta_m}{\delta_g}\right]M_t\right.\\
&\quad-\frac{(1-\delta_g)\,\sigma\delta_m}{\delta_g}\mathbb{E}_t\{M_{t+1}\}+\left(\sigma+\frac{\chi_y}{\chi_i}\right)(1-\delta_g)\,g_t-\sigma\,(1-\delta_g)\,\mathbb{E}_t\{g_{t+1}\}\\
&\quad\left.-\left(\sigma+\frac{\chi_y}{\chi_i}\right)\delta_\varphi\varphi_t+\sigma\delta_\varphi\mathbb{E}_t\{\varphi_{t+1}\}\right\},
\end{aligned}$$

where we have used the equality (16) to replace η by δ_m/δ_g. Iterating this equation forward to

$+\infty$ leads to the following value for the price level p_t in our MIU model under flexible prices:

$$\begin{aligned} p_t \;\; = \;\; & \frac{\delta_g}{\delta_g + (1-\delta_g)\,\sigma\delta_m}\mathbb{E}_t \left\{ \sum_{k=0}^{+\infty} (\omega_1^n)^{-k-1} \left\{ -\left(i_{t+k}^m - r_{t+k}\right) + (\omega_1^n - 1)\, M_{t+k} \right.\right. \\ & \left.\left. + \left[\frac{\chi_y}{\chi_i} - \sigma\,(\omega_1^n - 1)\right] \left[(1-\delta_g)\, g_{t+k} - \delta_\varphi \varphi_{t+k}\right] \right\}\right\} \\ & + \frac{\sigma\delta_g}{\delta_g + (1-\delta_g)\,\sigma\delta_m}\left[\frac{(1-\delta_g)\,\delta_m}{\delta_g} M_t + (1-\delta_g)\, g_t - \delta_\varphi \varphi_t\right], \end{aligned} \tag{C.9}$$

which implies in turn that the value of $\pi_t \equiv p_t - p_{t-1}$ in our MIU model under flexible prices coincides with the right-hand side of (C.5). In turn, using (C.8), (C.9), and the identity $m_t = M_t - p_t$, we get that the value of y_t in our MIU model under flexible prices coincides with the right-hand side of (C.6). Thus, our MIU model solves the paradox of flexibility: the limits of π_t and y_t as $\theta \to 0$ are finite and coincide with the values of π_t and y_t when $\theta = 0$.

C.3 Convergence to the Basic NK Model

We start with the separable specification

$$u(c_t, m_t) = u_1(c_t) + \gamma u_2(m_t),$$

where $\gamma > 0$ is a scale parameter. Under this specification, the steady-state value h of h_t, given by (B.13) with $u_c[f(h) - g, m] = u_1'[f(h) - g]$, is identical to the steady-state value of h_t in the basic NK model (with consumption-utility function u_1). The IS equation (13) and the Phillips curve (14) are also identical to the IS equation (1) and the Phillips curve (2) of the basic NK model, in the sense that their reduced-form parameters take the same values (in particular $\eta = 0$ and $\delta_m = 0$). The steady-state value m of m_t is given by (B.14), which can be rewritten as

$$u_2'(m) = \left(\frac{1-\beta I^m}{\gamma}\right)\left(\frac{\varepsilon}{\varepsilon - 1}\right)\frac{v'(h)}{f'(h)}. \tag{C.10}$$

If (I^m, γ) goes to $(\beta^{-1}, 0)$ with $(1 - \beta I^m)/\gamma$ bounded away from zero and infinity, as in the thought experiment of Subsection 4.1, then m is bounded away from zero and infinity. In this case, $\chi_y = (-u_2''m/u_2')^{-1}(-u_1''y/u_1')$ is also bounded away from zero and infinity, while $\chi_i = (-u_2''m/u_2')^{-1}\beta I^m/(1-\beta I^m)$ goes to infinity. Therefore, χ_i^{-1} and $\chi_y\chi_i^{-1}$ converge to zero, and the money-demand equation (15) converges to $i_t = i_t^m$. Alternatively, if I^m goes to β^{-1} holding γ constant, as in the policy experiment of Subsection 5.1, then m goes to infinity (asymptotic satiation). In that case, we still have χ_i^{-1} and $\chi_y\chi_i^{-1}$ converging to zero, and (15) converging to $i_t = i_t^m$, if the elasticity $-u_2''m/u_2'$ is bounded from above – a condition that is met, in particular, for isoelastic u_2 functions.

We now turn to the CES-based specification

$$u(c_t, m_t) = U\left\{\left[(1-\gamma)\, c_t^\alpha + \gamma m_t^\alpha\right]^{1/\alpha}\right\},$$

where $\alpha \in (-\infty, 1)$, $\gamma \in (0,1)$, and the function U, defined over the set of positive real numbers $\mathbb{R}_{>0}$, is twice differentiable, strictly increasing ($U' > 0$), and strictly concave ($U'' < 0$). In addition, we impose that $U''(x)x/U'(x) \leq 1-\alpha$ for any $x > 0$, which is the necessary and sufficient condition for $u_{cm} \geq 0$. Under this specification, (B.5) at the steady state can be rewritten as $m = \phi c = \phi[f(h) - g]$, where

$$\phi \equiv \left[\frac{\gamma}{(1-\gamma)(1-\beta I^m)}\right]^{\frac{1}{1-\alpha}},$$

which implies that (B.13) can in turn be rewritten as

$$(1-\gamma)\left[(1-\gamma)+\gamma\phi^\alpha\right]^{\frac{1-\alpha}{\alpha}} U'\left\{\left[(1-\gamma)+\gamma\phi^\alpha\right]^{\frac{1}{\alpha}}\left[f(h)-g\right]\right\} = \left(\frac{\varepsilon}{\varepsilon-1}\right)\frac{v'(h)}{f'(h)}. \tag{C.11}$$

If (I^m, γ) goes to $(\beta^{-1}, 0)$ with $(1-\beta I^m)/\gamma$ bounded away from zero and infinity, as in the thought experiment of Subsection 4.1, then ϕ is bounded away from zero and infinity, and (C.11) converges to

$$U'\left[f(h)-g\right] = \left(\frac{\varepsilon}{\varepsilon-1}\right)\frac{v'(h)}{f'(h)}.$$

This last equation, which characterizes the limit value of h, is the same as the equation implicitly and uniquely defining the steady-state value $h^\star$ of h_t in the basic NK model (with consumption-utility function U). Therefore, h, y, and c converge respectively to $h^\star$, $y^\star \equiv f(h^\star)$, and $c^\star \equiv f(h^\star) - g$, while $m = \phi[f(h) - g]$ is bounded away from zero and infinity. As a consequence, we have $C \equiv [(1-\gamma)c^\alpha + \gamma m^\alpha]^{1/\alpha} \to c^\star$ and

$$\begin{aligned}
\frac{-u_{cc}y}{u_c} &= \left(\frac{c}{C}\right)^\alpha \frac{y}{c}\left\{(1-\alpha)\gamma\phi^\alpha + (1-\gamma)\left[\frac{-U''(C)\,C}{U'(C)}\right]\right\} &\longrightarrow& \quad \frac{-U''(c^\star)\,y^\star}{U'(c^\star)},\\
\frac{u_{cm}m}{u_c} &= \gamma\phi^\alpha\left(\frac{c}{C}\right)^\alpha\left\{(1-\alpha) - \left[\frac{-U''(C)\,C}{U'(C)}\right]\right\} &\longrightarrow& \quad 0,\\
\frac{-u_{mm}m}{u_m} &= \left(\frac{c}{C}\right)^\alpha\left\{(1-\alpha)(1-\gamma) + \gamma\phi^\alpha\left[\frac{-U''(C)\,C}{U'(C)}\right]\right\} &\longrightarrow& \quad 1-\alpha,\\
\frac{u_{cm}y}{u_m} &= (1-\gamma)\left(\frac{c}{C}\right)^\alpha\frac{y}{c}\left\{(1-\alpha) - \left[\frac{-U''(C)\,C}{U'(C)}\right]\right\} &\longrightarrow& \quad (1-\alpha)\frac{y^\star}{c^\star} - \left[\frac{-U''(c^\star)\,y^\star}{U'(c^\star)}\right].
\end{aligned}$$

Using these limit results, we get that the reduced-form parameters σ, κ, δ_g, and δ_φ converge to their counterparts in the basic NK model, while η, δ_m, χ_i^{-1}, and $\chi_y\chi_i^{-1}$ converge to zero. We conclude that the steady state and reduced form of our MIU model, under the CES-based specification, converge to the steady state and reduced form of the basic NK model.

C.4 Corridor System

Under the corridor system considered in Subsection 6.5, the (log-deviation of the) spread $i_t - i_t^m$ is equal to zero, and the money-demand equation (15) becomes

$$m_t = \chi_y\left(y_t - g_t\right). \tag{C.12}$$

Using (C.12) to replace m_t and m_{t+1} in the IS equation (13), we get

$$y_t = \mathbb{E}_t\{y_{t+1}\} - \frac{1}{\widetilde{\sigma}}(i_t - \mathbb{E}_t\{\pi_{t+1}\} - r_t) + g_t - \mathbb{E}_t\{g_{t+1}\} \tag{C.13}$$

with

$$\widetilde{\sigma} \equiv \left(1 - \frac{\delta_m \chi_y}{\delta_g}\right)\sigma > 0,$$

where the inequality comes from

$$\begin{aligned} 1 - \frac{\delta_m \chi_y}{\delta_g} &= 1 - \left(\frac{u_{cm} m}{u_c}\right)\left(\frac{-u_{cc} y}{u_c}\right)^{-1}\left(\frac{u_{cm} m}{u_c} - \frac{u_{mm} m}{u_m}\right)^{-1}\left(\frac{u_{cm} y}{u_m} - \frac{u_{cc} y}{u_c}\right) \\ &= \left(\frac{-u_{cc} y}{u_c}\right)^{-1}\left(\frac{u_{cm} m}{u_c} - \frac{u_{mm} m}{u_m}\right)^{-1}\left(\frac{my}{u_c u_m}\right)\left[u_{cc} u_{mm} - (u_{cm})^2\right] \\ &> 0. \end{aligned}$$

Using (C.12) to replace m_t in the Phillips curve (14), we get

$$\pi_t = \beta \mathbb{E}_t\{\pi_{t+1}\} + \widetilde{\kappa}\left(y_t - \widetilde{\delta}_g g_t - \widetilde{\delta}_\varphi \varphi_t\right) \tag{C.14}$$

with

$$\begin{aligned} \widetilde{\kappa} &\equiv \frac{(1-\theta)(1-\beta\theta)}{\theta\left[1 - \frac{\varepsilon f f''}{(f')^2}\right]}\widetilde{\psi} > 0, \\ \widetilde{\delta}_g &\equiv \widetilde{\sigma}\widetilde{\psi}^{-1} \in (0,1), \\ \widetilde{\delta}_\varphi &\equiv \left\{\mathbb{1}_{\varphi_t = \varphi_{1,t}} + \left[1 + \frac{v'' h}{v'}\frac{f}{f' h} - \frac{f f''}{(f')^2}\right]\mathbb{1}_{\varphi_t = \varphi_{2,t}} + \mathbb{1}_{\varphi_t = \varphi_{3,t}} + \left(\frac{1}{\varepsilon - 1}\right)\mathbb{1}_{\varphi_t = \varphi_{4,t}}\right\}\widetilde{\psi}^{-1} > 0, \end{aligned}$$

where

$$\widetilde{\psi} \equiv \widetilde{\sigma} + \frac{v'' h}{v'}\frac{f}{f' h} - \frac{f f''}{(f')^2} > 0.$$

The rewritten IS equation (C.13) and Phillips curve (C.14) are isomorphic to the IS equation (1) and Phillips curve (2) of the basic NK model. More specifically, the reduced-form parameters of (C.13) and (C.14) can be derived from those of (1) and (2) simply by replacing $\sigma \equiv -u'' y/u'$ (where u denotes the consumption-utility function in the basic NK model) with $\widetilde{\sigma} \equiv (1 - \delta_m \chi_y/\delta_g)(-u_{cc} y/u_c) < -u_{cc} y/u_c$ (where u denotes the consumption-and-money-utility function in the MIU model). In this sense, moving from the basic NK model to the MIU model under the corridor system, holding constant the coefficient of relative risk aversion ($-u'' c/u' = -u_{cc} c/u_c$) and the steady-state share of government purchases in output (g/y), is equivalent to reducing the coefficient of relative risk aversion and/or the steady-state share of government purchases in output in the basic NK model.

Appendix D: Model With Banks

D.1 Root Analysis

We show that $0 < \rho < 1 < \omega_1 < \omega_2$. The polynomial $\mathcal{P}(X)$ can be rewritten as

$$\begin{aligned}\mathcal{P}(X) &= X^3 - \left(\frac{1+2\beta+\beta\Theta_1+\Theta_2}{\beta}\right)X^2 + \left[\frac{2+\beta+(1+\beta)\Theta_1+\Theta_2+\Theta_3}{\beta}\right]X - \left(\frac{1+\Theta_1}{\beta}\right)\\ &= (X-1-\Theta_1)\left[X^2 - \left(\frac{1+\beta+\Theta_2}{\beta}\right)X + \frac{1}{\beta}\right] - \left(\frac{\Theta_1\Theta_2-\Theta_3}{\beta}\right)X,\end{aligned}$$

where $\Theta_1 \equiv \chi_y/(\sigma\chi_i) > 0$, $\Theta_2 \equiv (1/\sigma - \delta_m)\kappa$, and $\Theta_3 \equiv (1-\delta_m\chi_y)\kappa/(\sigma\chi_i)$. The double inequality (21) implies $\Theta_2 > 0$, $\Theta_3 > 0$, and $\Theta_1\Theta_2 - \Theta_3 = (\chi_y - \sigma)\kappa/(\sigma^2\chi_i) > 0$. Therefore, we get $\mathcal{P}(0) = -(1+\Theta_1)/\beta < 0$, $\mathcal{P}(1) = \Theta_3/\beta > 0$, $\mathcal{P}(1+\Theta_1) = -(\Theta_1\Theta_2-\Theta_3)(1+\Theta_1)/\beta < 0$, and $\lim_{X\to+\infty}\mathcal{P}(X) = +\infty > 0$. As a consequence, the roots of $\mathcal{P}(X)$ are three real numbers ρ, ω_1, and ω_2 such that $0 < \rho < 1 < \omega_1 < 1+\Theta_1 < \omega_2$.

D.2 Resolution of the Paradox of Flexibility

Using the definition of Z_t, and after some simple algebra, we can rewrite (9) and (18) as

$$\begin{aligned}\pi_t =& -(1-\rho)p_{t-1} + \frac{\kappa}{\beta(\omega_2-\omega_1)}\mathbb{E}_t\left\{-\frac{1}{\sigma}\sum_{k=0}^{+\infty}\left(\omega_1^{-k-1}-\omega_2^{-k-1}\right)\left(i_{t+k}^m - r_{t+k}\right)\right.\\ &+\sum_{k=0}^{+\infty}\left(\xi_1^M\omega_1^{-k-1}-\xi_2^M\omega_2^{-k-1}\right)M_{t+k} - \sum_{k=0}^{+\infty}\left(\xi_1^g\omega_1^{-k-1}-\xi_2^g\omega_2^{-k-1}\right)g_{t+k}\\ &\left.+\sum_{k=0}^{+\infty}\left(\xi_1^\varphi\omega_1^{-k-1}-\xi_2^\varphi\omega_2^{-k-1}\right)\varphi_{t+k}\right\},\end{aligned} \tag{D.1}$$

$$\begin{aligned}y_t =& -\vartheta p_{t-1} + g_t + \frac{\mathbb{E}_t}{\beta(\omega_2-\omega_1)}\left\{\frac{1}{\sigma}\sum_{k=0}^{+\infty}\left(\xi_1\omega_1^{-k-1}-\xi_2\omega_2^{-k-1}\right)\left(i_{t+k}^m - r_{t+k}\right)\right.\\ &-\sum_{k=0}^{+\infty}\left(\xi_1\xi_1^M\omega_1^{-k-1}-\xi_2\xi_2^M\omega_2^{-k-1}\right)M_{t+k} + \sum_{k=0}^{+\infty}\left(\xi_1\xi_1^g\omega_1^{-k-1}-\xi_2\xi_2^g\omega_2^{-k-1}\right)g_{t+k}\\ &\left.-\sum_{k=0}^{+\infty}\left(\xi_1\xi_1^\varphi\omega_1^{-k-1}-\xi_2\xi_2^\varphi\omega_2^{-k-1}\right)\varphi_{t+k}\right\},\end{aligned} \tag{D.2}$$

where $\vartheta \equiv (1-\rho)(1-\beta\rho)/\kappa + \delta_m\rho$ and

$$\begin{aligned}\xi_j &\equiv \beta(\omega_j+\rho-1) + \kappa\delta_m - 1,\\ \xi_j^M &\equiv \delta_m(\omega_j - 1) + \frac{1-\delta_m\chi_y}{\sigma\chi_i},\\ \xi_j^g &\equiv (1-\delta_g)(\omega_j-1) + \frac{\delta_g\chi_y - \chi_g}{\sigma\chi_i},\\ \xi_j^\varphi &\equiv \delta_\varphi(\omega_j-1) + \frac{\chi_\varphi - \delta_\varphi\chi_y}{\sigma\chi_i}\end{aligned}$$

for $j \in \{1,2\}$.

The only parameter that depends on the degree of price stickiness θ in the structural equations (1), (19), and (20) is the slope κ of the Phillips curve (19). We have $\lim_{\theta\to 0}\kappa = +\infty$ and hence

$$\left(\frac{-\beta\sigma}{1-\sigma\delta_m}\right)\lim_{\theta\to 0}\left[\frac{\mathcal{P}(X)}{\kappa}\right] = X\left(X-\omega_1^n\right)$$

for any $X \in \mathbb{R}$, where

$$\omega_1^n \equiv 1 + \frac{1-\delta_m\chi_y}{\left(1-\sigma\delta_m\right)\chi_i} > 1,$$

where in turn the inequality follows from (21). Therefore, we get

$$\lim_{\theta\to 0}\rho = 0, \quad \lim_{\theta\to 0}\omega_1 = \omega_1^n, \quad \text{and} \quad \lim_{\theta\to 0}\omega_2 = +\infty. \tag{D.3}$$

Using (D.3) and

$$\left(1-\rho\right)\left(\omega_1-1\right)\left(\omega_2-1\right) = \mathcal{P}(1) = \frac{\left(1-\delta_m\chi_y\right)\kappa}{\beta\sigma\chi_i},$$

we also get that

$$\lim_{\theta\to 0}\frac{\kappa}{\omega_2} = \frac{\beta\sigma}{1-\sigma\delta_m}. \tag{D.4}$$

Using (D.3) and (D.4), we can easily determine the limits of (D.1) and (D.2) as $\theta \to 0$:

$$\begin{aligned}\lim_{\theta\to 0}\pi_t \;=\;& -p_{t-1} + \frac{1}{1-\sigma\delta_m}\mathbb{E}_t\left\{\sum_{k=0}^{+\infty}\left(\omega_1^n\right)^{-k-1}\left\{-\left(i_{t+k}^m - r_{t+k}\right) + \left(\omega_1^n-1\right)M_{t+k}\right.\right.\\ & -\left[\sigma\left(1-\delta_g\right)\left(\omega_1^n-1\right) + \frac{\delta_g\chi_y-\chi_g}{\chi_i}\right]g_{t+k} + \left.\left.\left[\sigma\delta_\varphi\left(\omega_1^n-1\right) + \frac{\chi_\varphi-\delta_\varphi\chi_y}{\chi_i}\right]\varphi_{t+k}\right\}\right\}\\ & + \frac{\sigma}{1-\sigma\delta_m}\left[-\delta_m M_t + \left(1-\delta_g\right)g_t - \delta_\varphi\varphi_t\right],\end{aligned} \tag{D.5}$$

$$\begin{aligned}\lim_{\theta\to 0}y_t \;=\;& \frac{\delta_m}{1-\sigma\delta_m}\mathbb{E}_t\left\{\sum_{k=0}^{+\infty}\left(\omega_1^n\right)^{-k-1}\left\{i_{t+k}^m - r_{t+k} - \left(\omega_1^n-1\right)M_{t+k}\right.\right.\\ & +\left[\sigma\left(1-\delta_g\right)\left(\omega_1^n-1\right) + \frac{\delta_g\chi_y-\chi_g}{\chi_i}\right]g_{t+k} - \left.\left.\left[\sigma\delta_\varphi\left(\omega_1^n-1\right) + \frac{\chi_\varphi-\delta_\varphi\chi_y}{\chi_i}\right]\varphi_{t+k}\right\}\right\}\\ & + \frac{1}{1-\sigma\delta_m}\left[\delta_m M_t + \left(\delta_g-\sigma\delta_m\right)g_t + \delta_\varphi\varphi_t\right].\end{aligned} \tag{D.6}$$

These limits are finite, unlike their counterparts in the basic NK model.

We now show that the right-hand sides of (D.5) and (D.6) coincide with the values taken by π_t and y_t when prices are perfectly flexible ($\theta = 0$). The flexible-price value of y_t is straightforwardly obtained by setting to zero the last term in the Phillips curve (19), which is proportional to (the log-deviation of) firms' marginal cost of production:

$$y_t = \delta_m m_t + \delta_g g_t + \delta_\varphi\varphi_t. \tag{D.7}$$

Using the IS equation (1), the money-demand equation (20), the identity $m_t = M_t - p_t$, and the solution for flexible-price output (D.7), we get the following dynamic equation under flexible prices:

$$p_t = (\omega_1^n)^{-1} \mathbb{E}_t \{p_{t+1}\} + \frac{(\omega_1^n)^{-1}}{1-\sigma\delta_m} \left\{ -(i_t^m - r_t) + \left(\frac{1-\delta_m \chi_y}{\chi_i} - \sigma\delta_m \right) M_t + \sigma\delta_m \mathbb{E}_t \{M_{t+1}\} \right.$$
$$+ \left[\frac{\chi_g - \delta_g \chi_y}{\chi_i} + \sigma (1-\delta_g) \right] g_t - \sigma (1-\delta_g) \mathbb{E}_t \{g_{t+1}\}$$
$$\left. + \left(\frac{\chi_\varphi - \delta_\varphi \chi_y}{\chi_i} - \sigma\delta_\varphi \right) \varphi_t + \sigma\delta_\varphi \mathbb{E}_t \{\varphi_{t+1}\} \right\}.$$

Iterating this equation forward to $+\infty$ leads to the following value for the price level p_t in our model with banks under flexible prices:

$$p_t = \frac{1}{1-\sigma\delta_m} \mathbb{E}_t \left\{ \sum_{k=0}^{+\infty} (\omega_1^n)^{-k-1} \left\{ -\left(i_{t+k}^m - r_{t+k}\right) + (\omega_1^n - 1) M_{t+k} \right. \right.$$
$$\left. \left. - \left[\sigma (1-\delta_g)(\omega_1^n - 1) + \frac{\delta_g \chi_y - \chi_g}{\chi_i} \right] g_{t+k} + \left[\sigma\delta_\varphi (\omega_1^n - 1) + \frac{\chi_\varphi - \delta_\varphi \chi_y}{\chi_i} \right] \varphi_{t+k} \right\} \right\}$$
$$+ \frac{\sigma}{1-\sigma\delta_m} \left[-\delta_m M_t + (1-\delta_g) g_t - \delta_\varphi \varphi_t \right], \tag{D.8}$$

which implies in turn that the value of $\pi_t \equiv p_t - p_{t-1}$ in our model with banks under flexible prices coincides with the right-hand side of (D.5). In turn, using (D.7), (D.8), and the identity $m_t = M_t - p_t$, we get that the value of y_t in our model with banks under flexible prices coincides with the right-hand side of (D.6). Thus, our model with banks solves the paradox of flexibility: the limits of π_t and y_t as $\theta \to 0$ are finite and coincide with the values of π_t and y_t when $\theta = 0$.

D.3 Convergence to the Basic NK Model

In a previous version of this paper (Diba and Loisel, 2019), we show that the steady state and reduced form of our model with banks converge to the steady state and reduced form of the basic NK model, with the steady-state stock of real reserves m bounded away from zero and infinity, as the scale parameter of banking costs γ and the steady-state interest-rate spread $\beta^{-1} - I^m$ are shrunk to zero at the same speed (as in the thought experiment of Subsection 4.2). It is straightforward to check that we still get this steady-state and reduced-form convergence, this time with m going to infinity (asymptotic satiation), as we hold γ constant and shrink only $\beta^{-1} - I^m$ to zero (as in the policy experiment of Subsection 5.1), provided that two conditions are met. The first condition is that the marginal banking cost should go to zero as the stock of real reserves goes to infinity ($\lim_{m_t \to +\infty} \Gamma_\ell(\ell_t, m_t) = 0$, where ℓ_t denotes real loans and Γ the banking-cost function). The second condition is that banking-cost elasticities ($\Gamma_{\ell\ell}\ell/\Gamma_\ell$, $\Gamma_{mm}m/\Gamma_m$, $\Gamma_{\ell m}\ell/\Gamma_m$, and $\Gamma_{\ell m}m/\Gamma_\ell$) should be bounded from above. These two conditions are met, in particular, when the banking-cost function Γ is isoelastic, which happens when the loan-production and banker-labor-disutility functions are themselves isoelastic.

Appendix E: Discounting Models

E.1 Non-Resolution of the Paradox of Flexibility

In this appendix, we show that discounting models do not solve the paradox of flexibility. More specifically, we show that if $i_t = 0$ for $1 \leq t \leq T-1$, $i_T = i^* \neq 0$, and $y_{T+1} = \pi_{T+1} = 0$, then $\lim_{\theta \to 0} |\pi_1| = \lim_{\theta \to 0} |y_1| = +\infty$.

We start with the case in which $\xi_3(\theta) > 0$ or $\xi_4(\theta) > 0$. In this case, the system made of the IS equation (22) and the Phillips curve (23) can be rewritten as

$$\mathbb{E}_t \left\{ \begin{bmatrix} y_{t+1} \\ \pi_{t+1} \end{bmatrix} \right\} = \mathbf{P} \begin{bmatrix} y_t \\ \pi_t \end{bmatrix} + \mathbf{Z} i_t \tag{E.1}$$

with

$$\mathbf{P} \equiv \frac{1}{\varphi(\theta)} \begin{bmatrix} \kappa(\theta)\xi_2 + \beta\sigma\xi_3(\theta) & -\xi_2 \\ \kappa(\theta)\sigma[\xi_4(\theta) - \xi_1] & \sigma\xi_1 \end{bmatrix} \quad \text{and} \quad \mathbf{Z} \equiv \frac{\xi_2}{\varphi(\theta)} \begin{bmatrix} \beta\xi_3(\theta) \\ \kappa(\theta)\xi_4(\theta) \end{bmatrix},$$

where $\varphi(\theta) \equiv \beta\sigma\xi_1\xi_3(\theta) + \kappa(\theta)\xi_2\xi_4(\theta) > 0$. The characteristic polynomial of $\mathbf{P}$ is

$$\mathcal{P}(X) \equiv X^2 - \frac{\sigma\xi_1 + \kappa(\theta)\xi_2 + \beta\sigma\xi_3(\theta)}{\varphi(\theta)} X + \frac{\sigma}{\varphi(\theta)}.$$

Since $\mathcal{P}(0) \neq 0$, $\mathbf{P}$ is invertible. Iterating the dynamic equation (E.1) forward to date T, and using the terminal condition $y_{T+1} = \pi_{T+1} = 0$ and the invertibility of $\mathbf{P}$, we get

$$\begin{bmatrix} y_1 \\ \pi_1 \end{bmatrix} = -\mathbf{P}^{-T}\mathbf{Z} i^*.$$

For any $X \in \mathbb{R}$, we have

$$\frac{1}{\xi_2} \lim_{\theta \to 0} \left[\frac{\varphi(\theta)\mathcal{P}(X)}{\kappa(\theta)} \right] = \left[\lim_{\theta \to 0} \xi_4(\theta) \right] X^2 - X.$$

One root of the polynomial on the right-hand side of this equation is zero. Therefore, one root of $\mathcal{P}(X)$ converges towards zero as $\theta \to 0$, which implies in turn that $\lim_{\theta \to 0} ||\mathbf{P}^{-1}|| = +\infty$. Using the fact that $||\mathbf{Z}||$ is bounded away from zero as $\theta \to 0$, we conclude that $\lim_{\theta \to 0} |y_1| = \lim_{\theta \to 0} |\pi_1| = +\infty$.

In the alternative case in which $\xi_3(\theta) = \xi_4(\theta) = 0$, the system made of the IS equation (22) and the Phillips curve (23) implies the following dynamic equation in inflation:

$$\left[\xi_1 + \frac{\kappa(\theta)\xi_2}{\sigma} \right] \mathbb{E}_t \{\pi_{t+1}\} = \pi_t + \frac{\kappa(\theta)\xi_2}{\sigma} i_t. \tag{E.2}$$

Iterating this dynamic equation forward to date T, and using the terminal condition $\pi_{T+1} = 0$, we get

$$\pi_1 = -\left[\xi_1 + \frac{\kappa(\theta)\xi_2}{\sigma} \right]^{T-1} \frac{\kappa(\theta)\xi_2 i^*}{\sigma},$$

so that $\lim_{\theta \to 0} |\pi_1| = +\infty$. Using the Phillips curve (23) with $\xi_3(\theta) = \xi_4(\theta) = 0$, we then get $\lim_{\theta \to 0} |y_1| = +\infty$.

E.2 Discrete Departure From the Basic NK Model

In this appendix, we show that discounting models generate indeterminacy under a permanently exogenous policy rate when they are sufficiently close to the basic NK model, i.e. when $(\xi_1, \xi_2, \xi_3(\theta), \xi_4(\theta))$ is sufficiently close to $(1,1,1,0)$. We focus on the case in which $\xi_3(\theta)$ is sufficiently close to 1 for $\xi_3(\theta) > 0$. In this case, the system made of the IS equation (22) and the Phillips curve (23) can be rewritten as (E.1), and its characteristic polynomial under a permanently exogenous policy rate is $\mathcal{P}(X)$ (as defined in Appendix E.1). It is straightforward to check that as $(\xi_1, \xi_2, \xi_3(\theta), \xi_4(\theta))$ goes to $(1,1,1,0)$, $\mathcal{P}(X)$ converges to $\mathcal{P}_b(X)$ for any $X \in \mathbb{R}$, and therefore the two roots of $\mathcal{P}(X)$ converge to the two roots $\rho_b \in (0,1)$ and $\omega_b > 1$ of $\mathcal{P}_b(X)$. For $(\xi_1, \xi_2, \xi_3(\theta), \xi_4(\theta))$ sufficiently close to $(1,1,1,0)$, thus, only one root of $\mathcal{P}(X)$ lies outside the unit circle. With only one eigenvalue outside the unit circle for two non-predetermined variables ($\mathbb{E}_t\{y_{t+1}\}$ and $\mathbb{E}_t\{\pi_{t+1}\}$), discounting models then generate indeterminacy.

E.3 No Fisher Effect

In this appendix, we show that in discounting models, if setting exogenously the policy rate delivers local-equilibrium determinacy, then a permanent increase in the policy rate leads to a permanent decrease in the inflation rate.

We start with the case in which $\xi_3(\theta) > 0$ or $\xi_4(\theta) > 0$. In this case, under a permanent peg $i_t = i^*$, the system made of the IS equation (22) and the Phillips curve (23) can be rewritten as (E.1) with $i_t = i^*$. If the peg ensures local-equilibrium determinacy, then $\mathcal{P}(X)$, the characteristic polynomial of $\mathbf{P}$ (derived in Appendix E.1), must have no root inside the unit circle, because the system has no predetermined variable. In particular, $\mathcal{P}(X)$ must have no root inside the real-number interval $[0,1]$, which requires that $\mathcal{P}(0)\,\mathcal{P}(1) > 0$, i.e. equivalently

$$\sigma(1-\xi_1)\left[1-\beta\xi_3(\theta)\right] - \kappa(\theta)\,\xi_2\left[1-\xi_4(\theta)\right] > 0. \tag{E.3}$$

In the unique local equilibrium, the (constant) inflation rate is easily obtained as

$$\pi_t = \pi^* \equiv \frac{-\kappa(\theta)\,\xi_2\left[1-\xi_4(\theta)\right] i^*}{\sigma(1-\xi_1)\left[1-\beta\xi_3(\theta)\right] - \kappa(\theta)\,\xi_2\left[1-\xi_4(\theta)\right]}.$$

Given (E.3), π^* is negatively related to i^*.

In the alternative case in which $\xi_3(\theta) = \xi_4(\theta) = 0$, under a permanent peg $i_t = i^*$, the system made of the IS equation (22) and the Phillips curve (23) implies the dynamic equation (E.2) with $i_t = i^*$. Therefore, for the peg to ensure determinacy, we need

$$\sigma(1-\xi_1) - \kappa(\theta)\,\xi_2 > 0. \tag{E.4}$$

In the unique local equilibrium, the (constant) inflation rate is easily obtained as

$$\pi_t = \pi^* \equiv \frac{-\kappa(\theta)\,\xi_2 i^*}{\sigma(1-\xi_1) - \kappa(\theta)\,\xi_2}.$$

Given (E.4), π^* is negatively related to i^*.

Appendix F: Simple Model (Numerical Illustrations)

F.1 Additional Numerical Illustrations under our Benchmark Calibration

In Section 3 of the main text, Figure 1 illustrates numerically the effects of forward guidance (i.e. future policy-rate cuts) on inflation in our simple model, and compares these effects to the implications of the standard NK equilibrium. In this appendix, we illustrate and discuss the effects of forward guidance on output, and the effects of anticipated changes in fiscal policy on inflation and output − both in the equilibrium of our simple model and in the standard equilibrium of the basic NK model. We continue to use our benchmark calibration taken from Galí (2008, Chapter 3), which sets $\theta = 2/3$ (corresponding to "3-quarter price rigidity"); we also report the effects of cutting θ in half, step by step, to make prices more flexible.

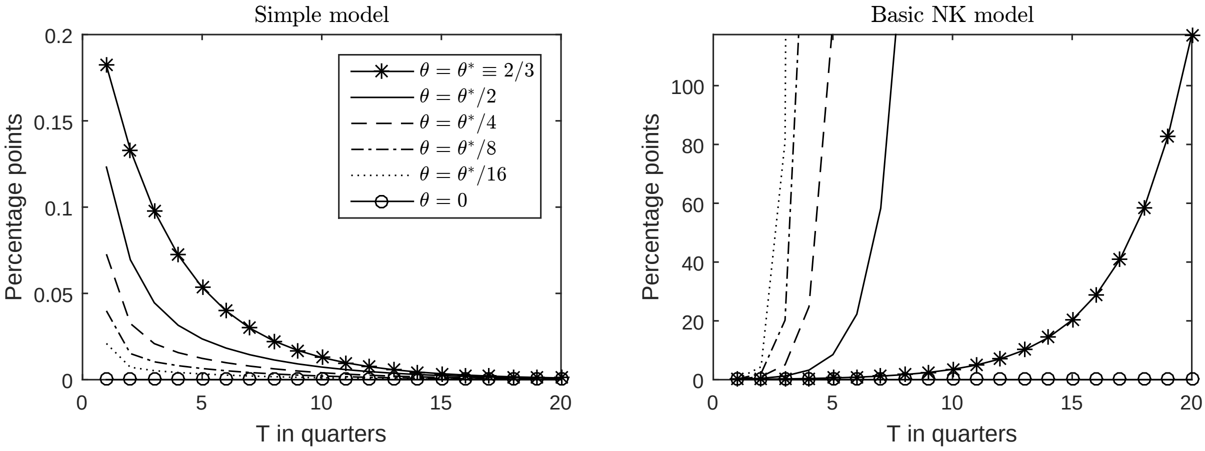
Figure F.1 shows the effects of forward guidance on output in the two models. As before, our policy experiment is to cut the policy rate by 25 basis points (one percentage point per annum) in Quarter T, and we display the effects on output in Quarter 1 (when the rate cut is announced). The right panel in Figure F.1 replicates the implausible implications of the basic NK model. The left panel shows that our model does not share these implications. The rate cut has small effects on output (less than 0.2 percent of steady-state output to begin with), and the effects die off quickly as we delay the rate cut. Moreover, these effects decline smoothly as we make prices more flexible; they converge to the flexible-price ($\theta = 0$) effects.

To analyze the effects of fiscal policy, we add government purchases to Galí's calibration. We set the share of government purchases in output to 0.3 in the steady state. We follow Galí's calibration for the structural parameters (like the intertemporal elasticity of substitution) and adjust the reduced-form parameters (like the coefficient $1/\sigma$ on the real interest rate in the IS equation) to reflect the introduction of government purchases. Our policy experiment is an increase in government purchases, amounting to one percent of steady-state output, occurring (only) in Quarter T and announced in Quarter 1. Figures F.2 and F.3 display the effects on inflation and output in Quarter 1. Once again, the comparison between the left and the right panels shows that our model's equilibrium does not share the puzzling implications of the basic NK model's standard equilibrium: the effects of anticipated fiscal policy die out as we delay the policy intervention, and they converge to the flexible-price values as we make prices more and more flexible.

Another notable difference between our model's equilibrium and the basic NK model's standard equilibrium under our benchmark calibration is that anticipated fiscal expansions have a contractionary effect on output in our model.[21] Several contributions (e.g., Christiano et al., 2011) suggest that anticipated fiscal expansions can have large positive output multipliers at the ZLB

[21] Our analytical derivations in the main text show that this is always the case when we use our model to go to the basic-NK-model limit. Figure F.3 makes the point numerically under our benchmark calibration, without taking the model to the basic-NK-model limit.

Figure F.1 – Effect of a policy-rate cut at date T on output at date 1

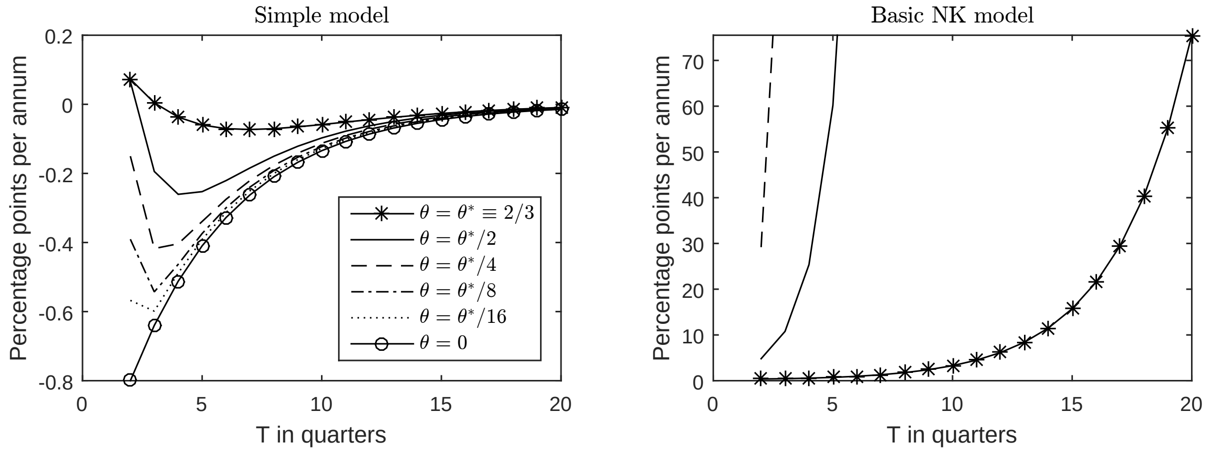


Note: The figure displays the effect on y_1 of announcing at date 1 a one-percentage-point-per-annum cut in i_T^m (for the simple model) or i_T (for the basic NK model), as a function of $T \in \{1, ..., 20\}$. Parameter values are the same as for Figure 1 in the main text. More specifically, benchmark parameter values are set as in Galí (2008, Chapter 3): $\beta = 0.99$, $\sigma = 1$, $\chi_y = 1$, $\chi_i = 4$, and $\kappa = \lambda[(1-\theta)(1-\beta\theta)/\theta] = 0.13$, where $\lambda = 3/4$ and $\theta = \theta^* \equiv 2/3$. As θ takes the values $\theta^*/2$, $\theta^*/4$, $\theta^*/8$, and $\theta^*/16$, κ takes respectively the values 1.00, 3.13, 7.57, and 16.54.

according to the basic NK model. The right-hand panel of Figure F.3 confirms this implication of the basic NK model. This implication arises from a feedback loop first described in Farhi and Werning (2016). As we explain in Subsection 3.2 of the main text, this feedback loop works back in time via the IS equation and the Phillips curve: given that $\pi_{T+1} = y_{T+1} = 0$, a fiscal expansion at date T raises inflation at date T, which lowers the real interest rate at date $T-1$, which raises output and inflation at date $T-1$, and so on. This feedback loop is also present in our model, but π_{T+1} and y_{T+1} are endogenously determined when the fiscal expansion is announced. As a result, expected future fiscal expansions can reduce current output in our model (as is the case under the calibration we use for Figure F.3). Intuitively, these contractionary effects of anticipated fiscal expansions may come from wealth effects that also arise in standard Real-Business-Cycle models: consumers realize that the future fiscal expansion reduces their permanent income, and they respond by lowering current consumption.

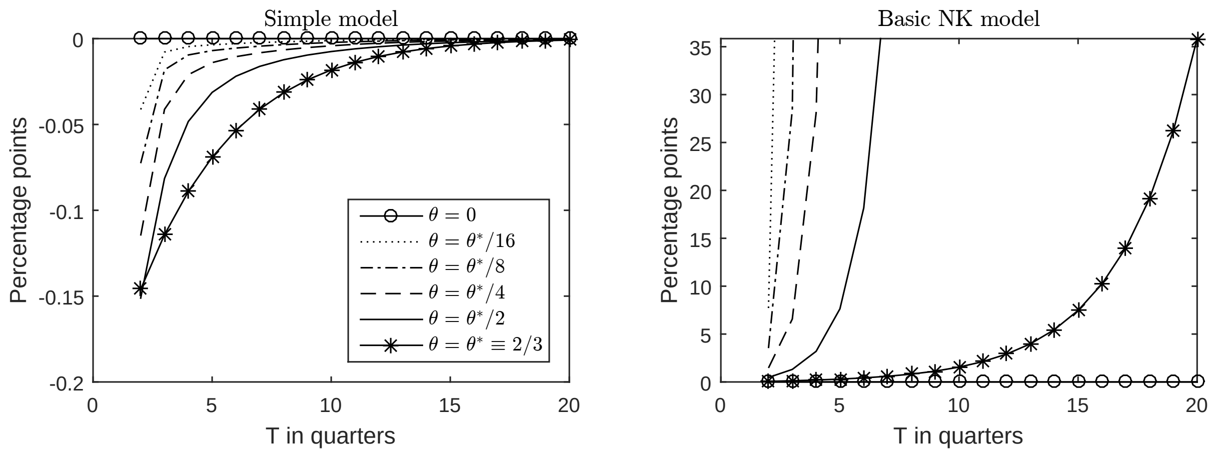
The effects of anticipated fiscal expansions on inflation may be dominated either by the wealth effect we mention above (which is deflationary) or by an inflationary effect that we can trace back to staggered price setting. The latter effect arises because the fiscal expansion is expected to raise prices in the future, and this motivates current price setters to set higher prices too. Under our benchmark calibration, the effects of anticipated fiscal expansions on inflation (displayed in the left panel of Figure F.2, for various quarters T and price-stickiness degrees θ) are small, and mostly negative.

Figure F.2 – Effect of government purchases at date T on inflation at date 1



Note: The figure displays the effect on π_1 of announcing at date 1 a one-percent-of-steady-state-output increase in g_T, as a function of $T \in \{2, ..., 20\}$. The steady-state share of government purchases in output is set to 0.3, and benchmark structural-parameter values are set as in Galí (2008, Chapter 3), implying $\beta = 0.99$, $\sigma = 1.43$, $\delta_g = 0.42$, $\chi_y = 1$, $\chi_i = 4$, and $\kappa = \lambda[(1-\theta)(1-\beta\theta)/\theta] = 0.15$, where $\lambda = 0.86$ and $\theta = \theta^* \equiv 2/3$. As θ takes the values $\theta^*/2$, $\theta^*/4$, $\theta^*/8$, and $\theta^*/16$, κ takes respectively the values 1.15, 3.58, 8.65, and 18.90.

Figure F.3 – Effect of government purchases at date T on output at date 1



Note: The figure displays the effect on y_1 of announcing at date 1 a one-percent-of-steady-state-output increase in g_T, as a function of $T \in \{2, ..., 20\}$. Parameter values are the same as for Figure F.2 above.

F.2 Numerical Sensitivity Analysis

The quantitative impressions conveyed by Figure 1 in the main text and Figures F.1-F.3 in the previous appendix are not particularly sensitive to Galí's (2008, Chapter 3) choices about the parameters of the basic NK model, nor to his (standard) assumption of a unitary income elasticity of money demand. The value taken by the interest semi-elasticity of money demand χ_i, however, does matter for the quantitative impression conveyed by our results about the effects of forward guidance. The value of χ_i affects both the magnitude and the persistence of the effects of future changes in the IOR rate on current inflation and output. Our choice of $\chi_i = 4$, following Galí, represents a middle-of-the-range value compared to estimates that we could take from the empirical literature on money demand.

Semi-log specifications of money demand typically yield small estimates of χ_i based on US data. The estimates in Stock and Watson (1993) and Cochrane (2018), for example, suggest semi-elasticities close to -0.1 on an annual basis.[22] Given the quarterly frequency of our model, these estimates correspond to $\chi_i = 0.4$ (one order of magnitude smaller than the value we use for Figure 1). By contrast, log-log specifications of money demand, estimated on US or cross-country data, suggest interest elasticities around $-1/4$ (e.g., Teles and Zhou, 2005) or $-1/3$ (e.g., Teles et al., 2016). If we set the opportunity cost of holding money to one percent per quarter, an elasticity of $-1/3$ implies $\chi_i = 33$ (one order of magnitude larger than the value we use for Figure 1). Figure F.4 shows how the quantitative effects of forward guidance on inflation vary when we set χ_i to 0.4 or 33.

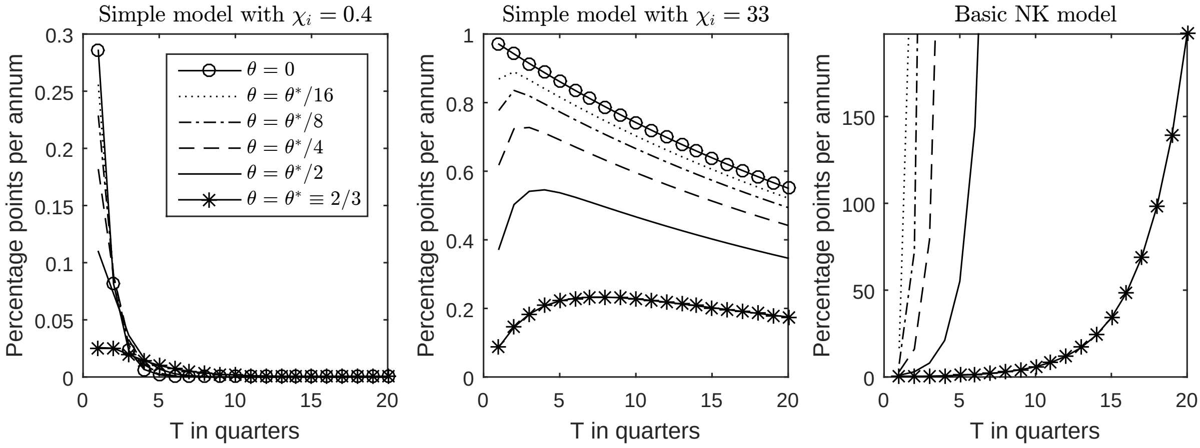
The policy experiment and the parameter values (other than the value of χ_i) used for Figure F.4 are the same as earlier for Figures 1 and F.1. The right panel in Figure F.4 replicates the implausible implications of the basic NK model. The left panel shows the results for our simple model with $\chi_i = 0.4$, and the middle panel shows the results with $\chi_i = 33$. The left panel suggests that the inflationary effects of anticipated IOR-rate cuts are tiny (below 3 basis points to begin with, and dying off quickly). The middle panel suggests that forward guidance has a sizable and more persistent effect on inflation (announcing that the IOR rate will be cut by one percentage point in 20 quarters raises current inflation by 17 basis points).

Beyond this quantitative difference, however, both the left and middle panels of Figure F.4 also illustrate the analytical results that are the main focus of our paper: the inflationary effects go to zero as we cut the IOR rate in the more distant future ($T \to +\infty$), and they converge to the flexible-price effects as we make prices more flexible ($\theta \to 0$). Whatever the calibration, our simple model exhibits neither the forward-guidance puzzle nor the paradox of flexibility.

Figure F.5 shows the effects of forward guidance on output (under the same policy experiment and parameter values we describe above). Again, the quantitative impressions we get are sen-

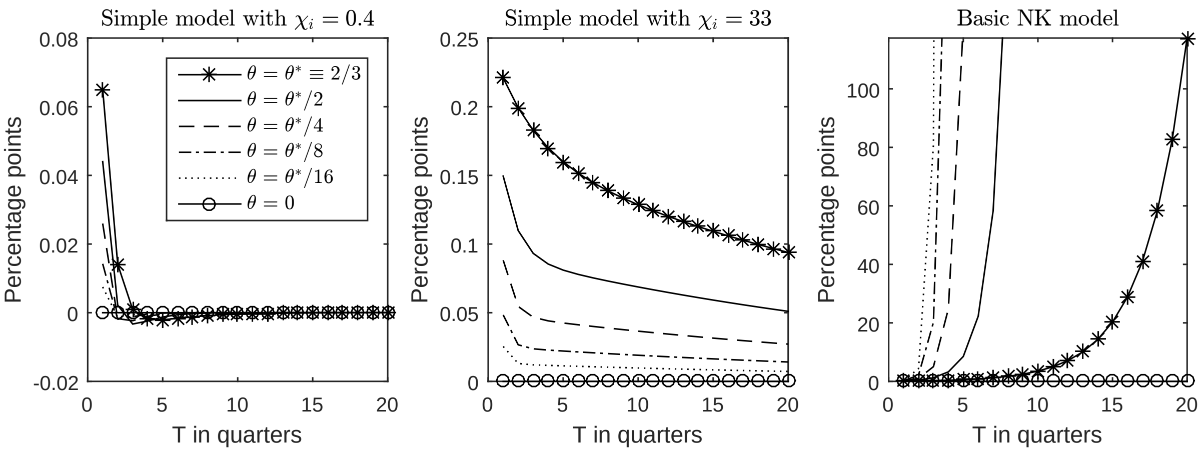
[22] Ball's (2001) estimate of -0.05 is even closer to zero.

Figure F.4 – Effect of a policy-rate cut at date T on inflation at date 1 for alternative values of χ_i



Note: The figure displays the effect on π_1 of announcing at date 1 a one-percentage-point-per-annum cut in i_T^m (for the simple model) or i_T (for the basic NK model), as a function of $T \in \{1, ..., 20\}$. Parameter values (except the value of χ_i) are the same as for Figure 1 in the main text and Figure F.1 above.

Figure F.5 – Effect of a policy-rate cut at date T on output at date 1 for alternative values of χ_i



Note: The figure displays the effect on y_1 of announcing at date 1 a one-percentage-point-per-annum cut in i_T^m (for the simple model) or i_T (for the basic NK model), as a function of $T \in \{1, ..., 20\}$. Parameter values (except the value of χ_i) are the same as for Figure 1 in the main text and Figures F.1 and F.4 above.

sitive to the value of the semi-elasticity χ_i. The effects are tiny if we set $\chi_i = 0.4$, but more noteworthy and persistent if we set $\chi_i = 33$.

Of course, our simple model is not really suitable for a quantitative assessment of the effects that one may associate with forward-guidance policies. Nonetheless, we suspect that the sensitivity of quantitative results to the specification of money demand may also be present in richer (larger-scale) models. So, we suspect that the unsettled state of empirical research on money demand may hinder sharp answers to interesting policy questions in this context.

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