Pegging the Interest Rate on Bank Reserves:  
A Resolution of New Keynesian Puzzles and Paradoxes

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Abstract

We develop a model of monetary policy with a simple departure from the basic New Keynesian (NK) model. In this model, the central bank sets independently the interest rate on bank reserves and the nominal stock of bank reserves. As long as demand for real reserves is not fully satiated, the model delivers local-equilibrium determinacy under permanently exogenous monetary-policy instruments. As a result, it does not share the puzzling and paradoxical implications of the basic NK model under a temporary interest-rate peg (e.g., in the context of a liquidity trap). More specifically, it offers a resolution of the “forward-guidance puzzle,” a related puzzle about fiscal multipliers, and the “paradox of flexibility,” even for an arbitrarily small departure from the basic NK model. It still solves or attenuates these puzzles and that paradox for a vanishingly small departure, and also solves the “paradox of toil” in that case. We argue that our non-satiation assumption is reasonable for analyzing the role of monetary policy during the Great Recession.
1 Introduction

The Great Recession led central banks to peg their policy rates near zero and provide forward guidance about their future policy rates. The recession also rekindled interest in the use of discretionary fiscal policy as a stabilization tool, and, in Europe, sparked debate about implementing structural reforms. The mainstream New Keynesian (NK) literature is of limited help for economists and policymakers in this context, because standard NK models have puzzling and paradoxical implications about the consequences of forward guidance, fiscal policy, and structural reforms under a temporary interest-rate peg — e.g., in the context of a liquidity trap during which the interest rate is pegged to its zero lower bound (ZLB).

In this paper, we show that a simple and possibly minimal departure from the basic NK model presented in Woodford (2003) and Galí (2008) offers a resolution of these puzzles and paradoxes. This departure introduces banks and bank reserves into the model, in order to capture more precisely what central banks actually did: pegging more specifically the interest rate on bank reserves (the IOR rate), and switching to balance-sheet policies that set the nominal stock of bank reserves. Our resolution of the puzzles and paradoxes rests on the assumption that demand for real reserves was not fully satiated — an assumption that we argue is reasonable for understanding the role of monetary policy during the Great Recession.

In standard NK models, the effects of a marginal increase in the duration of an interest-rate peg on current inflation and output become unboundedly large as the duration of the peg goes to infinity (the so-called “forward-guidance puzzle”) or as prices become perfectly flexible (the so-called “paradox of flexibility”). Moreover, the current effects of a one-off fiscal expansion at the end of the peg also grow explosively as the duration of the peg goes to infinity (what we henceforth call the “fiscal-multiplier puzzle”). The forward-guidance and fiscal-multiplier puzzles make policy interventions in the vanishingly distant future have unboundedly large effects, instead of vanishingly small effects, on current outcomes; while the paradox of flexibility makes the effects of policy interventions grow explosively as prices become more and more flexible, instead of converging towards their finite flexible-price effects. In addition to these three “limit puzzles,” the basic NK model has another perplexing implication known as the “paradox of toil:” positive supply shocks — such as downward shifts in the labor-disutility function, labor-tax cuts, technology improvements, and reductions in market power — are not expansionary, but contractionary, under a temporary interest-rate peg.

The literature — cited below — has linked some of these NK puzzles and paradoxes to local-equilibrium indeterminacy (i.e., indeterminacy of equilibrium in the neighborhood of the steady state) under permanently exogenous policy rates. The basic NK model has a stable eigenvalue under exogenous policy rates, but no predetermined variable. Under permanently exogenous policy rates (e.g., under a permanent peg), this loose stable eigenvalue leads to indeterminacy. Under temporarily exogenous policy rates (e.g., under a temporary peg), determinacy is ensured
by future policy normalization; but as the model is iterated backward in time, the loose stable eigenvalue is inverted and magnifies the effects of future conditions (close to or at the end of the peg) on initial outcomes (at the start of the peg). These effects grow explosively as the duration of the peg goes to infinity, thus giving rise to the forward-guidance and fiscal-multiplier puzzles. Moreover, the indeterminacy property of the basic NK model is also behind the paradox of flexibility, as we highlight in the text: the stable eigenvalue converges to zero as prices become more and more flexible; this leads to explosive initial outcomes even for a peg of given (short) duration.

Indeterminacy under permanently exogenous policy rates, in turn, is a common feature of models that follow Sargent and Wallace (1975) in assuming that the central bank sets the nominal interest rate on a bond that serves only as a store of value (i.e., has no non-pecuniary “convenience yield”). This is the interest rate that appears in the IS equation of standard NK models. Once the central bank pegs this interest rate, it commits to buy or sell the bond at the implied price. This makes the money supply endogenous, and it makes any arbitrary price level (with the associated nominal money stock) consistent with an equilibrium.

During the Great Recession, however, central banks did not peg the interest rate on a (hypothetical) bond with no convenience yield; the lower bound on nominal interest rates forced them to peg the IOR rate, which is the interest rate that they directly control (their policy rate). Central banks also announced and implemented balance-sheet policies that set the nominal stock of bank reserves. In our model, setting exogenously these two monetary-policy instruments, the IOR rate and the nominal stock of bank reserves, delivers determinacy because bank reserves have a convenience yield (as they serve to reduce banking costs). In essence, in the steady-state equilibrium of our model, setting the IOR rate determines the demand for real reserves; and, given the outstanding nominal stock of reserves, this pins down the price level. In the dynamic model, under sticky prices à la Calvo (1983), setting exogenously the IOR rate and the stock of reserves amounts to following a “shadow Wicksellian rule” for the interest rate on a bond that serves only as a store of value: it is as if the central bank directly controlled this interest rate and set it as an increasing function of the price level and output. This is because an increase in the price level or output would raise demand for nominal reserves; and the interest rate on bonds would have to rise to restore equilibrium, given the prevailing nominal reserves and IOR rate set by policy. Wicksellian rules are well known to ensure determinacy in the basic NK model (as shown in Woodford, 2003, Chapter 4). We show that this specific shadow Wicksellian rule, given its implied coefficients, also ensures determinacy in our NK model with banks, for all functional forms of the utility and production functions and all values of the structural and (steady-state) policy parameters.

Since it delivers determinacy under exogenous monetary-policy instruments, our model solves the three limit puzzles: policy interventions in the vanishingly distant future have vanishingly small effects, instead of unboundedly large effects, on current outcomes; and, as prices become
more and more flexible, the effects of policy interventions converge to their finite flexible-price effects, instead of growing explosively. In particular, our model solves the three limit puzzles even with an arbitrarily small departure from the basic NK model, i.e. with arbitrarily small banking costs and convenience yield of bank reserves. In the limit, as we make the departure from the basic NK model vanishingly small, our model serves to select uniquely a particular equilibrium of the basic NK model under permanently exogenous policy rates. We show that our selected equilibrium exhibits neither the fiscal-multiplier puzzle, nor the paradox of flexibility, nor the paradox of toil; and that it exhibits an attenuated form of forward-guidance puzzle.

Our main points can be made using any model that attributes some non-pecuniary convenience yield to bank reserves or, more generally, to real money balances. This can range from realistic observations about how reserves help banks with liquidity management or in compliance with prudential-policy regulations to more abstract modeling devices like putting real money balances into consumers’ utility functions. For simplicity, we will first solve the NK puzzles and paradoxes in a generic setting adding a money-demand nexus to the basic NK model. As we explain in the text, however, this setting has some limitations for making our points, as do standard models with a cash-in-advance constraint or money in the utility function. For this reason, we will then move to our model with banks, which does not share these limitations because of its more concrete structure – in particular, because holding reserves reduces the banking costs associated with working-capital loans in this model.

Our resolution of NK puzzles and paradoxes assumes that demand for bank reserves is not fully satiated; this is in contrast to views often expressed about the U.S. economy in recent years (e.g., Cochrane, 2014). We will argue that a model assuming a positive (albeit, arbitrarily small) marginal convenience yield of reserves provides a useful framework for thinking about monetary policy during the Great Recession. It seems hard to discriminate, based on empirical evidence, between the view that the marginal convenience yield of reserves is exactly zero and our preferred view that it may be small and fairly flat, but still positive and inversely related to the amount of reserves. We will address two claims to the contrary in the text.

First, some observers may make a case for satiation noting that the federal-funds rate and T-bill returns have remained below the IOR rate since the early stages of the Great Recession. We don’t think this contradicts our claim that reserves still have a positive marginal convenience yield. Most of the trading activity in the federal-funds market in recent years has involved banks borrowing funds from entities that don’t have direct access to the IOR rate (particularly, from Federal Home Loan Banks, at the time of this writing). Given the presence of such eager lenders, the federal-funds rate may need to fall below the IOR rate to incentivize the borrowers (banks with direct access to the IOR rate). But the spread cannot be more than a few basis points because banks compete for funds, and the only costs of this transaction are small balance-sheet costs (as we will elaborate in the text). As to T-bill returns, the low rates may reflect strong demand by non-bank entities – holding T-bills to use as collateral, or in response to regulatory
constraints. We will sketch a model in the text (with formal derivations in the online appendix) to make our point.

The second argument making a case for satiation of demand for reserves is the fact that large increases in reserve balances after QE1 have had no apparent inflationary consequences — as Reis (2016) and Cochrane (2017b) point out. Our counter-argument is to illustrate how large increases in the money supply (say, doubling the stock of reserves) can have very small inflationary effects (under twenty basis points) in our model if: (1) the demand for reserves is “close to satiation” in a sense we will articulate, and (2) the monetary expansion is perceived as temporary (say, balance-sheet normalization is expected to occur in about five years).

Some brief remarks may serve to put our contribution in the context of the recent literature on NK puzzles and paradoxes. The phrases “forward-guidance puzzle,” “paradox of flexibility,” and “paradox of toil” were coined by, respectively, Del Negro et al. (2015), Eggertsson and Krugman (2012), and Eggertsson (2010). Farhi and Werning (2016) were the first to expose the fiscal-multiplier puzzle. Other early contributions related to at least one of the NK puzzles and paradoxes include Christiano et al. (2011), Eggertsson (2011, 2012), Eggertsson et al. (2014), Werning (2012), and Woodford (2011). Wieland (2017) presents empirical evidence against the paradox of toil. Carlstrom et al. (2015) and Cochrane (2017a) clearly show the link between the forward-guidance and fiscal-multiplier puzzles and indeterminacy under permanently exogenous policy rates.

A growing number of contributions propose departures from the basic NK model that solve or attenuate at least one of the NK puzzles and paradoxes (by “attenuate,” we mean here reducing the quantitative effects of policy interventions under an interest-rate peg for a given duration of the peg and a given degree of price stickiness). These departures may involve non-rational expectations (e.g. Farhi and Werning, 2017, Gabaix, 2016, García-Schmidt and Woodford, 2018); information frictions (e.g. Angeletos and Lian, 2018, Kiley, 2016, Wiederholt, 2015); incomplete markets (e.g. Bilbiie, 2018, McKay et al., 2016); overlapping generations (Del Negro et al., 2015); non-Ricardian fiscal policy (Cochrane, 2017a, 2017b); and government bonds with a convenience yield (e.g. Bredemeier et al., 2018, Hagedorn, 2018a, 2018b, Hagedorn et al., 2018, Michaillat and Saez, 2018).

Our work is distinct from most of these contributions in three respects. First, we are solving (not just attenuating) all the NK puzzles and paradoxes. In the literature, Cochrane (2017a, 2017b) solves them all as well. His resolution rests on the fiscal theory of the price level, while ours is purely monetary, based on what central banks actually did during the Great Recession. Monetary policy alone provides the nominal anchor in our model, fiscal policy plays no role in this respect. Second, we stay close to the NK paradigm: at the ZLB, we solve the puzzles and paradoxes even for an arbitrarily small departure from the basic NK model, and we still solve or attenuate them for a vanishingly small departure; away from the ZLB, our model is isomorphic
to the basic NK model if the central bank operates a corridor system. And third, our departure from the basic NK model is sufficiently simple to remain analytically tractable.

Some of these contributions, in particular, propose models that can solve the forward-guidance puzzle by “discounting” the IS equation or the Phillips curve of the basic NK model, i.e. by scaling down the coefficients of their expectational terms. In the online appendix, we generalize Cochrane’s (2016) comments on Gabaix (2016) to highlight three differences between these models and ours: (i) discounting models do not solve the paradox of flexibility; (ii) they require a discrete (sufficiently large) departure from the basic NK model to solve the forward-guidance puzzle; and (iii) their resolution of the forward-guidance puzzle comes at the expense of non-standard implications for equilibrium determinacy in normal times and of overturning the standard Fisher effect (i.e. replacing the one-to-one long-term relationship between the inflation rate and the nominal interest rate with a negative relationship).

The rest of the paper is organized as follows. Section 2 briefly exposes the puzzles and paradoxes in the basic NK model. Section 3 presents our main points in a simple setting that adds a money-demand nexus to the basic NK model. Section 4 confirms and strengthens our points in a fully fledged model with banks. Section 5 motivates our assumption that demand for reserves was not fully satiated in the U.S. over the past few years. Section 6 provides some concluding remarks about the implications of our model for the current normalization process and the future “new normal” of monetary policy. The Appendix contains determinacy proofs, while the online appendix essentially provides a detailed presentation of our model with banks and its extension with liquid government bonds, as well as a detailed comparison with other models.

2 The Basic NK Model

We start with a brief exposition of the puzzles and paradoxes in the basic NK model. The log-linearized IS equation and Phillips curve of this model are

\[ y_t = \mathbb{E}_t \{ y_{t+1} \} - \frac{1}{\sigma} (i_t - \mathbb{E}_t \{ \pi_{t+1} \} - r_t) + g_t - \mathbb{E}_t \{ g_{t+1} \} \]  
\[ \pi_t = \beta \mathbb{E}_t \{ \pi_{t+1} \} + \kappa (y_t - \delta_g g_t - \delta_\varphi \varphi_t) , \]  

where \( y_t \) denotes the output level, \( \pi_t \) the inflation rate, \( i_t \) the nominal interest rate on a bond (in zero net supply) that households can trade with each other, \( r_t \) a preference shock (stemming from variations in the discount factor), \( g_t \) a government-purchases shock, and \( \varphi_t \) a supply shock (stemming from shifts in the labor-disutility function, labor-tax modifications, technology changes, or variations in the elasticity of substitution between goods). All variables and shocks are expressed in log-deviation from their steady-state value, except \( g_t \) (which is proportional to the log-deviation of government purchases from steady state). The parameters satisfy \( \beta \in (0, 1) \), \( \sigma > 0 \), \( \delta_g \in (0, 1) \), \( \delta_\varphi > 0 \), \( \kappa > 0 \); and \( \kappa \) increases without bound as the degree of price stickiness \( \theta \) (i.e. the probability that a firm cannot change its price at a given date) goes to zero. Finally,
\( \mathbb{E}_t \{ \cdot \} \) denotes the rational-expectations operator at date \( t \).

The monetary-policy instrument, in this model, is the interest rate \( i_t \). Under permanently exogenous interest rates, the IS equation and the Phillips curve lead to the dynamic equation

\[
\mathbb{E}_t \{ P_b (L^{-1}) \pi_t \} = -\frac{\kappa}{\beta} (i_t - \pi_t) + \frac{(1 - \delta_y) \kappa}{\beta} (g_t - \mathbb{E}_t \{ g_{t+1} \}) - \frac{\delta_y \kappa}{\beta} (\varphi_t - \mathbb{E}_t \{ \varphi_{t+1} \}),
\]

where \( L \) denotes the lag operator and \( P_b(X) \equiv X^2 - [1 + 1/\beta + \kappa/(\beta \sigma)]X + 1/\beta \) (the subscript "b" stands for “basic”). Since \( P_b(0) = 1/\beta > 0 \), \( P_b(1) = -\kappa/(\beta \sigma) < 0 \), and \( \lim_{X \to +\infty} P_b(X) = +\infty > 0 \), the roots of \( P_b(X) \) are two real numbers \( \rho_b \) and \( \omega_b \) such that \( 0 < \rho_b < 1 < \omega_b \). The dynamic equation (3) expresses \( \pi_t \) as a function of \( \mathbb{E}_t \{ \pi_{t+1} \}, \mathbb{E}_t \{ \pi_{t+2} \} \), and current and expected future exogenous shocks. However, because \( 0 < \rho_b < 1 \), we cannot iterate this equation forward to \( +\infty \) and get \( \pi_t \) as a bounded function of current and expected future exogenous shocks. Nor can we iterate the equation backward to pin down a unique solution for \( \pi_t \), given the absence of \( \pi_{t-1} \) term in this equation. Local-equilibrium indeterminacy arises under permanently exogenous interest rates because the dynamic system has a stable eigenvalue (\( \rho_b \)) and no predetermined variable (no \( \pi_{t-1} \) term).

Under temporarily exogenous interest rates, local-equilibrium determinacy can be obtained by assuming that policy will switch in the future to a rule that sets a nominal anchor (e.g., a Taylor rule like \( i_t = \phi \pi_t \) with \( \phi > 1 \)). The literature about the NK puzzles and paradoxes typically assumes that after a finite date \( T \), in expectation, there are no shocks and the economy is back to its steady state: \( \mathbb{E}_t \{ r_{T+k} \} = \mathbb{E}_t \{ g_{T+k} \} = \mathbb{E}_t \{ \varphi_{T+k} \} = \mathbb{E}_t \{ \pi_{T+k} \} = \mathbb{E}_t \{ y_{T+k} \} = \mathbb{E}_t \{ i_{T+k} \} = 0 \) for all \( t \leq T \) and \( k \geq 1 \). Under this assumption, we can solve for \( \pi_t \) by iterating the dynamic equation (3) forward until date \( T \) and using the terminal conditions. As we show in the online appendix,\(^1\) we then get

\[
\pi_t = \frac{\kappa \mathbb{E}_t}{\beta (\omega_b - \rho_b)} \left\{ \frac{-1}{\sigma} \sum_{k=0}^{T-t} \left( \rho_b^{-k-1} - \omega_b^{-k-1} \right) (i_{t+k} - r_{t+k}) \right. \\
+ \left. \sum_{k=0}^{T-t} \left[ (1 - \rho_b) \rho_b^{-k} + (\omega_b - 1) \omega_b^{-k} \right] \left[ (1 - \delta_y) g_{t+k} - \delta_y \varphi_{t+k} \right] \right\},
\]

and, using the Phillips curve (2),

\[
y_t = \frac{\kappa \mathbb{E}_t}{\beta (\omega_b - \rho_b)} \left\{ \frac{-1}{\sigma} \sum_{k=0}^{T-t} \left( \rho_b^{-k} + \frac{\omega_b^{-k}}{1 - \rho_b} \right) (i_{t+k} - r_{t+k}) \right. \\
+ \left. \sum_{k=1}^{T-t} \left( \rho_b^{-k} - \omega_b^{-k} \right) \left[ (1 - \delta_y) g_{t+k} - \delta_y \varphi_{t+k} \right] \right\} + g_t.
\]

Equations (4)-(5) form what Cochrane (2017a) calls the “standard equilibrium” of the basic NK model. Because the stable eigenvalue \( \rho_b \) has been inverted to obtain these equations, the

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\(^1\)We have relegated the derivations to the online appendix because they follow earlier work. Our presentation in this section is a discrete-time version of the presentations in Werning (2012), Farhi and Werning (2016), and Cochrane (2017a).
The inversion of the stable eigenvalue $\rho_b$ in the horizon $k$ of the shocks. In particular, a negative preference shock ($r_{t+k} < 0$ for all $k \in \{0, ..., T - t\}$) exerts deflationary and contractionary pressures that grow exponentially in the duration of the shock ($\lim_{T \to +\infty} \partial \pi_t / \partial r_T = \lim_{T \to +\infty} \partial y_t / \partial r_T = +\infty$). Setting the current policy rate $i_t$ to its ZLB may not be enough to offset these pressures. But the central bank can easily offset or even overturn them by promising a low policy rate in the sufficiently distant future (i.e. a low $i_{t+T}$ for a sufficiently large $T$), because the power of forward guidance grows exponentially in the guidance horizon ($\lim_{T \to +\infty} \partial \pi_t / \partial i_t = \lim_{T \to +\infty} \partial y_t / \partial i_t = -\infty$). Similarly, the government can easily offset or overturn these pressures by promising a fiscal expansion in the sufficiently distant future (i.e. $g_{t+T} > 0$ for a sufficiently large $T$), because the effectiveness of future fiscal expansions grows exponentially in their implementation horizon ($\lim_{T \to +\infty} \partial \pi_t / \partial g_T = \lim_{T \to +\infty} \partial y_t / \partial g_T = +\infty$). The equilibrium, thus, exhibits the forward-guidance and fiscal-multiplier puzzles.

The inversion of the stable eigenvalue $\rho_b$, leading to the terms $\rho_b^{-k}$ in Equations (4)-(5), is also responsible for the paradox of flexibility. Indeed, as prices become more and more flexible ($\theta \to 0$), we have $\kappa \to +\infty$ and therefore $\omega_b = [1 + \beta + \kappa / \sigma + \sqrt{(1 + \beta + \kappa / \sigma)^2 - 4\beta}] / (2\beta) \to +\infty$ and $\rho_b = \mathcal{P}_b(0)/\omega_b = 1/((\beta \omega_b) \to 0$. As a consequence, the terms $\rho_b^{-k}$ in Equations (4)-(5) make current inflation and output explode in response to future shocks as prices are made more and more flexible ($\lim_{\theta \to 0} \partial \pi_t / \partial i_{t+k} = \lim_{\theta \to 0} \partial y_t / \partial i_{t+k} = -\infty$ and $\lim_{\theta \to 0} \partial \pi_t / \partial y_{t+k} = \lim_{\theta \to 0} \partial y_t / \partial y_{t+k} = +\infty$ for any $k \in \{t + 1, ..., T\}$). Finally, in addition to these three limit puzzles and for a related reason, the standard equilibrium also exhibits the paradox of toil: current and future positive supply shocks are not expansionary; instead, current ones are neutral ($\partial y_t / \partial \varphi_t = 0$) and future ones are contractionary ($\partial y_t / \partial \varphi_{t+k} < 0$ for any $k \in \{1, ..., T - t\}$, given the presence of $\rho_b^{-k} > \omega_b^{-k}$ in the second line of (5)).

3 A Simple Model

We now add a money-demand nexus to the basic NK model, and show how setting exogenously the interest rate on money and the stock of money solves the NK puzzles and paradoxes. The model is not micro-founded; we will discuss a fully articulated model with banks – and bank reserves as money – in the next section. Here, we consider a standard money-demand equation, of the kind estimated in the empirical literature, linking demand for real money balances $m_t$ positively to output $y_t$ and negatively to the opportunity cost $i_t - i_t^m$ of holding money:

$$m_t = \chi_y y_t - \chi_i (i_t - i_t^m),$$ (6)

where $i_t^m$ denotes the interest rate on money, $\chi_y > 0$, and $\chi_i > 0$. Real money balances $m_t$ are also linked to nominal money balances $M_t$ and the price level $p_t$ through the identity $m_t = M_t - p_t$. The monetary-policy instruments are, now, $i_t^m$ and $M_t$. When these instruments are permanently exogenous, the IS equation (1), the Phillips curve (2), the money-demand
equation (6), and the identities \( m_t = M_t - p_t \) and \( \pi_t = p_t - p_{t-1} \) lead to the dynamic equation

\[
\mathbb{E}_t \{ LP \ (L^{-1}) \ p_t \} = Z_t = \frac{-\kappa}{\beta \sigma} (i^m_t - r_t) + \frac{\kappa}{\beta \sigma \chi_i} M_t + \left[ 1 - \left( \frac{\chi_y}{\sigma \chi_i} \right) \delta_y \right] \frac{\kappa}{\beta} g_t \\
- \left( 1 - \delta_y \right) \frac{\kappa}{\beta} \mathbb{E}_t \{ g_{t+1} \} - \left( 1 + \frac{\chi_y}{\sigma \chi_i} \right) \frac{\delta_y \kappa}{\beta} \varphi_t + \frac{\delta_y \kappa}{\beta} \mathbb{E}_t \{ \varphi_{t+1} \},
\]

where \( P(X) \) is a polynomial of degree 3 defined in the Appendix. We show in the Appendix that \( P(X) \) has one root inside the unit circle (\( \rho \in (0, 1) \)), and two roots outside the unit circle (\( \omega_1 \) and \( \omega_2 \) with \( |\omega_1| \leq |\omega_2| \)), either positive real numbers or complex conjugates. We assume that the latter roots are distinct real numbers, so that \( \omega_2 > \omega_1 > 1 \), and postpone the discussion of this assumption to the next section. Using the roots of \( P(X) \), we can then rewrite the dynamic equation as

\[
\mathbb{E}_t \left\{ (L^{-1} - \omega_1) (L^{-1} - \omega_2) (1 - \rho L) p_t \right\} = Z_t,
\]

and use the method of partial fractions to solve this equation forward and get the unique bounded solution for \( p_t - \rho p_{t-1} \):

\[
p_t - \rho p_{t-1} = \mathbb{E}_t \left\{ \frac{Z_t}{(L^{-1} - \omega_1)(L^{-1} - \omega_2)} \right\} = \frac{\mathbb{E}_t}{\omega_2 - \omega_1} \left\{ \omega_1^{-1} Z_t \left( 1 - (\omega_1 L)^{-1} \right) - \omega_2^{-1} Z_t \left( 1 - (\omega_2 L)^{-1} \right) \right\}
\]

\[
= \frac{\mathbb{E}_t}{\omega_2 - \omega_1} \sum_{k=0}^{+\infty} \left( \omega_1^{-k-1} - \omega_2^{-k-1} \right) Z_{t+k}.
\]

(7)

At any date \( t \), \( p_{t-1} \) is known, and (7) pins down \( p_t \) uniquely. If time starts at \(-\infty\), we can iterate (7) backward and get \( p_t \) as a unique bounded function only of the exogenous forcing variables \( \mathbb{E}_{t-j} \{ Z_{t-j+k} \} \) for all \((j, k) \in \mathbb{N}^2\). Thus, the model delivers local-equilibrium determinacy under permanently exogenous monetary-policy instruments. The dynamic system still has one stable eigenvalue (\( \rho \)), but this eigenvalue is now matched by a predetermined variable (\( p_{t-1} \)).

Our determinacy result can be interpreted as follows. Setting exogenously the monetary-policy instruments \( i^m_t \) and \( M_t \) makes the interest rate on bonds \( i_t \) a strictly increasing function of output and, crucially, the price level: \( i_t = (\chi_y/\chi_i) y_t + (1/\chi_i) p_t + [i^m_t - (1/\chi_i) M_t] \), as follows from (6). The reason is that if output or the price level rises, then the demand for nominal money balances increases; given that both the supply of nominal money balances and the interest rate that they pay are fixed, the interest rate on bonds has then to increase to clear the money market. Thus, our model with exogenous monetary-policy instruments is isomorphic to the basic NK model with a "Wicksellian rule" for \( i_t \) (which is the policy rate in the latter model); and Wicksellian rules are well known to ensure determinacy in the basic NK model (as shown in Woodford, 2003, Chapter 4). At the ZLB, the central bank has to peg the IOR rate \( i^m_t \); but it is as if it could set the interest rate on bonds \( i_t \) according to this (shadow) Wicksellian rule ensuring determinacy.

\(^2\)Confining our analysis to bounded solutions amounts to following the common practice to set aside the global indeterminacy inherent in models with fiat money.

\(^3\)If time starts at some finite date, say date 0, we cannot assume that prices are sticky (\( \theta > 0 \)) at date 0, since there is no date \(-1\). In the online appendix, we consider an alternative specification assuming that prices are flexible at date 0 and sticky afterwards; we show that this delivers local-equilibrium determinacy as well.
Using the price-level solution (7), the identity \( \pi_t = p_t - p_{t-1} \), and the Phillips curve (2), we get

\[
\pi_t = - (1 - \rho) p_{t-1} + \frac{\mathbb{E}_t}{\omega_2 - \omega_1} \left\{ \sum_{k=0}^{+\infty} \left( \omega_1^{k-1} - \omega_2^{k-1} \right) Z_{t+k} \right\},
\]

(8)

\[
y_t = - \vartheta p_{t-1} + \delta_t g_t + \delta \varphi \varphi_t - \frac{\mathbb{E}_t}{(\omega_2 - \omega_1) \kappa} \left\{ \sum_{k=0}^{+\infty} \left( \xi_1 \omega_1^{k-1} - \xi_2 \omega_2^{k-1} \right) Z_{t+k} \right\},
\]

(9)

where \( \vartheta \equiv (1 - \rho)(1 - \beta \rho)/\kappa \) and \( \xi_j \equiv \beta(\omega_j + \rho - 1) - 1 \) for \( j \in \{1, 2\} \). One key difference between our simple model’s equilibrium (8)-(9) and the basic NK model’s standard equilibrium (4)-(5) is that the former involves only \( \omega_1^{-k} \) and \( \omega_2^{-k} \) terms, where \( \omega_2 > \omega_1 > 1 \), while the latter involves \( p_b^{-k} \) terms, where \( 0 < p_b < 1 \). As a consequence, the implications of our simple model for the response of inflation and output to anticipated future shocks are in sharp contrast to the corresponding implications of the basic NK model: later future shocks have smaller current effects in our model, not bigger ones, regardless of the (preference, monetary, fiscal, or supply) type of shocks. More specifically, shocks occurring at date \( t + k \) and announced at date \( t \) do not affect \( p_{t-1} \); their effects on inflation and output at date \( t \) decay at an exponential rate, converging to zero essentially like \( \omega_1^{-k} \). So, neither the forward-guidance puzzle nor the fiscal-multiplier puzzle arises in our model. The central bank can now provide forward guidance not only about low future policy rates \( (i_t^{n}) \), but also about large future balance sheets \( (M_{t+k}) \), in order to offset the deflationary pressures exerted by the negative preference shock \( \varphi_t \); but the effectiveness of both types of forward guidance decreases in the guidance horizon \( k \).

Moreover, because we have \( 0 < \rho < 1 < |\omega_1| \leq |\omega_2| \) whatever the degree of price stickiness \( \theta \in (0, 1) \) and in particular as \( \theta \to 0 \), the paradox of flexibility does not arise either in our simple model. In the online appendix, we show that the limits of \( \pi_t \) and \( y_t \) as \( \theta \to 0 \) take finite values, unlike their counterparts in the basic NK model, and that these values coincide with the values that \( \pi_t \) and \( y_t \) take under perfectly flexible prices (in particular, \( \lim_{\theta \to 0} y_t = \delta_t g_t + \delta \varphi \varphi_t \)). So, the model involves no discontinuity at the \( \theta = 0 \) point, in contrast to the basic NK model.

To illustrate graphically our resolution of the forward-guidance puzzle and the paradox of flexibility, Figure 1 shows the effects of cutting the policy rate by 25 basis points (one percentage point per annum) in Quarter \( T \) on the annualized inflation rate in Quarter 1 (when the rate cut is announced). We start from a benchmark calibration borrowed from Galí (2008, Chapter 3), which sets the degree of price stickiness \( \theta \) to 2/3 (corresponding to “3-quarter price rigidity”), and we then cut \( \theta \) in half step by step to make prices more flexible. The right panel in Figure 1 replicates the implausible implications of the basic NK model: cutting the policy rate in a later quarter leads to an exponentially larger effect on current inflation; and making prices more flexible accelerates these explosive effects. The left panel shows the results for our simple model: with our benchmark value of \( \theta = 2/3 \), the inflationary effects of the policy-rate cut are modest (about 10 basis points for a cut in one of the first five quarters) and die off relatively

\[^{4}\text{The roots } \omega_1 \text{ and } \omega_2 \text{ are positive real numbers under all these calibrations, consistently with our assumption above.}\]
quickly with the horizon of the cut; as we make prices more flexible, these inflationary effects smoothly converge to the effects under perfect price flexibility ($\theta = 0$). Figure 1 is, of course, only illustrative, as our simple model is not really suitable for a quantitative assessment of the effects of forward guidance or other policies.\(^5\) Our contribution here is to show that this model offers a qualitative resolution of the three limit puzzles under any calibration.

In our model, demand for money is not satiated as long as $\chi_i$ takes a finite value. As we make $\chi_i$ go to infinity, we get a sequence of money-demand equations that converge to $i_t = i^m_t$, and hence a sequence of models that converge to the basic NK model. The corresponding sequence of unique bounded solutions converges to a particular equilibrium (out of an infinity of equilibria) of the basic NK model under permanently exogenous policy rates. Using $\lim_{\chi_i \to +\infty}(\rho, \omega_1, \omega_2) = (\rho_b, 1, \omega_b)$, we get that the equilibrium we select uniquely in this way is characterized by

$$
\pi_t = -(1 - \rho_b) p_{t-1} + E_t \left\{ - (1 - \rho_b) \sum_{k=0}^{+\infty} \left( 1 - \omega_b^{-k} \right) (t_{t+k} - r_{t+k}) + \kappa \rho_b \sum_{k=0}^{+\infty} \omega_b^{-k} [(1 - \delta_g) g_{t+k} - \delta \varphi_i t_{t+k}] \right\},
$$

(10)

$$
y_t = -\frac{\rho_b}{\sigma} p_{t-1} - \frac{\rho_b}{\sigma} E_t \left\{ \sum_{k=0}^{+\infty} \left[ 1 + \beta (1 - \rho_b) \omega_b^{-k} \right] (i_{t+k} - r_{t+k}) + \kappa \sum_{k=0}^{+\infty} \omega_b^{-k} [(1 - \delta_g) g_{t+k} - \delta \varphi_i t_{t+k}] \right\} + g_t.
$$

(11)

\(^5\)For this reason, we relegate the sensitivity analysis to the online appendix – with a particular focus on the sensitivity to the value of the interest-rate semi-elasticity of money demand $\chi_i$, given the wide range of estimates for $\chi_i$ in the empirical literature (Galí’s value standing in the middle of the range).
We can highlight two main differences between our selected equilibrium (10)-(11) and the standard equilibrium (4)-(5) of the basic NK model. First, there are no $\rho_b^{-k}$ terms in (10)-(11), and as a consequence our selected equilibrium does not exhibit any of the three limit puzzles—at least in the same form as the standard equilibrium. More specifically, the effects of announcing at date $t$ a fiscal expansion at date $t + k$ on inflation and output at date $t$ decrease at rate $\omega_b^{-k}$ in the horizon $k$ of the fiscal expansion, and asymptotically $\lim_{k \to +\infty} \partial \pi_t / \partial g_{t+k} = \lim_{k \to +\infty} \partial y_t / \partial g_{t+k} = 0$ (no fiscal-multiplier puzzle). Moreover, given that $\lim_{\theta \to 0} (\rho_b, \omega_\theta, \kappa/\omega_\theta) = (0, +\infty, \beta \sigma)$ and $\rho_b \omega_\theta = 1 / \beta$, (10) and (11) imply that $\lim_{\theta \to 0} \pi_t$ and $\lim_{\theta \to 0} y_t$ are finite, and in particular that $\lim_{\theta \to 0} y_t$ takes the same value (i.e., $\delta g_t + \delta \varphi_t$) as $y_t$ under perfectly flexible prices (no paradox of flexibility). Finally, in our selected equilibrium (10)-(11), the effects of announcing at date $t$ a policy-rate cut at date $t + k$ on inflation and output at date $t$ do not explode as the horizon $k$ of the policy-rate cut goes to infinity (as is the case in the standard equilibrium), nor do they converge to zero (as is the case in our simple model outside its basic-NK-model limit). Instead, they converge to non-zero finite values: $\lim_{k \to +\infty} \partial \pi_t / \partial i_{t+k} = -(1 - \rho_b)$ and $\lim_{k \to +\infty} \partial y_t / \partial i_{t+k} = -\rho_b / \sigma$ (attenuated forward-guidance puzzle). In the online appendix, we relate this last feature to price-level stationarity.

Second, our selected equilibrium and the standard equilibrium have opposite implications about the sign of the effects of future government-purchases and supply shocks on current output. In the standard equilibrium (5), anticipated fiscal expansions and negative supply shocks are expansionary ($\partial y_t / \partial g_T > 0$ and $\partial y_t / \partial \varphi_T < 0$ for $t < T$) — suggesting that fiscal policy is a potent stabilization tool at the ZLB and that structural reforms of labor and product markets may be best put on hold during a ZLB episode. These implications of the standard equilibrium arise from a feedback loop, first described in Farhi and Werning (2016), that works back in time via the IS equation and the Phillips curve: given that $\pi_{T+1} = y_{T+1} = 0$, a fiscal expansion or a negative supply shock at date $T$ raises inflation at date $T$, which lowers the real interest rate at date $T - 1$, which raises output and inflation at date $T - 1$, etc. This feedback loop is also present in our selected equilibrium, but it is counteracted by the presence of the state variable $p_{t-1}$ in (10)-(11): the starting point of the loop, $\pi_{T+1}$ and $y_{T+1}$, now reacts endogenously to prior developments. More specifically, (10) and the identity $\pi_t = p_t - p_{t-1}$ imply that a pre-announced fiscal expansion or negative supply shock at date $T$ ($g_T > 0$ or $\varphi_T < 0$) raises the price level at all dates starting from the announcement date, and in particular at date $T$ ($p_T > 0$); in turn, (10) and (11) then imply that $\pi_{T+1} = -(1 - \rho_b) p_T < 0$ and $y_{T+1} = -\rho_b / \sigma) p_T < 0$. As a result, expected future fiscal expansions and current or expected future negative supply shocks are now contractionary, reflecting the familiar wealth effect present in standard Real-Business-Cycle models (through a reduction in permanent income). Our selected equilibrium, thus, does not exhibit the paradox of toil: for example, structural reforms that reduce market power are

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6In the online appendix, we also compare our selected equilibrium with equilibria discussed in Cochrane (2017a).

7We cannot compare $\lim_{\theta \to 0} \pi_t$ with $\pi_t$ at $\theta = 0$, since $\pi_t$ at $\theta = 0$ is indeterminate in the basic NK model.
expansionary during a liquidity trap, not contractionary.

4 A Model with Banks

We have so far presented our main points in the context of a simple but ad hoc setup. It remains to show that these points can be made in a more structured model generating a demand for money. As we show in the online appendix, we could use standard models with a cash-in-advance (CIA) constraint or money in the utility function (MIU). But these standard models have some limitations for making our points. First, money in these models looks like household cash, rather than interest-bearing bank reserves. Second, setting exogenously the stock of money and the interest rate on money does not always deliver determinacy in the CIA model. Third, setting these two variables exogenously can lead to a negative real eigenvalue in the CIA model, and to complex eigenvalues in the MIU model (as was also the case in the simple setup of the previous section). These possibilities are awkward for our purposes because they imply deterministic cycles in the effects of future shocks as we change the horizon (i.e., recurrent sign reversals in the effect of any given shock — e.g., an interest-rate cut — at date $T$ on initial inflation and output, as we change $T$). And fourth, we cannot make the CIA model converge to the basic NK model (in order to use it as an equilibrium-selection device), and we can only make the MIU model converge to the basic NK model in some specific cases (like separable or constant-elasticity-of-substitution utility over money and consumption).

For all these reasons, we consider in this section a fully fledged model with banks in which money is explicitly made of bank reserves. We show that setting exogenously the interest rate on bank reserves (the IOR rate) and the stock of bank reserves — two monetary-policy instruments under the direct control of central banks — always delivers determinacy with positive real eigenvalues in this model under general assumptions. We also show how to use this model to uniquely select an equilibrium of the basic NK model under permanently exogenous policy rates — exactly the same equilibrium as in the previous section.

In this model, which is presented in detail in the online appendix, firms must borrow the wage bill (or some fraction of it) from banks. Banks incur costs making loans, and holding reserves mitigates these costs. The model makes weak standard assumptions about utility and production functions, like monotonicity and concavity, without specifying any functional form. The key log-linearized equilibrium conditions are, again, an IS equation, a Phillips curve, and a money-demand equation. The IS equation is still (1), but the Phillips curve and the money-demand equation are now

$$\pi_t = \beta \mathbb{E}_t \{ \pi_{t+1} \} + \kappa \left( y_t - \delta_m m_t - \delta_g g_t - \delta_\varphi \varphi_t \right), \quad (12)$$

$$m_t = \chi_y y_t - \chi_i \left( i_t - i^m_t \right) - \chi_g g_t - \chi_\varphi \varphi_t, \quad (13)$$

where $\beta \in (0, 1)$, $\delta_g \in (0, 1)$, $\chi_\varphi \geq 0$, and all the other parameters are positive (for convenience,
we have kept the same notations $\kappa$, $\delta_y$, $\delta_\varphi$, $\chi_y$, and $\chi_\varphi$ as previously, although these reduced-form parameters have changed). The reason why real reserves $m_t$ appear in the Phillips curve (12) is that they reduce banking costs, which in turn lowers the borrowing costs of firms and hence their marginal cost of production. The money-demand equation (13) now involves the government-purchases shock $g_t$, because money demand depends on (the marginal utility of) consumption, which we have eliminated using the goods-market-clearing condition. It also involves the supply shock $\varphi_t$, because the demand for reserves depends on the volume of loans, which in turn depends on firms’ wage bill, which in turn depends on the supply shock for a given output level — except when this shock is a markup shock (in which case $\chi_\varphi = 0$).

Our model with banks, given its structure, implies in particular that

$$\sigma < \chi_y < \frac{1}{\delta_m},$$

(14)

as we show in the online appendix. This double inequality will play a key role in our results below (as we will see). The first inequality in (14) arises from the fact that bank loans serve to finance the wage bill (or some fraction of it). If output $y_t$ increases by 1% for given government purchases $g_t$, the marginal utility of consumption decreases by $\sigma\%$; so, the wage, the wage bill, and loans all increase by more than $\sigma\%$; and, in turn, so does the demand for reserves $m_t$ for a given spread $i_t - i_t^m$ (i.e., $\chi_y > \sigma$). The second inequality in (14) reflects how holding reserves mitigates the costs of banking. For a given spread $i_t - i_t^m$, a rise in output $y_t$ has two opposite effects on firms’ marginal cost of production (i.e., on the term in factor of $\kappa$ in the Phillips curve): a standard positive direct effect (with elasticity $1$), and a negative indirect effect via the implied rise in reserves $m_t$ (with elasticity $\chi_y \delta_m$). The inequality states that the direct effect dominates the indirect one (i.e., $\chi_y \delta_m < 1$).

Under permanently exogenous monetary-policy instruments $i_t^m$ and $M_t$, the IS equation (1), the Phillips curve (12), the money-demand equation (13), and the identities $m_t = M_t - p_t$ and $\pi_t = p_t - p_{t-1}$ lead to the dynamic equation

$$\mathbb{E}_t \{LP^* (L^{-1}) p_t \} = Z_t^* \equiv \frac{-K}{\beta \sigma} (i_t^m - r_t) + \left[ \frac{1}{\sigma \chi_i} - \left( 1 + \frac{\chi_y}{\sigma \chi_i} \right) \delta_m \right] \frac{\kappa}{\beta} M_t + \frac{\delta m \kappa}{\beta} \mathbb{E}_t \{M_{t+1} \} + \left[ \left( 1 + \frac{\chi_\varphi}{\sigma \chi_i} \right) - \left( 1 + \frac{\chi_y}{\sigma \chi_i} \right) \delta_\varphi \right] \frac{\kappa}{\beta} g_t + \frac{1 - \delta_\varphi}{\beta} \mathbb{E}_t \{g_{t+1} \} + \left[ \left( 1 + \frac{\chi_\varphi}{\sigma \chi_i} \right) \delta_\varphi \right] \frac{\kappa}{\beta} \varphi_t + \frac{\delta \varphi \kappa}{\beta} \mathbb{E}_t \{\varphi_{t+1} \},$$

where $P^* (X)$ is a polynomial of degree 3 defined in the Appendix. Using (14), we show in the Appendix that the roots of $P^* (X)$ are three real numbers $\rho$, $\omega_1$, and $\omega_2$ such that $0 < \rho < 1 < \omega_1 < \omega_2$ (again, we keep for convenience the same notations for the roots as previously, even though these roots have changed). So, our model has a unique bounded solution, which is characterized by (7)-(9) with $Z_{t+k}$, $\vartheta$, $\delta_y g_t + \delta_\varphi \varphi_t$, and $\xi_j$ replaced by $Z_{t+k}^*$, $\vartheta^* \equiv (1 - \rho)(1 - \beta \rho)/\kappa + \delta m \rho$, $\delta m M_t + \delta_y g_t + \delta_\varphi \varphi_t$, and $\xi_j^* \equiv \beta (\omega_j + \rho - 1) + \kappa \delta m - 1$ for $j \in \{1, 2\}$.
This determinacy result can be interpreted essentially in the same way as the previous section’s determinacy result. In our model with banks, setting exogenously $i_t^m$ and $M_t$ also amounts to following a “shadow Wicksellian rule” for $i_t$: if the price level rises (making real reserves fall, given that nominal reserves are fixed), or if output rises, then the marginal utility of real reserves increases. Given that the IOR rate is fixed, the interest rate on bonds has then to increase for private agents to remain indifferent between holding reserves and holding bonds. Existing results for Wicksellian rules in the basic NK model (e.g., Woodford, 2003, Chapter 4) do not apply to our model with banks, and not all Wicksellian rules would ensure determinacy in our model. But because its coefficients are disciplined by (14), the specific shadow Wicksellian rule that arises under permanently exogenous monetary-policy instruments in our model always delivers determinacy and always does so with positive real eigenvalues $\omega_1$ and $\omega_2$. And since it does, our model with banks solves the three limit puzzles in the same way as the previous section’s simple setup (without even having to assume that $\omega_1$ and $\omega_2$ are positive real numbers).

Moreover, we show in the online appendix that by shrinking banking costs and the steady-state spread to zero (at suitable rates, to give us a finite level of steady-state real reserve balances in the limit), we make the steady state and log-linear approximation of our model converge to those of the basic NK model. We can thus consider a sequence of models converging to the basic NK model, and its corresponding sequence of unique bounded solutions converging to a particular equilibrium (out of an infinity of equilibria) of the basic NK model under permanently exogenous policy rates. This particular equilibrium is exactly the same as in the previous section; therefore, in particular, it does not exhibit the paradox of toil.

Our model with banks, thus, confirms and strengthens the points we have made in the previous section in the context of a simpler model. It will also serve, in the next section, for our simulation of the effects of the second round of quantitative easing in the US (“QE2”) and our explanation of why T-bills had a lower return than bank reserves during the crisis in the US.

5 Discussion of the Non-Satiation Assumption

Our resolution of the NK puzzles and paradoxes in the previous section rests on the assumption that demand for reserves is not fully satiated, i.e. that reserves still carry a positive convenience yield. The convenience yield of reserves is measured by the spread $I_t - I_t^m$, where $I_t$ denotes the gross nominal interest rate on household bonds (serving as a pure store of value) and $I_t^m$ the gross nominal IOR rate. So our non-satiation assumption is that this spread is positive – possibly very close to zero, but not exactly zero. In reality, holding reserves can reduce banking costs (and lead to a convenience yield) by helping banks manage the liquidity risk associated with short-term deposits, or by facilitating compliance with regulatory constraints imposed by prudential policy. So, we should ask if these considerations still imply a non-pecuniary benefit from holding reserves, at current levels. In this section, we address two types of arguments that
go against our (non-satiation) view.

The first type of argument picks an observable proxy for the spread $I_t - I^m_0$ and asks if the spread has been positive in the data. In our model, however, $I_t$ is the interest rate on a hypothetical bond that serves purely as a store of value. Del Negro et al. (2017) suggest that no observable interest rate corresponds to this shadow rate, because money-market assets have attributes that give them a convenience yield (e.g., they may serve as collateral or facilitate compliance with regulations for some financial entities).

In particular, the fact that the federal-funds rate has remained below the IOR rate in recent years does not contradict our claim that reserves still have a positive marginal convenience yield. Since the early stages of the Great Recession, banks have been borrowing federal funds from lenders who do not have direct access to the IOR rate, like Government-Sponsored Enterprises; banks earn the spread between the two rates. At the time of this writing, Federal Home Loan Banks (FHLBs) account for almost all the lending volume in the federal-funds market. FHLBs must comply with regulations to have a portion of their portfolios readily available for lending; so, lending federal funds (typically, overnight) presumably has a convenience yield for them.\(^8\)

This argues against identifying the federal-funds rate as the interest rate $I_t$ of our model. More to the point, the federal-funds rate may have to fall below the IOR rate to incentivize the banks who must borrow the funds and earn the IOR rate. The spread is small and does not vary much because banks compete for funds, and the only costs involved are small balance-sheet costs like deposit-insurance fees (which depend on total assets) and capital costs from increased leverage.

Similarly, evidence that T-bill returns have been below the IOR rate does not contradict our claim that reserves still have a positive marginal convenience yield. The reason is that T-bills also provide a convenience yield to many non-bank entities, e.g. by serving as collateral or international reserve asset. In the online appendix, we present a simple extension of our model with banks and construct an equilibrium in which T-bills have a lower return than reserves. In our extended model, workers get utility from holding government bonds, as a proxy for pension funds and money-market funds holding bonds and providing a service to households. Banks can use bonds instead of reserves for liquidity management, but they choose not to do so in equilibrium. So, adding liquidity services of bonds does not change at all the equilibrium that we analyze in the previous section.

The second type of argument suggesting that money demand is satiated has to do with the apparent lack of inflationary pressures in the aftermath of large expansions of the Federal Reserve’s balance sheet. This observation, however, is not inconsistent with our non-satiation assumption. In our model, a large increase in nominal-money supply can have small effects on inflation (and output) under two conditions. First, the interest-rate semi-elasticity of the demand for reserves $\chi_i$ has to be large. In this case, as follows from the log-linearized money-demand equation (13)\(^8\)See Gissler and Narajabad (2017) for a recent discussion of FHLB activities, and how regulatory constraints and incentives lead to asset preferences that imply (what we call) a convenience yield.
Note: The figure displays the effects of announcing at date 1 a large temporary balance-sheet expansion (left panel), starting from an already large balance-sheet size, on the spread (middle panel) and inflation (right panel) between dates 1 and 30. Nominal reserves $M_t$ are here expressed in levels, not in log-deviations from steady state as elsewhere in the paper.

and the identity $m_t = M_t - p_t$, a large increase in nominal-money supply $M_t$ can be absorbed by a small drop in the spread $i_t - i_t^m$, without changing the price level $p_t$ (and output $y_t$) by much. In turn, a large $\chi_t$ requires that the spread is close to zero to begin with, and demand for reserves is “almost satiated” in the sense that the money-demand curve (with the spread on the vertical axis) is very flat. The second condition is that the balance-sheet expansion should be expected to be temporary. Indeed, in our model, a permanent increase in nominal reserves (not accompanied by a permanent change in the IOR rate) would raise the price level by the same amount so as to leave the steady-state real reserve balances unchanged.

To make the point quantitatively, for a large monetary expansion, we need to work with the non-linear version of our model with banks. We report the details of our non-linear simulation in the online appendix, and show its outcome in Figure 2. In essence, our simulation considers four alternative monetary expansions. One, like QE2 in the US, raises the balance-sheet size from an already large value ($1$ trillion) to a substantially larger one ($1.6$ trillion) in the course of 3 quarters (solid line with asterisks in Figure 2). The others raise the balance-sheet size by two, three, or four times as much, i.e. from $1$ to $2.2$, $2.8$, or $3.4$ trillion (solid, dashed, and dotted lines in Figure 2). We assign standard values to the parameters that appear in standard models, and we set the IOR rate to $25$ basis points per annum (its value during QE2). To set our banking-cost parameters, we match US figures for the ratio of reserves to loans and the prime loan rate in November 2010 (at the start of QE2). We set the $I - I^m$ spread to $10$ basis points per annum, which assumes that Nagel’s (2016) estimate of the liquidity premium on T-bills in November 2010 also applies to bank reserves. The four monetary expansions that we consider are temporary: the balance-sheet size rises over 3 quarters, remains at its new value for 15 quarters, and goes back to its initial value over 3 quarters. As shown in Figure 2, the “single
QE2” expansion makes the $I_t - I_t^{m}$ spread fall from 10 to 6.2 basis points, and raises annualized inflation by only 16 basis points upon impact. And the “multiple QE2” expansions do not have much larger inflationary effects, given the decreasing returns of quantitative easing: following the “double, triple, and quadruple QE2” expansions, the spread falls to 4.6, 3.6, and 2.9 basis points, and inflation rises by only 24, 28, and 31 basis points.

Even if we observed the interest rate $I_t$ on our model’s hypothetical bonds, it would be hard to distinguish empirically between our preferred view and the alternative view setting the marginal convenience yield of reserves exactly to zero. Our view substitutes a narrative in which small shocks may lead to large changes in the demand for reserves for a narrative in which banks are truly indifferent across a range of values for their reserve balances; or, equivalently, a narrative in which quantitative easing has, on the margin, some (possibly very small) effects on the economy, for a narrative in which it has no effects at all. We don’t think a model with literal satiation (and no alternative mechanism to set a nominal anchor) provides a useful framework for thinking about the US economy. Such a model would imply that real reserve balances and the price level are indeterminate. If we contemplate any heterogeneity — like the overlapping-generations structure of Sargent and Wallace (1985) — we would have to conclude that indeterminacy permeates to other real variables. Casual observations don’t seem consistent with such a model: bank characteristics (like size or charter) seem to be good predictors of the distribution of reserves, inflation has been remarkably stable (as Cochrane, 2017b, emphasizes), and there are no signs of massive indeterminacy of real allocations.

6 Concluding Remarks

Our discussion so far has focused on episodes during which the ZLB forces the central bank to peg the IOR rate, and on how setting exogenously the second monetary-policy instrument (bank reserves) delivers determinacy and solves the NK puzzles and paradoxes. But our model also provides a simple framework to think about two important monetary-policy issues away from the ZLB: the impending monetary-policy normalization process, and the future “new normal” of monetary policy. On the first issue, our model highlights in particular the deflationary pressures that arise from expected future balance-sheet contractions and interest-rate hikes during the normalization process. For this reason, we think it may serve as a useful starting point for addressing questions about the timing and sequencing of balance-sheet contractions and interest-rate hikes.

On the second issue, our model can shed light on different ways of conducting monetary policy in normal times. In the online appendix, we consider the main two alternative options currently discussed for the Federal Reserve, and show that they have contrasting implications for determinacy. More specifically, if the central bank follows a rule for $I_t^{m}$, and adjusts the supply of reserves to keep the spread $I_t - I_t^{m}$ fixed, then our model inherits all the determinacy implica-
tions of the basic NK model — in particular, the so-called Taylor Principle. Alternatively, if the central bank follows a rule for \( I_t^m \) and sets reserves exogenously, then, if the IOR-rate rule reacts only to inflation, we get determinacy with any non-negative response to inflation. If the rule also reacts to output, then a sufficient condition for determinacy is that the response to output is below some threshold (which does not depend on the non-negative response to inflation). The value of this threshold is about 15 under the calibration used for Figure 2, which seems comfortably large enough to get determinacy for any empirically relevant Taylor rule.

7 Appendix

7.1 Proof that \( 0 < \rho < 1 < |\omega_1| \leq |\omega_2| \) in the Simple Model

The characteristic polynomial of the simple model’s dynamic equation is

\[
P(X) = X^3 - \left(2 + \frac{1}{\beta} + \frac{\kappa}{\beta \sigma} + \frac{\chi_y}{\sigma \chi_i}\right)X^2 + \ldots
\]

\[
\left[1 + \frac{2}{\beta} + \left(1 + \frac{1}{\chi_i}\right)\frac{\kappa}{\beta \sigma} + \left(1 + \frac{1}{\beta}\right)\frac{\chi_y}{\sigma \chi_i}\right]X - \left(\frac{1}{\beta} + \frac{\chi_y}{\beta \sigma \chi_i}\right).
\]

Since \( P(0) = -1/\beta - \chi_y/(\beta \sigma \chi_i) < 0 \) and \( P(1) = \kappa/(\beta \sigma \chi_i) > 0 \), \( P(X) \) has either one or three real roots inside \((0,1)\). Moreover, since \( P(X) < 0 \) for all \( X < 0 \), \( P(X) \) has no negative real roots. Therefore, \( P(X) \) has at least one real root inside \((0,1)\), which we denote by \( \rho \), and its other two roots, which we denote by \( \omega_1 \) and \( \omega_2 \) with \( |\omega_1| \leq |\omega_2| \), are either (i) both real and inside \((0,1)\), or (ii) both real and higher than 1, or (iii) both complex and conjugates of each other. Now, given that \( P(X) \) is of type \( X^3 - a_2X^2 + a_1X - a_0 \), we have \( \rho + \omega_1 + \omega_2 = a_2 \equiv 2 + 1/\beta + \kappa/(\beta \sigma) + \chi_y/(\sigma \chi_i) > 3 \). Therefore, Case (i) is impossible, and in Case (iii) the common real part of \( \omega_1 \) and \( \omega_2 \) is higher than 1. As a consequence, in the remaining two possible cases, namely Cases (ii) and (iii), \( \omega_1 \) and \( \omega_2 \) lie outside the unit circle.

7.2 Proof that \( 0 < \rho < 1 < \omega_1 < \omega_2 \) in the Model With Banks

The characteristic polynomial of the dynamic equation of the model with banks is

\[
P^*(X) = X^3 - \left(\frac{1 + 2\beta + \beta \Theta_1 + \Theta_2}{\beta}\right)X^2 + \left[\frac{2 + \beta + (1 + \beta) \Theta_1 + \Theta_2 + \Theta_3}{\beta}\right]X - \left(\frac{1 + \Theta_1}{\beta}\right)
\]

\[
= (X - 1 - \Theta_1) \left[X^2 - \left(\frac{1 + \beta + \Theta_3}{\beta}\right)X + \frac{1}{\beta}\right] - \left(\frac{\Theta_1 \Theta_2 - \Theta_3}{\beta}\right)X,
\]

where \( \Theta_1 \equiv \chi_y/(\sigma \chi_i) > 0 \), \( \Theta_2 \equiv (1/\sigma - \delta_m)\kappa \), and \( \Theta_3 \equiv (1 - \delta_m \chi_y)\kappa/(\sigma \chi_i) \). The double inequality (14) implies \( \Theta_2 > 0 \), \( \Theta_3 > 0 \), and \( \Theta_1 \Theta_2 - \Theta_3 = (\chi_y - \sigma)\kappa/(\sigma \chi_i) > 0 \). Therefore, we get \( P^*(0) = -(1 + \Theta_1)/\beta < 0 \), \( P^*(1) = \Theta_3/\beta > 0 \), \( P^*(1 + \Theta_1) = -(\Theta_1 \Theta_2 - \Theta_3)(1 + \Theta_1)/\beta < 0 \), and \( \lim_{X \to +\infty} P^*(X) = +\infty > 0 \). As a consequence, the roots of \( P^*(X) \) are three real numbers \( \rho, \omega_1, \) and \( \omega_2 \) such that \( 0 < \rho < 1 < \omega_1 < 1 + \Theta_1 < \omega_2 \).
References

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Appendix A: Basic NK Model

In this appendix, which complements Section 2 in the paper, we derive Equations (4) and (5), which characterize the standard equilibrium of the basic NK model.

Using the roots $\rho_b$ and $\omega_b$ of $P_b(X)$, we can rewrite the dynamic equation (3) as

$$\mathbb{E}_t \{(L^{-1} - \omega_b)(L^{-1} - \rho_b) \pi_t\} = Z^b_t,$$  \hspace{1cm} (A.1)

where

$$Z^b_t \equiv -\kappa \beta \sigma (i_t - r_t) + \frac{(1 - \delta_g) \kappa}{\beta} (g_t - \mathbb{E}_t \{g_{t+1}\}) - \frac{\delta_g \kappa}{\beta} (\varphi_t - \mathbb{E}_t \{\varphi_{t+1}\}).$$

Under the assumption $\mathbb{E}_t \{i_{T+k}\} = \mathbb{E}_t \{r_{T+k}\} = \mathbb{E}_t \{g_{T+k}\} = \mathbb{E}_t \{\varphi_{T+k}\} = \mathbb{E}_t \{\pi_{T+k}\} = 0$ for all $t \leq T$ and $k \geq 1$, we can use the method of partial fractions to iterate the dynamic equation (A.1) forward until date $T$ and get

$$\pi_t = \mathbb{E}_t \left\{ \frac{Z^b_t}{(L^{-1} - \omega_b)(L^{-1} - \rho_b)} \right\}$$

$$= \frac{\mathbb{E}_t}{\omega_b - \rho_b} \left\{ \frac{\rho_b^{-1} Z^b_t}{1 - (\rho_b L)^{-1}} - \frac{\omega_b^{-1} Z^b_t}{1 - (\omega_b L)^{-1}} \right\}$$

$$= \frac{\mathbb{E}_t}{\omega_b - \rho_b} \left\{ \sum_{k=0}^{T-t} \left( \rho_b^{-k-1} - \omega_b^{-k-1} \right) Z^b_{t+k} \right\}.$$
Using the definition of $Z_t^i$, we then get

$$
\pi_t = \frac{\kappa E_t}{\beta (\omega_b - \rho_b)} \left\{ \frac{-1}{\sigma} \sum_{k=0}^{T-t} (\rho_b)^{k-1} - \omega_b^{k-1} \right\} (i_{t+k} - r_{t+k}) + \sum_{k=0}^{T-t} \left[ (1 - \rho_b) \rho_b^{k-1} + (\omega_b - 1) \omega_b^{k-1} \right] [(1 - \delta_g) g_{t+k} - \delta \varphi \varphi_{t+k}] \\
$$

which is Equation (4). Finally, using (4) to replace $\pi_t$ and $\pi_{t+1}$ in the Phillips curve (2), and using $\beta \rho_b \omega_b = 1$ and $P_b(\rho_b) = P_b(\omega_b) = 0$, we get

$$
y_t = \frac{\kappa E_t}{\beta \sigma (\omega_b - \rho_b)} \left\{ \frac{-1}{\sigma} \sum_{k=0}^{T-t} \left( \frac{\rho_b}{1 - \rho_b} + \frac{\omega_b}{\omega_b - 1} \right) \right\} (i_{t+k} - r_{t+k}) + \sum_{k=1}^{T-t} \left( \rho_b^{k-1} - \omega_b^{k-1} \right) [(1 - \delta_g) g_{t+k} - \delta \varphi \varphi_{t+k}] + g_t,
$$

which is Equation (5).

**Appendix B: Simple Model**

In this appendix, which complements Section 3 in the paper, we do the following: (i) we show formally that our simple model solves the paradox of flexibility; (ii) we provide additional numerical illustrations; (iii) we conduct a numerical sensitivity analysis; (iv) we further comment upon our selected equilibrium of the basic NK model; and (v) we show that our simple model delivers determinacy under exogenous monetary-policy instruments when time starts at a finite date (rather than at date $-\infty$).

**B.1 Resolution of the Paradox of Flexibility**

In this subsection, we formally show that our simple model solves the paradox of flexibility. Using the definition of $Z_t$, and after some simple algebra, we can rewrite (8) and (9) as

$$
\pi_t = -(1 - \rho) p_{t-1} + \frac{\kappa}{\beta (\omega_2 - \omega_1)} E_t \left\{ \frac{-1}{\sigma} \sum_{k=0}^{+\infty} (\omega_1^{k-1} - \omega_2^{k-1}) \right\} \left( i_{t+k} - r_{t+k} - \frac{M_{t+k}}{\chi_t} \right) - \sum_{k=0}^{+\infty} \left( \xi_1 \omega_1^{k-1} - \xi_2 \omega_2^{k-1} \right) g_{t+k} + \sum_{k=0}^{+\infty} \left( \xi_1 \omega_1^{k-1} - \xi_2 \omega_2^{k-1} \right) \delta \varphi \varphi_{t+k} \right\}, \quad (B.1)
$$

$$
y_t = -\theta p_{t-1} + g_t + \frac{E_t}{\beta (\omega_2 - \omega_1)} \left\{ \frac{1}{\sigma} \sum_{k=0}^{+\infty} \left( \xi_1 \omega_1^{k-1} - \xi_2 \omega_2^{k-1} \right) \right\} \left( i_{t+k} - r_{t+k} - \frac{M_{t+k}}{\chi_t} \right) + \sum_{k=0}^{+\infty} \left( \xi_1 \omega_1^{k-1} - \xi_2 \omega_2^{k-1} \right) g_{t+k} - \sum_{k=0}^{+\infty} \left( \xi_1 \omega_1^{k-1} - \xi_2 \omega_2^{k-1} \right) \varphi_{t+k} \right\}, \quad (B.2)
$$

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where \( \vartheta \equiv (1 - \rho)(1 - \beta \rho)/\kappa \) and

\[
\begin{align*}
\xi_j &\equiv \beta (\omega_j + \rho - 1) - 1, \\
\xi^g_j &\equiv \left(1 - \delta_g\right)(\omega_j - 1) + \frac{\delta_g \chi y}{\sigma \chi_i}, \\
\xi^\varphi_j &\equiv \delta_\varphi(\omega_j - 1) - \frac{\delta_\varphi \chi y}{\sigma \chi_i}
\end{align*}
\]

for \( j \in \{1, 2\} \).

The only parameter that depends on the degree of price stickiness \( \theta \) in the structural equations (1), (2), and (6) is the slope \( \kappa \) of the Phillips curve (2). We have \( \lim_{\theta \to 0} \kappa = +\infty \) and therefore

\[
\lim_{\theta \to 0} \left[-\frac{\beta_0 P(X)}{\kappa}\right] = X (X - \omega^\theta_1)
\]

for any \( X \in \mathbb{R} \), where \( \omega^\theta_1 \equiv (1 + \chi_i)/\chi_i > 1 \), which implies in turn that

\[
\lim_{\theta \to 0} \rho = 0, \quad \lim_{\theta \to 0} \omega_1 = \omega^\theta_1, \quad \text{and} \quad \lim_{\theta \to 0} \omega_2 = +\infty. \tag{B.3}
\]

Using (B.3) and

\[
(1 - \rho)(\omega_1 - 1)(\omega_2 - 1) = P (1) = \frac{\kappa}{\beta_0 \chi_i},
\]

we also get that

\[
\lim_{\theta \to 0} \frac{\kappa}{\omega_2} = \beta_0. \tag{B.4}
\]

Using (B.3) and (B.4), we can easily determine the limits of (B.1) and (B.2) as \( \theta \to 0 \):

\[
\begin{align*}
\lim_{\theta \to 0} \pi_t &= -\rho_{t-1} - \mathbb{E}_t \left\{ \sum_{k=0}^{+\infty} (\omega^\theta_1)^{-k-1} \left\{ t_{t+k}^m - r_{t+k} - \frac{M_{t+k}}{\chi_i} \right\} \right\} + \sigma (1 - \delta_g) g_t - \sigma \delta_\varphi \varphi_t, \\
\lim_{\theta \to 0} y_t &= \delta_g g_t + \delta_\varphi \varphi_t. \tag{B.5}
\end{align*}
\]

These limits are finite, unlike their counterparts in the basic NK model. Moreover, the right-hand side of (B.6) coincides with the value taken by \( y_t \) when prices are perfectly flexible \( (\theta = 0) \) in the basic NK model and, therefore, in our simple model as well.

Finally, we can use the IS equation (1) and the money-demand equation (6), in which we replace the output level \( y_t \) by its flexible-price value \( \delta_g g_t + \delta_\varphi \varphi_t \), to get the following dynamic equation under flexible prices:

\[
\begin{align*}
pt &= (\omega^\theta_1)^{-1} \mathbb{E}_t \{ pt_{t+1} \} - (\omega^\theta_1)^{-1} \left\{ t^m_t - r_t - \frac{M_t}{\chi_i} - \left[ \sigma (1 - \delta_g) - \frac{\chi y \delta_g}{\chi_i} \right] g_t \right\} \\
&\quad + \sigma (1 - \delta_g) \mathbb{E}_t \{ gt_{t+1} \} + \left( \sigma + \frac{\chi y}{\chi_i} \right) \delta_\varphi \varphi_t - \sigma \delta_\varphi \mathbb{E}_t \{ \varphi_{t+1} \}.
\end{align*}
\]
Iterating this equation forward to $+\infty$ leads to the following value for the price level $p_t$ in our simple model under flexible prices:

$$p_t = -\mathbb{E}_t \left\{ \sum_{k=0}^{+\infty} (\omega_1^n)^{-k-1} \left\{ t^{m_{t+k}} - r_{t+k} - \frac{M_{t+k}}{\chi_i} + \left[ \frac{\sigma (1 - \delta_g) + \chi y}{\chi_i} \right] g_{t+k} \right. \right. \right.$$

$$+ \left. \left. \left[ \frac{\left( \chi y - \sigma \right) \delta_f}{\chi_i} \right] \phi_{t+k} \right\} \right\} + \sigma \left( 1 - \delta_g \right) g_t - \sigma \delta_f \phi_t,$$

which implies in turn that the value of $\pi_t \equiv p_t - p_{t-1}$ in our simple model under flexible prices coincides with the right-hand side of (B.5). Thus, our simple model solves the paradox of flexibility: the limits of $\pi_t$ and $y_t$ as $\theta \to 0$ are finite and coincide with the values of $\pi_t$ and $y_t$ when $\theta = 0$.

### B.2 Additional Numerical Illustrations under our Benchmark Calibration

In Section 3 of the main text, Figure 1 illustrates numerically the effects of forward guidance (i.e. future policy-rate cuts) on inflation in our simple model, and compares these effects to the implications of the standard NK equilibrium. In this subsection of the online appendix, we illustrate and discuss the effects of forward guidance on output, and the effects of anticipated changes in fiscal policy on inflation and output – both in the equilibrium of our simple model and in the standard equilibrium of the basic NK model. We continue to use our benchmark calibration taken from Galí (2008, Chapter 3), which sets $\theta = 2/3$ (corresponding to “3-quarter price rigidity”); we also report the effects of cutting $\theta$ in half, step by step, to make prices more flexible.

Figure B.1 shows the effects of forward guidance on output in the two models. As before, our policy experiment is to cut the policy rate by 25 basis points (one percentage point per annum) in Quarter $T$, and we display the effects on output in Quarter 1 (when the rate cut is announced). The right panel in Figure B.1 replicates the implausible implications of the basic NK model. The left panel shows that our model does not share these implications. The rate cut has small effects on output (less than 0.2 percent of steady-state output to begin with), and the effects die off quickly as we delay the rate cut. Moreover, these effects decline smoothly as we make prices more flexible; they converge to the flexible-price ($\theta = 0$) effects.

To analyze the effects of fiscal policy, we add government purchases to Galí’s calibration. We set the share of government purchases in output to 0.3 in the steady state. We follow Galí’s calibration for the structural parameters (like the intertemporal elasticity of substitution) and adjust the reduced-form parameters (like the coefficient $1/\sigma$ on the real interest rate in the IS equation) to reflect the introduction of government purchases. Our policy experiment is an increase in government purchases, amounting to one percent of steady-state output, occurring (only) in Quarter $T$ and announced in Quarter 1. Figures B.2 and B.3 display the effects on inflation and output in Quarter 1. Once again, the comparison between the left and the
Figure B.1 – Effect of a policy-rate cut at date $T$ on output at date 1

![Figure B.1](image)

Note: The figure displays the effect on $y_1$ of announcing at date 1 a one-percentage-point-per annum cut in $i^m_T$ (for the simple model) or $i_T$ (for the basic NK model) for $T \in \{1, \ldots, 20\}$. Parameter values are the same as for Figure 1 in the main text. More specifically, benchmark parameter values are set as in Galí (2008, Chapter 3): $\beta = 0.99$, $\sigma = 1$, $\chi_y = 1$, $\chi_i = 4$, and $\kappa = \lambda[(1 - \theta)(1 - \beta \theta)/\theta] = 0.13$, where $\lambda = 3/4$ and $\theta = \theta^* \equiv 2/3$. As $\theta$ takes the values $\theta^*/2$, $\theta^*/4$, $\theta^*/8$, and $\theta^*/16$, $\kappa$ takes respectively the values 1.00, 3.13, 7.57, and 16.54.

right panels shows that our model’s equilibrium does not share the puzzling implications of the standard NK equilibrium: the effects of anticipated fiscal policy die out as we delay the policy intervention, and they converge to the flexible-price values as we make prices more and more flexible.

Another notable difference between our model’s equilibrium and the standard NK equilibrium under our benchmark calibration is that anticipated fiscal expansions have a contractionary effect on output in our model.\(^1\) Several contributions (e.g., Christiano et al., 2011) suggest that anticipated fiscal expansions can have large positive output multipliers at the ZLB according to the basic NK model. The right-hand panel of Figure B.3 confirms this implication of the basic NK model. This implication arises from a feedback loop described in Farhi and Werning (2016), which works back in time via the IS equation and the Phillips curve: given that $\pi_{T+1} = y_{T+1} = 0$, a fiscal expansion at date $T$ raises inflation at date $T$, which lowers the real interest rate at date $T - 1$, which raises output and inflation at date $T - 1$, and so on. This feedback loop is also present in our model, but $\pi_{T+1}$ and $y_{T+1}$ are endogenously determined when the fiscal expansion is announced. As a result, expected future fiscal expansions can reduce current output in our model (as is the case under the calibration we use for Figure B.3). Intuitively, these contractionary effects of anticipated fiscal expansions may come from wealth effects that also arise in standard Real-Business-Cycle models: consumers realize that the future fiscal expansion reduces their permanent income, and they respond by lowering current consumption.

\(^1\)Our analytical derivations in the main text show that this is always the case when we use our model to go to the basic-NK-model limit. Figure B.3 makes the point numerically under our benchmark calibration, without taking the model to the basic-NK-model limit.
Figure B.2 – Effect of government purchases at date $T$ on inflation at date 1

Note: The figure displays the effect on $\pi_1$ of announcing at date 1 a one-percent-of-steady-state-output increase in $g_T$ for $T \in \{2, \ldots, 20\}$. The steady-state share of government purchases in output is set to 0.3, and benchmark structural-parameter values are set as in Galí (2008, Chapter 3), implying $\beta = 0.99, \sigma = 1.43, \delta_g = 0.42, \chi_y = 1, \chi_i = 4$, and $\kappa = \lambda[(1-\theta)(1-\beta\theta)]/\theta = 0.15$, where $\lambda = 0.86$ and $\theta = \theta^* \equiv 2/3$. As $\theta$ takes the values $\theta^*/2$, $\theta^*/4$, $\theta^*/8$, and $\theta^*/16$, $\kappa$ takes respectively the values 1.15, 3.58, 8.65, and 18.90.

Figure B.3 – Effect of government purchases at date $T$ on output at date 1

Note: The figure displays the effect on $y_1$ of announcing at date 1 a one-percent-of-steady-state-output increase in $g_T$ for $T \in \{2, \ldots, 20\}$. Parameter values are the same as for Figure B.2 above.
The effects of anticipated fiscal expansions on inflation may be dominated either by the wealth effect we mention above (which is deflationary) or by an inflationary effect that we can trace back to staggered price setting. The latter effect arises because the fiscal expansion is expected to raise prices in the future, and this motivates current price setters to set higher prices too. Under our benchmark calibration, the effects of anticipated fiscal expansions on inflation (displayed in the left panel of Figure B.2, for various quarters $T$ and price-stickiness degrees $\theta$) are small, and mostly negative.

B.3 Numerical Sensitivity Analysis

The quantitative impressions conveyed by Figure 1 in the main text and Figures B.1-B.3 in the preceding subsection are not particularly sensitive to Galí’s (2008, Chapter 3) choices about the parameters of the basic NK model, nor to his (standard) assumption of a unitary income elasticity of money demand. The value taken by the interest semi-elasticity of money demand $\chi_i$, however, does matter for the quantitative impression conveyed by our results for the effects of forward guidance. The value of $\chi_i$ affects both the magnitude and the persistence of the effects of future changes in the IOR rate on current inflation and output. Our choice of $\chi_i = 4$, following Galí, represents a middle-of-the-range value compared to estimates that we could take from the empirical literature on money demand.

Semi-log specifications of money demand typically yield small estimates of $\chi_i$ based on US data. The estimates in Stock and Watson (1993) and Cochrane (2017b), for example, suggest semi-elasticities close to $-0.1$ on an annual basis. Given the quarterly frequency of our model, these estimates correspond to $\chi_i = 0.4$ (one order of magnitude smaller than the value we use for Figure 1). By contrast, log-log specifications of money demand, estimated on US or cross-country data, suggest interest elasticities around $-1/4$ (e.g., Teles and Zhou, 2005) or $-1/3$ (e.g., Teles et al., 2016). If we set the opportunity cost of holding money to one percent per quarter, an elasticity of $-1/3$ implies $\chi_i = 33$ (one order of magnitude larger than the value we use for Figure 1). Figure B.4 shows how the quantitative effects of forward guidance on inflation vary when we set $\chi_i$ to $0.4$ or $33$.

The policy experiment and the parameter values (other than the value of $\chi_i$) used for Figure B.4 are the same as earlier for Figures 1 and B.1. The right panel in Figure B.4 replicates the implausible implications of the basic NK model. The left panel shows the results for our simple model with $\chi_i = 0.4$, and the middle panel shows the results with $\chi_i = 33$. The left panel suggests that the inflationary effects of anticipated IOR-rate cuts are tiny (below 3 basis points to begin with, and dying off quickly). The middle panel suggests that forward guidance has a sizable and more persistent effect on inflation (announcing that the IOR rate will be cut by one percentage point in 20 quarters raises current inflation by 17 basis points).

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Ball’s (2001) estimate of $-0.05$ is even closer to zero.
Figure B.4 – Effect of a policy-rate cut at date $T$ on inflation at date 1 for alternative values of $\chi_i$

Simple model with $\chi_i = 0.4$

Simple model with $\chi_i = 33$

Basic NK model

Note: The figure displays the effect on $\pi_1$ of announcing at date 1 a one-percentage-point-per-annum cut in $i_T^m$ (for the simple model) or $i_T$ (for the basic NK model) for $T \in \{1, \ldots, 20\}$. Parameter values (except the value of $\chi_i$) are the same as for Figure 1 in the main text and Figure B.1 above.

Figure B.5 – Effect of a policy-rate cut at date $T$ on output at date 1 for alternative values of $\chi_i$

Simple model with $\chi_i = 0.4$

Simple model with $\chi_i = 33$

Basic NK model

Note: The figure displays the effect on $y_1$ of announcing at date 1 a one-percentage-point-per-annum cut in $i_T^m$ (for the simple model) or $i_T$ (for the basic NK model) for $T \in \{1, \ldots, 20\}$. Parameter values (except the value of $\chi_i$) are the same as for Figure 1 in the main text and Figures B.1 and B.4 above.
Of course, both the left and middle panels of Figure B.4 also illustrate the analytical results that are the main focus of our paper: the inflationary effects go to zero as we cut the IOR rate in the more distant future \( (T \to +\infty) \), and they converge to the flexible-price effects as we make prices more flexible \( (\theta \to 0) \). Whatever the calibration, our simple model exhibits neither the forward-guidance puzzle nor the paradox of flexibility.

Figure B.5 shows the effects of forward guidance on output (under the same policy experiment and parameter values we describe above). Again, the quantitative impressions we get are sensitive to the value of the semi-elasticity \( \chi_i \). The effects are tiny if we set \( \chi_i = 0.4 \), but more noteworthy and persistent if we set \( \chi_i = 33 \).

We have relegated this numerical sensitivity analysis to the online appendix because our simple model is not really suitable for a quantitative assessment of the effects that one may associate with forward-guidance policies. Nonetheless, we suspect that the sensitivity of quantitative results to the specification of money demand may also be present in richer (more quantitative) models. So, we suspect that the unsettled state of empirical research on money demand may hinder sharp answers to interesting policy questions in this context.

### B.4 Selected Equilibrium of the Basic NK Model

In the main text, we use our simple model to select uniquely a particular equilibrium of the basic NK model under a permanently exogenous policy rate, and we discuss the main properties of this equilibrium. In this subsection, we extend our discussion of this equilibrium in three directions.

First, we briefly specify how we derive Equations (10) and (11), which characterize this equilibrium. As \( \chi_i \to +\infty \), we have \( \mathcal{P}(X) \to (X-1)\mathcal{P}_b(X) \) for any \( X \in \mathbb{R} \), where \( \mathcal{P}(X) \) is defined in the Appendix of the paper. Therefore, we get \( \lim_{\chi_i \to +\infty}(\rho, \omega_1, \omega_2) = (\rho_b, 1, \omega_b) \). Using this result, \( \beta \rho_b \omega_b = 1 \), \( \rho_b + \omega_b = 1 + 1/\beta + \kappa/(\beta \sigma) \), and \( \mathcal{P}_b(\rho_b) = 0 \), we then easily obtain that the limits of (B.1) and (B.2) as \( \chi_i \to +\infty \) are (10) and (11).

Second, we relate the fact that our selected equilibrium exhibits an attenuated form of the forward-guidance puzzle to price-level stationarity. More generally, we show that any equilibrium of the basic NK model under exogenous policy rates in which the price level is stationary in response to temporary policy-rate shocks (in particular, our selected equilibrium) may attenuate but cannot fully solve the forward-guidance puzzle. To see this, note that our simple model implies price-level stationarity in response to a one-off IOR-rate change \( (p_\infty = p_0 \text{ when } i^T_m = i^* \text{ and } i^m_t = 0 \text{ for } t \geq 1 \text{ and } t \neq T) \). Therefore, in our selected equilibrium of the basic NK model, the price level is also stationary in response to a one-off interest-rate change \( (p_\infty = p_0 \text{ when } i_T = i^* \text{ and } i_t = 0 \text{ for } t \geq 1 \text{ and } t \neq T) \). Now, iterating the IS equation (1) forward to \( +\infty \) under this interest-rate change at date \( T \), using price-level stationarity, and using the terminal
condition \( y_\infty = 0 \), leads to

\[ y_1 = -\frac{i^*}{\sigma} - \frac{\pi_1}{\sigma}. \]

This relationship is consistent with \( y_1 \) and \( \pi_1 \) converging towards non-zero finite values as \( T \to +\infty \) (puzzle attenuation), but inconsistent with \( y_1 \) and \( \pi_1 \) both converging towards zero (puzzle resolution). Thus, any equilibrium of the basic NK model in which the price level is stationary in response to a temporary interest-rate change may attenuate but cannot fully solve the forward-guidance puzzle.

Third, we briefly compare our selected equilibrium with some equilibria studied in the literature. Cochrane (2017a) characterizes the set of all equilibrium paths, in the basic NK model, with \( i_t = i^* \) for \( 1 \leq t \leq T \) and \( i_t = 0 \) for \( t \geq T + 1 \). He shows that any of these paths can be obtained as the unique local equilibrium under a temporary interest-rate peg followed by a suitably designed interest-rate rule. He considers a “local-to-frictionless” equilibrium-selection criterion, which requires that equilibrium outcomes converge towards flexible-price equilibrium outcomes as prices become more and more flexible. This criterion does not select a unique equilibrium, but rules out some equilibria (including the standard NK equilibrium). Our selected equilibrium satisfies this criterion, since it does not exhibit the paradox of flexibility.

Among the equilibria satisfying the local-to-frictionless criterion, Cochrane (2017a) describes more specifically two particular equilibria that do not exhibit any of the puzzles and paradoxes: (i) the “backward-stable” equilibrium, in which inflation goes to zero backward in time (\( \lim_{t \to -\infty} \pi_t = 0 \)) when the interest-rate peg between 1 and \( T \) is announced at date \( -\infty \); and (ii) the “no-inflation-jump” equilibrium, in which inflation is zero at the start of the peg (\( \pi_1 = 0 \)). Two differences between these equilibria and our selected equilibrium are worth emphasizing. First, unlike these equilibria, our equilibrium still exhibits (a weak form of) the forward-guidance puzzle. Second, at the start of a liquidity trap, inflation is negative in our equilibrium (\( \partial \pi_1 / \partial i^* < 0 \)), while it is positive in the backward-stable equilibrium (\( \partial \pi_1 / \partial i^* > 0 \)) and, by construction, zero in the no-inflation-jump equilibrium (\( \partial \pi_1 / \partial i^* = 0 \)).

Another interesting parallel is between our selected equilibrium of the basic NK model and the equilibrium of Mankiw and Reis’s (2002) sticky-information model. Carlstrom, Fuerst, and Paustian (2015) and Kiley (2016) show that Mankiw and Reis’s (2002) model solves the fiscal-multiplier puzzle, the paradox of flexibility, and the paradox of toil, and attenuates the forward-guidance puzzle — exactly like the basic NK model with our selected equilibrium. Thus, our selected equilibrium brings the canonical sticky-price model at par with its sticky-information cousin in terms of their ability to solve or attenuate all four NK puzzles and paradoxes. Kiley (2016) also points out that Mankiw and Reis’s (2002) model implies price-level stationarity when

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Note: The result \( \partial \pi_1 / \partial i^* > 0 \) in the backward-stable equilibrium is straightforwardly obtained from Cochrane’s (2017a) Equation (34) by setting \( C = 0 \) and \( t = T_l \); the result \( \partial \pi_1 / \partial i^* < 0 \) in our equilibrium is straightforwardly obtained from (10).

Note: Of course, the paradox of flexibility solved by Mankiw and Reis’s (2002) model is about the effects of information flexibility, not price flexibility.
the central bank follows a non-inertial interest-rate rule after the temporary interest-rate peg; in this case, we can attribute the inability of that model to (fully) solve the forward-guidance puzzle to price-level stationarity, for the same reason as in the basic NK model with our selected equilibrium.

B.5 Local-Equilibrium Determinacy With an Initial Date

In this subsection, we assume that time starts at date 0, rather than date \(-\infty\). We cannot assume that prices are sticky \((\theta > 0)\) at date 0, since there is no date \(-1\). Instead, we assume that prices are flexible at date 0 and sticky afterwards. We show that, under this assumption, our simple model still delivers local-equilibrium determinacy under exogenous monetary-policy instruments. Whether it delivers determinacy or not does not depend on the presence or the absence of exogenous shocks; so, for simplicity and without any loss in generality, we focus on the case in which there are no exogenous shocks \((\bar{z}_t^m = \bar{m}_t = g_t = \bar{r}_t = \bar{\varphi}_t = 0 \text{ for } t \geq 0)\).

From date 1 onwards, the economy is characterized by the same equilibrium conditions as previously, so that \(\pi_t\) and \(y_t\) for \(t \geq 1\) are given by (8) and (9) with \(Z_{t+k} = 0\) for all \(k \geq 0\). Applying the operator \(E_0\{\cdot\}\) to the left- and right-hand sides of these two equations taken at date 1, we get

\[
E_0\{\pi_1\} = -(1 - \rho)p_0, \quad (B.7)
\]
\[
E_0\{y_1\} = -\varphi p_0. \quad (B.8)
\]

At date 0, the IS equation (1) and the money-demand equation (6) still hold, and give

\[
y_0 = E_0\{y_1\} - \frac{1}{\sigma}(i_0 - E_0\{\pi_1\}), \quad (B.9)
\]
\[
p_0 = -\chi_y y_0 + \chi_i i_0. \quad (B.10)
\]

The Philips curve (2), however, no longer holds at date 0, both because there is no date \(-1\) and because prices are flexible at date 0. The following lemma indicates what becomes of the date-0 Phillips curve:

Lemma 1 (Phillips Curve at Date 0 in the Simple Model): When prices are flexible at date 0 and sticky afterwards, the Phillips curve at date 0 is

\[
0 = \beta E_0\{\pi_1\} + \kappa y_0 \quad (B.11)
\]

in the simple model in the absence of exogenous shocks.


This date-0 Phillips curve is identical to the equation that would be obtained by replacing \(\pi_0\) by 0 in the Phillips curve (2) taken at date 0. The reason is simple. At any date \(t \geq 1\), we
have $\pi_t = (1 - \theta)(p_t^* - p_{t-1})$ and $p_t = \theta p_{t-1} + (1 - \theta)p_t^*$, where $p_t^*$ denotes the newly set price at date $t$, so that $\pi_t = \frac{(1-\theta)^2}{\theta}(p_t^* - p_t)$: inflation differs from zero if and only if the newly set price differs from the average price. The Phillips curve (2) at dates $t \geq 1$ can thus be rewritten as $(1-\theta)^2(p_t^* - p_t) = \beta \mathbb{E}_t \{ \pi_{t+1} \} + \kappa y_t$: the newly set price departs from the average price by a factor proportional to expected future inflation and the current output level. This equilibrium condition, written this way, still holds at date 0; only, the newly set price cannot depart from the average price at that date, since all prices are set at that date, so that the equilibrium condition at date 0 becomes (B.11).

The equilibrium conditions (B.7)-(B.11) are five linear equations in the five unknowns $p_0$, $y_0$, $i_0$, $\mathbb{E}_0 \{ \pi_1 \}$, and $\mathbb{E}_0 \{ y_1 \}$, without any forcing term. They imply

$$\Theta p_0 = 0,$$

where

$$\Theta \equiv \left[ 1 + \beta (1 - \rho) + \frac{\kappa}{\sigma} + \beta \lambda y \right] (1 - \rho) + \frac{\kappa}{\sigma \chi_i} > 0.$$

Since $\Theta \neq 0$, we get $p_0 = 0$: the initial price level is uniquely pinned down. Using (B.7) and (B.11), we then obtain a unique solution for $y_0 (y_0 = 0)$. Using (8) and (9) recursively for $t \geq 1$, we also obtain a unique solution for $\pi_t$ and $y_t$ for $t \geq 1$ ($\pi_t = y_t = 0$). We conclude that the model delivers local-equilibrium determinacy.

### B.6 Proof of Lemma 1

In this subsection, we prove Lemma 1. We use capital letters to denote nominal variables and lower-case letters to denote real variables. Variables without time subscript denote steady-state values, and variables with hats denote log-deviations from steady-state values.

At date 0, firm $i$ chooses its price $P_0^*(i)$ to maximize

$$\mathbb{E}_0 \left\{ \sum_{t=0}^{+\infty} (\beta \theta)^t \frac{\lambda_t P_0}{\lambda_0 P_t} \left[ P_0^*(i) y_{t|0} (i) - W_t f^{-1} (y_{t|0} (i)) \right] \right\}$$

subject to

$$y_{t|0} (i) = \left[ \frac{P_0^* (i)}{P_t} \right]^{-\varepsilon} y_t,$$

where $\lambda_t$ denotes the marginal utility of consumption at date $t$, $P_t$ the aggregate price index at date $t$, $W_t$ the nominal wage at date $t$, $y_t$ the aggregate output level at date $t$, $\beta$ the discount factor, $\theta$ the probability for a firm not to be able to change its price at any given date (except date 0), $f$ the production function, $\varepsilon$ the elasticity of substitution between differentiated goods, and $y_{t|0} (i)$ the output level of firm $i$ at date $t$ in the case where firm $i$ did last set its price at date 0. The first-order condition is

$$\mathbb{E}_0 \left\{ \sum_{t=0}^{+\infty} (\beta \theta)^t \frac{\lambda_t P_0}{\lambda_0 P_t} \left[ P_0^*(i) - \left( \frac{\varepsilon}{\varepsilon - 1} \right) \frac{P_t w_t}{f' (y_{t|0} (i))} \right] y_{t|0} (t) \right\} = 0.$$
where \( w_t \) denotes the real wage at date \( t \) and \( h_{t|0}(i) \) denotes hours worked in firm \( i \) at date \( t \) in the case where firm \( i \) did last set its price at date \( 0 \). In a symmetric equilibrium, all firms set the same price: \( P^*_0(i) = P^*_0 \), implying that \( y_{t|0}(i) = y_{t|0} \) and \( h_{t|0}(i) = h_{t|0} \). Therefore, the first-order condition becomes

\[
E_0 \left\{ +\infty \sum_{t=0}^{\infty} (\beta \theta)^t \frac{\lambda t P_0}{\lambda t P^*_0} \left[ P^*_0 - \left( \frac{\epsilon}{\epsilon - 1} \frac{P_t w_t}{f'(h_{t|0})} \right) y_{t|0} \right] \right\} = 0.
\]

Log-linearizing this equation around the unique steady state leads to

\[
\hat{P}^*_0 = (1 - \beta \theta) E_0 \left\{ +\infty \sum_{t=0}^{\infty} (\beta \theta)^t \left( \hat{w}_t + \hat{P}_t - \hat{m}p_{t|0} \right) \right\}
\]

\[
= (1 - \beta \theta) \left( \hat{w}_0 + \hat{P}_0 - \hat{m}p_{0|0} \right) + \beta \theta E_0 \left\{ \hat{P}^*_1 \right\}
\]

\[
+ \beta \theta (1 - \beta \theta) E_0 \left\{ +\infty \sum_{t=1}^{\infty} (\beta \theta)^{t-1} \left( \hat{m}p_{t|1} - \hat{m}p_{t|0} \right) \right\},
\]

(B.12)

where \( P^*_1 \) denotes the newly set price at date 1 and \( mp_{t|\tau} \) for \( 0 \leq \tau \leq t \) denotes the marginal productivity at date \( t \) of the firms that did last (re)set their price at date \( \tau \).

Let us rewrite in turn the terms \( \hat{P}^*_0, \hat{P}^*_1, \hat{w}_0, \hat{m}p_{0|0}, \) and \( \hat{m}p_{t|1} - \hat{m}p_{t|0} \) featuring in (B.12) as functions of \( \hat{y}_0, \hat{P}_0, \) and \( \hat{P}_1 \). First, since all firms set their price at date 0, we have

\[
\hat{P}^*_0 = \hat{P}_0.
\]

Second, we can log-linearize the aggregate price index at date 1 to get

\[
\hat{P}_1 = (1 - \theta) \hat{P}^*_1 + \theta \hat{P}_0.
\]

Third, to rewrite \( \hat{w}_0 \), we log-linearize the production function \( y_t = f(h_t) \), the goods-market-clearing condition \( y_t = c_t + g \), and the intra-temporal first-order condition of households’ optimization problem \( w_t = v'(h_t)/u'(c_t) \):

\[
\hat{y}_t = \frac{f' h_t}{f} \hat{h}_t,
\]

(B.15)

\[
\hat{y}_t = \frac{c_t}{y} \hat{c}_t,
\]

(B.16)

\[
\hat{w}_t = - \frac{u'' c_t}{u'} \hat{c}_t + \frac{v'' h_t}{v'} \hat{h}_t,
\]

(B.17)

where \( c_t \) denotes consumption at date \( t \), \( g \) government purchases (constant by assumption), \( h_t \) aggregate hours worked at date \( t \), \( u \) the consumption-utility function, and \( v \) the labor-disutility function. Using (B.15) and (B.16), we can then rewrite (B.17) at date 0 as

\[
\hat{w}_0 = \left( \frac{u'' y}{u'} + \frac{v'' h}{v'} \frac{f}{f h} \right) \hat{g}_0.
\]

(B.18)

Fourth, to rewrite \( \hat{m}p_{0|0} \), we use the production function \( y_t = f(h_t) \) to get

\[
\hat{m}p_t = \frac{f f''}{(f')^2} \hat{y}_t.
\]

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where $mp_t$ denotes the average marginal productivity at date $t$. Since all firms set the same price at date 0, we then get
\[
\hat{mp}_{0|0} = \hat{mp}_0 = \frac{ff''}{(f')^2} \hat{y}_0. \tag{B.19}
\]
And fifth, we rewrite $\hat{mp}_{1|1} - \hat{mp}_{t|0}$ as
\[
\hat{mp}_{1|1} - \hat{mp}_{t|0} = \frac{ff''}{(f')^2} (\hat{y}_{t|1} - \hat{y}_{t|0}) = \frac{ff''}{(f')^2} \left[ (\hat{y}_{t|1} - \hat{y}_t) - (\hat{y}_{t|0} - \hat{y}_t) \right] = \frac{-\varepsilon ff''}{(f')^2} \left( \hat{P}_t - \hat{P}_t^* \right)
\]
\[
= \frac{-\varepsilon ff''}{(1 - \theta) (f')^2} \left( \hat{P}_t - \hat{P}_0 \right), \tag{B.20}
\]
where $P_t^*$ denotes the price set by price-(re)setting firms at date $t$, and where the last equality is obtained using (B.13) and (B.14).

Using (B.13)-(B.14), (B.18)-(B.20), and $\pi_1 = \hat{P}_1 - \hat{P}_0$, we can rewrite (B.12) as
\[
0 = \beta \mathbb{E}_0 \{ \pi_1 \} + \kappa \hat{y}_0.
\]
where
\[
\kappa \equiv \frac{(1 - \theta) (1 - \beta \theta)}{\theta} \left[ \frac{w''y}{u'} + \frac{v''h}{v'} f' \right] \frac{ff''}{(f')^2} > 0.
\]
Equation (B.11) corresponds to this equation in which, for simplicity and consistency with the main text, the notation $\hat{y}_0$ is replaced by the notation $y_0$.

### Appendix C: Model With Banks – Presentation

In this appendix, we present our model with banks, which is used in Sections 4 and 5 of the paper. In this model, monopolistic firms use labor to produce goods. They need to pay the wage bill (or some fraction of it) before they can produce and sell their output. They borrow the corresponding amount from banks. Banks incur costs making loans; and holding reserves mitigates these costs. The central bank sets both the interest rate on bank reserves and the quantity of bank reserves. The model is partly non-parametric, as we do not specify any functional form for the utility and production functions, in order to broaden the scope of our results.

We begin by highlighting the equilibrium conditions that provide the main intuition for our analysis. These conditions arise from the optimization problem of households operating banks. Then we describe the optimization problem of firms, our specification of the public sector, and the market-clearing conditions. In the last subsections, we go back to households’ optimization problem and articulate the foundations of this problem. These foundations will serve us in formal proofs in the rest of the online appendix. Readers who are not interested in the technical aspects of our proofs may skip the last subsections.
C.1 Households (Reduced-Form Setup)

Each household consists of production workers and bankers. In this first subsection, we start from households’ reduced-form utility function, whose arguments are consumption \((c_t)\), hours worked by production workers \((h_t)\), real loans \((\ell_t)\), and real reserves \((m_t)\):

\[
U_t = \mathbb{E}_t \left\{ \sum_{k=0}^{\infty} \beta^k \left[ u(c_{t+k}) - v(h_{t+k}) - \frac{\Gamma(\ell_{t+k}, m_{t+k})}{\varphi_{1,t+k}} \right] \right\}, \tag{C.1}
\]

where \(\beta \in (0, 1)\). The consumption-utility function \(u\) and the labor-disutility function \(v\) are assumed to satisfy standard minimal conditions that we specify in Subsection C.5. The term \(-\Gamma(\ell_t, m_t)\) comes from households acting as bankers. In Subsection C.5, we will articulate how bankers produce loans using reserves and their own labor effort as inputs. In the current subsection, we take a lighter approach to convey intuition: we simply work with the implied utility cost of making loans \(\Gamma(\ell_t, m_t)\), reflecting the bankers’ disutility from work. The derivations in Subsection C.5 imply that the cost of banking is increasing in loans \((\Gamma_{\ell} > 0)\) and decreasing in reserves \((\Gamma_m < 0)\). Finally, the stochastic exogenous labor-disutility shock \(\varphi_{1,t}\), of mean one, is the first of the four alternative supply shocks that we consider. We assume that this shock affects both production workers and bankers, but none of our results would be affected in any substantial way if we assumed instead that it affects only production workers.\(^5\)

In addition to making loans and holding reserve balances at the central bank, households hold bonds \(b_t\) (or issue bonds when \(b_t < 0\)), which serve only as a store of value. Loans, reserves, and bonds are one-period non-contingent assets. We let \(I_{\ell}^t\), \(I_m^t\), and \(I_t\) denote the corresponding gross nominal interest rates. We let \(P_t\) denote the price level, and \(\Pi_t \equiv P_t / P_{t-1}\) the gross inflation rate. The household budget constraint, expressed in real terms, is then

\[
c_t + b_t + \ell_t + m_t \leq \frac{I_{\ell}^{t-1}}{\Pi_t} b_{t-1} + \frac{I_{\ell}^t}{\Pi_t} \ell_{t-1} + \frac{I_m^{t-1}}{\Pi_t} m_{t-1} + w_t h_t + \tau_t, \tag{C.2}
\]

where \(w_t\) represents the real wage and \(\tau_t\) captures firm profits and the government’s lump-sum taxes or transfers.

Households choose \(b_t, c_t, h_t, \ell_t,\) and \(m_t\) to maximize their reduced-form utility function (C.1) subject to their budget constraint (C.2), taking all prices \((I_t, I_{\ell}^t, I_m^t, P_t,\) and \(w_t\)) as given. Letting \(\lambda_t\) denote the Lagrange multiplier on the period-\(t\) budget constraint, the first-order conditions of households’ optimization problem are

\[
\lambda_t = u'(c_t), \tag{C.3}
\]

\[
\lambda_t = \beta I_t \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\Pi_{t+1}} \right\}, \tag{C.4}
\]

\[
\lambda_t w_t = \frac{v'(h_t)}{\varphi_{1,t}}, \tag{C.5}
\]

\(^5\)In particular, the equilibrium of the basic NK model that we would select under this alternative assumption would be exactly the same, so that our resolution of the paradox of toil would be unchanged.
\[
\frac{\Gamma_\ell (\ell_t, m_t)}{\varphi_{1,t}} + \lambda_t = \beta I^\ell_t \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\Pi_{t+1}} \right\}, \\
\frac{\Gamma_m (\ell_t, m_t)}{\varphi_{1,t}} + \lambda_t = \beta I^m_t \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\Pi_{t+1}} \right\}.
\]

Using (C.4), we can rewrite the last two conditions as
\[
\frac{I^\ell_t}{I_t} = 1 + \frac{\Gamma_\ell (\ell_t, m_t)}{\varphi_{1,t} \lambda_t}, \quad \text{(C.6)} \\
\frac{I^m_t}{I_t} = 1 + \frac{\Gamma_m (\ell_t, m_t)}{\varphi_{1,t} \lambda_t}. \quad \text{(C.7)}
\]

Condition (C.6) implies that loans pay more interest than bonds, because the marginal banking cost is positive ($\Gamma_\ell > 0$). Condition (C.7) implies that reserves pay less interest than bonds, because they serve to reduce banking costs ($\Gamma_m < 0$). Assuming that households’ optimization problem is also subject to a standard no-Ponzi-game condition, the transversality condition is
\[
\lim_{k \to +\infty} \mathbb{E}_t \left\{ \beta^{t+k} \lambda_{t+k} a_{t+k} \right\} = 0, \quad \text{(C.8)}
\]
where $a_t = b_t + \ell_t + m_t$ denotes households’ total assets. The second-order conditions of households’ optimization problem are met under the assumptions we articulate in Subsection C.5.

### C.2 Firms

There is a continuum of monopolistically competitive firms owned by households and indexed by $i \in [0, 1]$. Each firm $i$ uses $h_t(i)$ units of labor to produce
\[
y_t (i) = \varphi_{2,t} f [h_t (i)] \quad \text{(C.9)}
\]
units of output. The production function $f$, defined over $\mathbb{R}_{\geq 0}$, is twice differentiable, strictly increasing ($f’ > 0$), weakly concave ($f'' \leq 0$), and such that $f(0) = 0$. The stochastic exogenous technology shock $\varphi_{2,t}$, of mean one, is the second of the four alternative supply shocks that we consider. The third supply shock that we consider, $\varphi_{3,t}$, also of mean one, captures a labor subsidy received by firms (when $\varphi_{3,t} > 1$) or labor tax paid by firms (when $\varphi_{3,t} < 1$): if $W_t$ denotes the pre-subsidy or pre-tax nominal wage, then the after-subsidy or after-tax nominal wage paid by firms is $W_t/\varphi_{3,t}$. To generate a demand for bank loans, we assume that firm $i$ has to borrow a fraction $\phi \in (0, 1]$ of its after-subsidy or after-tax nominal wage bill $W_t h_t(i)/\varphi_{3,t}$ from banks, at the gross nominal interest rate $I^\ell_t$, before it can produce and sell its output. Thus, the nominal value of firm $i$’s loan $L_t(i)$ must satisfy
\[
\frac{\phi W_t h_t (i)}{\varphi_{3,t}} \leq L_t (i). \quad \text{(C.10)}
\]

Following Calvo (1983), we assume that each firm, whatever its history, has the probability $\theta \in [0, 1)$ not to be allowed to reset its price at any date. If allowed to reset its price at date $t$, the...
firm $i$ chooses its new price $P_t^*(i)$ to maximize the current market value of the profits that this price will generate:

$$
\mathbb{E}_t \left\{ \sum_{k=0}^{+\infty} (\beta \theta)^k \frac{\lambda_{t+k}}{\lambda_t \Pi_{t+k}} \left[ P_t^* (i) y_{t+k} (i) - \frac{\beta \lambda_{t+k+1} I_{t+k}^f}{\lambda_{t+k} \Pi_{t+k+1}} - [W_{t+k} h_{t+k} (i) - L_{t+k} (i)] \right] \right\},
$$

subject to the production function (C.9), the borrowing constraint (C.10), and the demand schedule

$$
y_{t+k} (i) = \left[ \frac{P_t^* (i)}{P_{t+k}} \right]^{-\varepsilon \phi_{4,t+k}} y_{t+k},
$$

where $\Pi_{t+k} \equiv P_{t+k}/P_t$ for any $k \in \mathbb{N}$, $\varepsilon > 0$ denotes the steady-state elasticity of substitution between differentiated goods, and $y_t \equiv \int_0^1 y_t (i)^{\phi_{4,t-1} / \phi_{4,t}} \mathrm{d} i]^{\phi_{4,t} / (\phi_{4,t} - 1)}$. The stochastic exogenous shock $\varphi_{4,t}$, of mean one, affecting the elasticity of substitution between differentiated goods is the last of the four alternative supply shocks that we consider.

Since households’ first-order condition (C.6) implies $I_t^f > I_t$, firms’ borrowing constraint (C.10) is binding. Therefore, using (C.9) and (C.10) holding with equality, we can rewrite the current market value of the profits generated by $P_t^*(i)$ as

$$
\mathbb{E}_t \left\{ \sum_{k=0}^{+\infty} (\beta \theta)^k \frac{\lambda_{t+k}}{\lambda_t \Pi_{t+k}} \left[ P_t^* (i) y_{t+k} (i) - \left( \phi \frac{\beta \lambda_{t+k+1} I_{t+k}^f}{\lambda_{t+k} \Pi_{t+k+1}} + (1 - \phi) \right) \frac{W_{t+k}}{\varphi_{3,t+k} f'} [h_{t+k} (i)] \right] \right\}.
$$

Choosing $P_t^*(i)$ to maximize this market value subject to (C.11) leads to the first-order condition

$$
\mathbb{E}_t \left\{ \sum_{k=0}^{+\infty} (\beta \theta)^k \frac{\lambda_{t+k}}{\lambda_t \Pi_{t+k}} \left[ P_t^* (i) - \left( \phi \frac{\beta \lambda_{t+k+1} I_{t+k}^f}{\lambda_{t+k} \Pi_{t+k+1}} + (1 - \phi) \right) \frac{W_{t+k}}{\varphi_{3,t+k} f' [h_{t+k} (i)]} \right] y_{t+k} (i) \right\} = 0.
$$

Using the law of iterated expectations and the Euler equation (C.4), we can rewrite this first-order condition as

$$
\mathbb{E}_t \left\{ \sum_{k=0}^{+\infty} (\beta \theta)^k \frac{\lambda_{t+k}}{\lambda_t \Pi_{t+k}} \left[ P_t^* (i) - \left( \phi \frac{\beta \lambda_{t+k+1} I_{t+k}^f}{\lambda_{t+k} \Pi_{t+k+1}} + (1 - \phi) \right) \frac{W_{t+k}}{\varphi_{2,t+k} \varphi_{3,t+k} f' [h_{t+k} (i)]} \right] y_{t+k} (i) \right\} = 0. \tag{C.12}
$$

In the limit case of perfectly flexible prices ($\theta = 0$), and in a symmetric equilibrium ($P_t^*(i) = P_t$ and $h_t(i) = h_t$), this first-order condition becomes

$$
P_t = \frac{\varepsilon \varphi_{4,t}}{\varepsilon \varphi_{4,t} - 1} \left[ \frac{I_t^f}{I_t} + (1 - \phi) \right] \frac{W_t}{\varphi_{2,t} \varphi_{3,t} f'(h_t)}. \tag{C.13}
$$

### C.3 Government

The government consists of a fiscal authority and a monetary authority. The fiscal authority consumes an exogenous quantity $g_t \geq 0$ of goods, does not issue bonds, and sets lump-sum

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taxes on households so as to balance its budget (making fiscal policy Ricardian). We assume for simplicity that government expenditures $g_t$ are wasted, but the results would be unchanged if they entered households’ utility function in a separable way.

The monetary authority – i.e., the central bank – has two independent instruments: the (gross) nominal interest rate on reserves $I^m_t \geq 0$, and the stock of nominal reserves $M_t > 0$. We assume that the central bank injects reserves via lump-sum transfers. The consolidated budget constraint of the government is thus

$$M_t = I^m_{t-1}M_{t-1} + P_tg_t - T_t,$$

where $T_t$ denotes the net lump-sum tax imposed by the government (the fiscal authority’s tax minus the monetary authority’s transfer).

To capture a lower bound on $I^m_t$ in a simple and stark way, we assume that vault cash (with no interest payments) is a perfect substitute for deposits at the central bank in terms of reducing banking costs. This introduces a zero lower bound (ZLB) for the net nominal IOR rate $I^m_t - 1$ in our model. In an equilibrium with $I^m_t > 1$, banks will hold no cash. In an equilibrium with $I^m_t = 1$, the composition of reserve balances will be indeterminate, but also inconsequential; so, we will assume that banks hold no cash in equilibrium.

### C.4 Market Clearing

The bond-market-clearing condition is

$$b_t = 0,$$

(C.15)

the reserve-market-clearing condition is

$$m_t = M_t / P_t,$$

(C.16)

and the goods-market-clearing condition is

$$c_t + g_t = y_t.$$  

(C.17)

### C.5 Households (Primitive Setup)

In this subsection, we describe the primitive setup for households that leads to the reduced-form utility function (C.1). More specifically, we derive the banking-cost function $\Gamma$ and its main properties. We also specify the assumptions on the consumption-utility function $u$ and the labor-disutility function $v$.

---

6It would be straightforward to modify our model and allow changes in reserve balances to be matched by changes in the monetary authority’s holdings of bonds issued by households or the fiscal authority; such features, however, would not play a role in our analysis. We will introduce government bonds in Appendix G.

7A more realistic model in which vault cash is substitutable to some extent for deposits at the central bank could imply a negative lower bound for the net nominal IOR rate $I^m_t - 1$. Our analysis below does not rely in any substantive way on whether the effective lower bound is zero or negative.
In this primitive setup, households get utility from consumption \((c_t)\) and disutility from labor \((h_t\) for production workers, \(h^b_t\) for bankers). Their intertemporal utility function is

\[
U_t = E_t \left\{ \sum_{k=0}^{\infty} \beta^k \left[ u(c_{t+k}) - \frac{v(h_{t+k})}{\varphi_{1,t+k}} - \frac{v^b(h^b_{t+k})}{\varphi_{1,t+k}} \right] \right\}
\]

where \(\beta \in (0, 1)\). The consumption-utility function \(u\), defined over the set of positive real numbers \(\mathbb{R}_{>0}\), is twice differentiable, strictly increasing \((u' > 0)\), strictly concave \((u'' < 0)\), and satisfies the usual Inada conditions

\[
\lim_{c_t \to 0} u'(c_t) = +\infty, \quad (C.18)
\]

\[
\lim_{c_t \to +\infty} u'(c_t) = 0. \quad (C.19)
\]

The labor-disutility functions \(v\) and \(v^b\), defined over the set of non-negative real numbers \(\mathbb{R}_{\geq 0}\), are twice differentiable, strictly increasing \((v' > 0 \text{ and } v^{br} > 0)\), and weakly convex \((v'' \geq 0 \text{ and } v^{brr} \geq 0)\). Again, \(\varphi_{1,t}\) denotes a stochastic exogenous labor-disutility shock of mean one – the first of the four alternative supply shocks that we consider.

Bankers use their own labor \(h^b_t\) and (real) reserves at the central bank \(m_t\) to produce (real) loans \(\ell_t\) according to the technology

\[
\ell_t = f^b(h^b_t, m_t).
\]

The production function \(f^b\), defined over \((\mathbb{R}_{\geq 0})^2\), is twice differentiable, strictly increasing \((f^b_h > 0 \text{ and } f^b_m > 0)\), homogeneous of degree \(d \in (0, 1]\), and such that \(f^b_{hh} < 0, f^b_{mm} < 0, f^b_{hm} \geq 0, \forall h^b_t \in \mathbb{R}_{\geq 0}\)

\[
\lim_{m_t \to +\infty} f^b_m(h^b_t, m_t) = 0, \quad (C.20)
\]

\[
\lim_{m_t \to 0} f^b_h(h^b_t, m_t) = 0. \quad (C.21)
\]

Assumption (C.20) is a standard Inada condition, while Assumption (C.21) articulates a sense in which holding reserves is essential for banking. The assumption of decreasing or constant returns to scale \((d \leq 1)\) is not necessary for our results, but simplifies our non-parametric analysis.\(^8\)

Similarly, the assumption that labor and reserves are complements \((f^b_{hm} \geq 0)\) could be relaxed to some extent without affecting our results. These assumptions imply that \(f^b\) is concave:

**Lemma 2 (Concavity of Function \(f^b\)):** The production function \(f^b\) is concave.

**Proof:** See Subsection C.6. ■

The set of functions \(f^b\) satisfying all these assumptions is broad enough to include, for instance, any constant-elasticity-of-substitution (CES) function, or more generally any CES function raised to a power \(d\) such that \(\max[(s - 1)/s, 0] < d \leq 1\), where \(s\) denotes the elasticity of substitution.

\(^8\)In a previous version of this paper (Diba and Loisel, 2017), we allow for increasing returns to scale \((d > 1)\) in the context of a parametric version of our model.
The function $f^b$ is, of course, a convenient short cut to capture the role of bank reserves—which in reality may come, for example, from a maturity mismatch between banks’ assets and liabilities. Our results will not depend on the quantitative importance of this role: the elasticity of loans to reserves, $m_t f^b_m (h^b_t, m_t) / f^b(h^b_t, m_t)$, may be arbitrarily small for any $(h^b_t, m_t)$ in $(\mathbb{R}_{\geq 0})^2$. What we need for our results, however, is that this elasticity is not exactly zero in equilibrium, i.e. that the demand for reserves is not fully satiated (which is necessarily the case under our assumptions).

Given the properties of $f^b$, we can invert it and get

$$h^b_t = g^b(\ell_t, m_t),$$

where the function $g^b$ is implicitly and uniquely defined over $(\mathbb{R}_{\geq 0})^2$ by $\ell_t = f^b[g^b(\ell_t, m_t), m_t]$. The utility cost of banking, as a function of loans and reserves, is therefore defined over $(\mathbb{R}_{\geq 0})^2$ by

$$\Gamma(\ell_t, m_t) \equiv v^b\left[g^b(\ell_t, m_t)\right].$$

The following lemma states some properties of the function $\Gamma$:

**Lemma 3 (Properties of Function $\Gamma$):** The banking-cost function $\Gamma$ is strictly increasing in loans ($\Gamma_\ell > 0$); strictly decreasing in reserves ($\Gamma_m < 0$); convex ($\Gamma_{\ell \ell} > 0$, $\Gamma_{mm} > 0$, $\Gamma_{\ell \ell} \Gamma_{mm} - (\Gamma_{\ell m})^2 \geq 0$); and such that $\Gamma_{\ell m} < 0$,

$$\forall \ell_t \in \mathbb{R}_{> 0}, \lim_{m_t \to +\infty} \Gamma_m(\ell_t, m_t) = 0,$$

$$\forall \ell_t \in \mathbb{R}_{> 0}, \lim_{m_t \to 0} \Gamma_{\ell}(\ell_t, m_t) = +\infty.\tag{C.23}$$

**Proof:** See Subsection C.7. $\blacksquare$

The property that $\Gamma_m$ is not zero (except asymptotically, as $m_t \to +\infty$) reflects our assumption that there is no finite satiation point in the demand for reserves. The negative-cross-derivative property ($\Gamma_{\ell m} < 0$) says that a marginal increase in reserves decreases costs by more the larger are loans, while Property (C.23) reflects our assumption that holding reserves is essential for banking.\(^9\)

### C.6 Proof of Lemma 2

In this subsection, we prove Lemma 2. To lighten the notation, we omit time subscripts (as well as function arguments when no ambiguity results), thus writing $h^b$ and $m$ instead of $h^b_t$ and $m_t$.

---

\(^9\)These properties of $\Gamma$ are quite similar to the banking-cost assumptions made by Cúrdia and Woodford (2011). Three differences between the two models are worth mentioning: (i) our model has no finite satiation level in the demand for reserves, unlike their model; (ii) we link banking costs to time spent on banking activities, in order to make our steady-state analysis tractable, while they link them to goods consumed in banking activities; and (iii) the borrowers in our model are firms (borrowing the wage bill or some fraction of it), while they are impatient households in their model.
Since $f^b$ is homogeneous of degree $d$, we have $\forall x \in \mathbb{R}_{\geq 0}$, $f^b(xh^b, xm) = x^d f^b(h^b, m)$. Computing the first derivative of the left- and right-hand sides of this equation with respect to $x$ at $x = 1$ leads to

$$df^b = h^b f^b_h + m f^b_m.$$  \hfill (C.24)

In turn, computing the first derivative of the left- and right-hand sides of the last equation with respect to $h^b$ and $m$ leads to

$$f^b_{hh} = -(1-d) f^b_h + m f^b_{hm},$$

$$f^b_{mm} = -(1-d) f^b_m + h^b f^b_{hm}.$$  

Using these expression for $f^b_{hh}$ and $f^b_{hm}$, as well as (C.24), we get

$$f^b_{hh} f^b_{mm} - \left(f^b_{hm}\right)^2 = \frac{1-d}{h^b m} \left[(1-d) f^b_{hh} f^b_{mm} + f^b_{hm} \left(h^b f^b_h + m f^b_m\right)\right]$$

$$= \frac{1-d}{h^b m} \left[(1-d) f^b_{hh} f^b_{mm} + df^b f^b_{hm}\right]$$

$$\geq 0,$$

which implies (together with $f^b_{hh} \leq 0$ and $f^b_{mm} \leq 0$) that the function $f^b$ is (weakly) concave.

C.7 Proof of Lemma 3

In this subsection, we prove Lemma 3. As in the previous subsection, to lighten the notation, we omit time subscripts (as well as function arguments when no ambiguity results), thus writing $\ell$ and $m$ instead of $\ell_t$ and $m_t$. Computing the first and second derivatives of the left- and right-hand sides of $\ell = f^b[y^b(\ell, m), m]$ with respect to $\ell$ and $m$ gives

$$1 = f^b_{h\ell \ell},$$

$$0 = f^b_{h\ell m} + f^b_m,$$

$$0 = f^b_{hh} \left(g^b_\ell\right)^2 + f^b_{h\ell \ell},$$

$$0 = f^b_{hh} g^b_m + f^b_{hm} g^b_\ell + f^b_{h\ell m},$$

$$0 = f^b_{hh} \left(g^b_m\right)^2 + 2 f^b_{hm} g^b_m + f^b_{h\ell \ell} + f^b_{mm}. $$
Using these equations and \( f^h_m > 0, f^h_m < 0, f^h_{mm} \geq 0, \) and \( f^h_{mm} < 0, \) we sequentially get

\[
\begin{align*}
g^b_\ell &= \frac{1}{f^h_\ell} > 0, \\
g^b_m &= -\frac{f^b_m}{f^h_m} < 0, \\
g^b_{\ell\ell} &= -\frac{f^h_{hh}}{(f^h_\ell)^2} > 0, \\
g^b_{\ell m} &= \frac{f^h_{m h h} (f^h_m)^2}{(f^h_\ell)^4} - \frac{f^h_{km}}{f^h_\ell} < 0, \\
g^b_{mm} &= -\frac{f^h_{hh}}{(f^h_m)^2} + 2 \frac{f^h_m f^h_{hm}}{(f^h_\ell)^2} - \frac{f^h_{mm}}{f^h_\ell} > 0.
\end{align*}
\]

Then, using these expressions for \( g^b_\ell, g^b_{mm}, g^b_{\ell m}, \) and the concavity of \( f^h, \) we easily get

\[
g^b_\ell g^b_{mm} - \left( g^b_{\ell m} \right)^2 = \frac{f^h_{h h} f^h_{mm} - (f^h_{km})^2}{(f^h_\ell)^4} \geq 0,
\]

which implies (together with \( g^b_\ell > 0 \) and \( g^b_{mm} > 0 \)) that the function \( g^b \) is (weakly) convex.

Moreover, since \( f^h \) is homogeneous of degree \( d, \) we have \( \forall x \in \mathbb{R}_{\geq 0}, \) \( g^b(x^\ell, x^m) = xy^b(\ell, m). \)

Computing the first derivative of the left- and right-hand sides of this equation with respect to \( x \) at \( x = 1 \) leads to

\[
g^b = d \ell g^b_\ell + m g^b_m. \tag{C.25}
\]

In turn, computing the first derivative of the left- and right-hand sides of the last equation with respect to \( \ell \) and \( m \) leads to

\[
\begin{align*}
g^b_\ell &= \frac{(1 - d) g^b_\ell - m g^b_{\ell m}}{d \ell}, \tag{C.26} \\
g^b_{mm} &= -\frac{d \ell g^b_{\ell m}}{m}. \tag{C.27}
\end{align*}
\]

Computing the first and second derivatives of the left- and right-hand sides of \( \Gamma(\ell, m) \equiv v^b[g^b(\ell, m)] \) with respect to \( \ell \) and \( m \) gives

\[
\begin{align*}
\Gamma_\ell &= v^{b \ell} g^b_\ell > 0, \\
\Gamma_m &= v^{b m} g^b_m < 0, \\
\Gamma_{\ell\ell} &= v^{b \ell \ell} \left( g^b_\ell \right)^2 + v^{b \ell} g^b_\ell > 0, \\
\Gamma_{\ell m} &= v^{b \ell m} g^b_\ell g^b_m + v^{b m} g^b_m < 0, \\
\Gamma_{mm} &= v^{b m m} \left( g^b_m \right)^2 + v^{b m} g^b_m > 0.
\end{align*}
\]

where the inequalities follow from \( v^{b \ell} > 0, v^{b m} \geq 0, g^b_\ell > 0, g^b_m < 0, g^b_\ell > 0, g^b_{mm} > 0, \) and
\( g_{b\ell m} < 0 \). In addition, using first (C.26)-(C.27) and then (C.25), we easily get

\[
\Gamma_{\ell\ell}\Gamma_{mm} - (\Gamma_{\ell m})^2 = (v^b)^2 \left[ g^b_v g^b_m - (g^b_{\ell m})^2 \right] + v^b \sum v^b_r \left[ \left( g^b_\ell \right)^2 g^b_m + \left( g^b_m \right)^2 g^b_\ell - 2g^b_v g^b_r g^b_{\ell m} \right] = - (1 - d) \left( v^b \right)^2 \sum g^b_{\ell m} - v^b \sum v^b_r \left[ \left( g^b_\ell \right)^2 g^b_m + \left( g^b_m \right)^2 g^b_\ell - 2g^b_v g^b_r g^b_{\ell m} \right] \geq 0,
\]

which implies (together with \( \Gamma_{\ell\ell} > 0 \) and \( \Gamma_{mm} > 0 \)) that the function \( \Gamma \) is (weakly) convex.

Finally, as a consequence of (C.20) and (C.21), we have \( \lim_{m \to +\infty} g^b_{m}(\ell, m) = 0 \) and \( \lim_{m \to 0} \sum g^b_{\ell}(\ell, m) = +\infty \) for all \( \ell \in \mathbb{R}_{\geq 0} \), from which we straightforwardly get (C.22) and (C.23).

**Appendix D: Model With Banks – Steady State**

In this appendix, we prepare the ground for the next appendices by showing that our model with banks has a unique steady state when the central bank sets exogenously the IOR rate and the stock of reserves. We start by showing that global equilibrium dynamics under flexible prices can be summarized by a single equation relating \( h_t \) to \( h_{t+1} \). We then use this dynamic equation to establish the existence and uniqueness of a steady state under sticky prices.

**D.1 Dynamic Equation Under Flexible Prices**

In this subsection, we consider the case in which prices are flexible (\( \theta = 0 \)). To derive the key equation summarizing global equilibrium dynamics in this case, we first use (C.3), (C.5), (C.9), (C.10) holding with equality, and (C.17), to express loans \( \ell_t \) as a function of employment \( h_t \):

\[
\ell_t = \mathcal{L}(h_t) = \frac{\phi h_t \varphi' \left( h_t \right)}{\varphi_1 \varphi_3 \varphi' \left[ \varphi_2 f(h_t) - g_t \right]}.
\]

The function \( \mathcal{L} \) is defined over \((h_t, +\infty)\), where \( h_t \geq 0 \) is implicitly and uniquely defined by \( \varphi_2 f(h_t) = g_t \). It is strictly increasing (\( \mathcal{L}' > 0 \)) with

\[
\lim_{h_t \to h^*} \mathcal{L}(h_t) = 0, \quad \lim_{h_t \to +\infty} \mathcal{L}(h_t) = +\infty.
\]
Next, we note that under flexible prices, the pricing equation (C.13) gives the real wage
\[ w_t = \frac{\varepsilon \varphi_{4,t} - 1}{\varepsilon \varphi_{4,t}} \varphi_{2,t} \varphi_{3,t} f'(h_t) \left( \frac{\phi}{T_t} + (1 - \phi) \right)^{-1}. \]  
(D.4)

We then use households' first-order condition (C.6), together with (C.3), (C.5), (C.9), (C.17), (D.1), and (D.4), to get a relationship between reserves \( m_t \) and employment \( h_t \):
\[ \Gamma_t [\mathcal{L}(h_t), m_t] = \mathcal{A}(h_t) = \frac{\varphi_{1,t} u'[\varphi_{2,t} f(h_t) - g_t]}{\phi} \left\{ \left( \frac{\varepsilon \varphi_{4,t} - 1}{\varepsilon \varphi_{4,t}} \right) - \frac{\varphi_{1,t} \varphi_{2,t} \varphi_{3,t} u'[\varphi_{2,t} f(h_t) - g_t]}{v'(h_t)} f'(h_t) \right\}. \]  
(D.5)

Because \( \Gamma_t > 0 \), we restrict the domain of definition of \( \mathcal{A} \) to \((h_t, h_t^*)\), where, given the Inada conditions (C.18) and (C.19), \( h_t^* > h_t \) is implicitly and uniquely defined by
\[ \frac{\varphi_{1,t} \varphi_{2,t} \varphi_{3,t} u'[\varphi_{2,t} f(h_t^*) - g_t]}{v'(h_t^*)} = \frac{\varepsilon \varphi_{4,t}}{\varepsilon \varphi_{4,t} - 1}. \]

The function \( \mathcal{A} \) is strictly decreasing (\( \mathcal{A}' < 0 \)) with
\[ \lim_{h_t \to h_t^*} \mathcal{A}(h_t) = +\infty, \]  
(D.6)
\[ \lim_{h_t \to h_t^*} \mathcal{A}(h_t) = 0. \]  
(D.7)

Note that \( h_t^* \) represents the value that \( h_t \) would take in the absence of financial frictions, i.e. if the marginal banking cost \( \Gamma_t \) were zero.

Since \( \Gamma_{\ell\ell} > 0 \), \( \Gamma_{\ell m} < 0 \), \( \mathcal{L}' > 0 \), and \( \mathcal{A}' < 0 \), Equation (D.5) implicitly and uniquely defines a function \( \mathcal{M} \) such that
\[ m_t = \mathcal{M}(h_t). \]  
(D.8)

The function \( \mathcal{M} \) is strictly increasing (\( \mathcal{M}' > 0 \)). The reason is that under flexible prices, firms’ profit maximization makes their real marginal cost equal to the inverse of their mark-up \( (\varepsilon \varphi_{4,t} - 1)/\varepsilon \varphi_{4,t} \); since real marginal cost depends positively on employment and negatively on reserves (through borrowing costs), real reserves need to react positively to employment to keep real marginal cost equal to \( (\varepsilon \varphi_{4,t} - 1)/\varepsilon \varphi_{4,t} \). Moreover, given (C.23), \( \mathcal{M} \) is defined over \((h_t, \bar{h}_t)\), where \( \bar{h}_t \in (h_t, h_t^*) \) is implicitly and uniquely defined by
\[ \lim_{m_t \to +\infty} \Gamma_t [\mathcal{L}(\bar{h}_t), m_t] = \mathcal{A}(\bar{h}_t). \]  
(D.9)

The uniqueness of \( \bar{h}_t \) follows from \( \mathcal{A}' < 0 \), \( \mathcal{L}' > 0 \), and \( \Gamma_{\ell\ell} > 0 \), while its existence is ensured by (D.6) and (D.7). Finally, given (C.23) and (D.9), we have
\[ \lim_{h_t \to h_t} \mathcal{M}(h_t) = 0, \]  
(D.10)
\[ \lim_{h_t \to \bar{h}_t} \mathcal{M}(h_t) = +\infty. \]  
(D.11)

Thus, in our model with no satiation point in the demand for reserves, real reserve balances grow without bound as employment rises towards its upper bound \( \bar{h}_t \). This upper bound coincides with

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the frictionless employment level \( h^*_t \) in the case where the marginal banking cost \( \Gamma_t \) converges to zero as real reserves tend to infinity. In general, however, we allow the marginal banking cost to converge to a positive value — in which case we have \( \bar{h}_t < h^*_t \), and our economy with the financial friction cannot attain the employment level of the frictionless economy.

Finally, we use households’ first-order conditions (C.4) and (C.7), together with (C.3), (C.9), (C.16), (C.17), (D.1), and (D.8), to get the dynamic equation in employment:

\[
1 + \frac{\Gamma_m}{\varphi_1 t^u} \left[ \varphi_2 t f(h_t) - g_t \right] = \beta I_t^{m} \mathbb{E}_t \left\{ \frac{u' \left[ \varphi_2 t f(h_{t+1}) - g_{t+1} \right] M(h_{t+1})}{\mu_{t+1} u' \left[ \varphi_2 t f(h_t) - g_t \right] M(h_t)} \right\},
\]

(D.12)

where \( \mu_t \equiv M_t/M_{t-1} \) denotes the (gross) growth rate of nominal reserves. In the next subsection, we use this dynamic (flexible-price) equation to establish the existence and uniqueness of a steady state in our (sticky-price) model.

### D.2 Steady-State Existence and Uniqueness

We now allow prices to be sticky (\( \theta \geq 0 \)) and assume that \( I_t^m \) can vary exogenously around a given value \( I^m \geq 1 \), \( \mu_t \) around the value \( \mu = 1 \), and \( g_t \) around a given value \( g \geq 0 \) (variables without time subscript denote steady-state values). In any steady state, both nominal reserves (because \( \mu = 1 \)) and real reserves (by definition of a steady state) are constant over time. Therefore, prices are also constant over time, and the set of steady states is the same under sticky prices (\( \theta > 0 \)) as under flexible prices (\( \theta = 0 \)). To characterize this set, we can therefore use the flexible-price dynamic equation (D.12). When \( h_{t+1} = h_t \) and \((I_t^m, \mu_t, g_t, \varphi_{1,t}, \varphi_{2,t}, \varphi_{3,t}, \varphi_{4,t}) = (I^m, 1, g, 1, 1, 1, 1)\), this dynamic equation boils down to the static equation

\[
\mathcal{F}(h_t) \equiv \frac{\Gamma_m}{u'[f(h_t) - g]} \left[ L(h_t), M(h_t) \right] = \beta I^m - 1,
\]

(D.13)

where \( L(h_t) \) and \( M(h_t) \) are evaluated at \((g_t, \varphi_{1,t}, \varphi_{2,t}, \varphi_{3,t}, \varphi_{4,t}) = (g, 1, 1, 1, 1)\). The function \( \mathcal{F} \) is defined over \((\underline{h}, \bar{h})\), where \( \underline{h} \) and \( \bar{h} \) denote the values of \( h^*_t \) and \( \bar{h}_t \) respectively when \((g_t, \varphi_{1,t}, \varphi_{2,t}, \varphi_{3,t}, \varphi_{4,t}) = (g, 1, 1, 1, 1)\). We establish the following lemma:

**Lemma 4 (Properties of Function \( \mathcal{F} \)):** The function \( \mathcal{F} \) is strictly increasing (\( \mathcal{F}' > 0 \)), with

\[
\lim_{h_t \to \underline{h}} \mathcal{F}(h_t) = -\infty \quad \text{and} \quad \lim_{h_t \to \bar{h}} \mathcal{F}(h_t) = 0.
\]

**Proof:** See Subsection D.3. ■

The static equation (D.13) and Lemma 4 straightforwardly imply the following proposition:

**Proposition 1 (Steady-State Existence and Uniqueness):** In our model with banks, when \( I_t^m, \mu_t, \) and \( g_t \) vary exogenously around the values \( I^m \geq 1, \mu = 1, \) and \( g \geq 0 \), there is a unique steady state if and only if \( I^m < \beta^{-1} \).
Since bonds serve only as a store of value, we have $I = \beta^{-1}$, so that the condition for steady-state existence and uniqueness stated in Proposition 1 is that the steady-state IOR rate $I^m$ be set strictly below the steady-state interest rate on bonds $I$. When $I^m \geq I$, there is no steady state because banks would be tempted to issue infinite amounts of debt and deposit the proceeds at the central bank. When $I^m < I$, the first-order condition (C.7) implies that the convenience yield of bank reserves is positive (i.e., we have $\Gamma_m < 0$ in equilibrium), and this basically pins down the demand for real reserves. Since the nominal stock of reserves is exogenous, pinning down the demand for real reserves also pins down the price level.

Proposition 1 thus implies that the type of indeterminacy discussed in Sargent and Wallace (1975) does not arise in our model. This type of indeterminacy associates any value in a continuum of initial price levels with the same constant values for real variables (and inflation). In our setup, the initial price level is uniquely pinned down at the steady state.

Given Lemma 4, the steady-state employment level $h = F^{-1}(\beta I^m - 1)$ is strictly increasing in the steady-state IOR rate $I^m$. This is because an increase in $I^m$ reduces the opportunity cost of holding reserves $I/I^m = (\beta I^m)^{-1}$. The lower opportunity cost, in turn, decreases the banking cost $\Gamma$ and the banking spread $I^\ell/I$. The lower spread (borrowing cost) increases the real wage, which stimulates employment and output.

### D.3 Proof of Lemma 4

In this subsection, we prove Lemma 4. Using (D.5), we can rewrite $F(h_t)$ as

$$F(h_t) = \frac{1}{\phi} F_1(h_t) F_2(h_t),$$

where the functions $F_1$ and $F_2$ are defined over $(h, \tilde{h})$ by

$$F_1(h_t) = \frac{\Gamma_m [L(h_t), M(h_t)]}{\Gamma_\ell [L(h_t), M(h_t)]} = \frac{g_{bl}^m [L(h_t), M(h_t)]}{g_{bl}^\ell [L(h_t), M(h_t)]},$$

$$F_2(h_t) = \frac{\varepsilon - 1}{\varepsilon} \frac{u' [f(h_t) - g]}{v' (h_t)} - 1.$$

We have

$$\left( g_{bl} \right)^2 F_1' = g_{bl} \left( g_{bl} L' + g_{bl} M' \right) - g_{bl} \left( g_{bl} L' + g_{bl} M' \right)$$

$$= -g_{bl} \left( dL g_{bl} + M g_{bl} \right) \left( \frac{M}{M} - \frac{L'}{dL} \right) - (1 - d) g_{bl} g_{bl} L' \frac{L'}{dL},$$

where the second equality is obtained by using (C.26)-(C.27), and the third equality by using (C.25). Now, deriving the left- and right-hand sides of (D.5) with respect to $h_t$ gives

$$\Gamma_{\ell m} M' + \Gamma_{\ell \ell} L' = A' < 0.$$

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Moreover, using (C.25) and (C.26), we get

\[
d\Gamma^\ell + M\Gamma^m = dL\left[\hat{v}^b g_L^b (g_L^b)^2 + v^b g_L^b v_m^b + \hat{v}^b g_L^b + \hat{v}^b (dLg_L^b + Mg_m^b)\right]
\]

\[
= v^b g_L^b (dLg_L^b + Mg_m^b) + \hat{v}^b (dLg_L^b + Mg_m^b)
\]

\[
= v^b g_L^b + (1 - d) v^b g_L^b
\]

\[
\geq 0.
\]

The last two inequalities together imply

\[
\frac{\mathcal{M}'}{\mathcal{M}} > \frac{L'}{dL}, \tag{D.14}
\]

from which we conclude that \(F_1' > 0\). Then, using \(F_1' > 0, F_1 < 0, F_2 < 0,\) and \(F_2 > 0,\) we get that the function \(F\) is strictly increasing \((F' > 0)\). Moreover, \(F_1' > 0 \) and \(F_1 < 0\) imply that \(\lim_{h_t \to h} F_1(h_t) < 0,\) while the Inada condition (C.18) implies that \(\lim_{h_t \to h} F_2(h_t) = +\infty,\) so that

\[
\lim_{h_t \to h} F(h_t) = -\infty.
\]

Finally, both \(\lim_{h_t \to \bar{h}} F_1(h_t)\) and \(\lim_{h_t \to \bar{h}} F_2(h_t)\) are finite, since \(F_1\) is increasing and negative, and \(F_2\) decreasing and positive. If \(\bar{h} < h^*,\) then the Inada condition (C.22) implies \(\lim_{h_t \to \bar{h}} F_1(h_t) = 0.\) Alternatively, if \(\bar{h} = h^*,\) then \(\lim_{h_t \to \bar{h}} F_2(h_t) = 0.\) We conclude that, in both cases,

\[
\lim_{h_t \to \bar{h}} F(h_t) = 0.
\]

Appendix E: Model With Banks – Log-Linearized Model

In this appendix, which complements Section 4 in the paper, we study our model with banks log-linearized around its unique steady state. More specifically, we do the following: (i) we derive the log-linearized Phillips curve (12) and reserves-demand equation (13); (ii) we prove the double inequality (14); (iii) we show formally that the model solves the paradox of flexibility; and (iv) we show that the model delivers determinacy under exogenous monetary-policy instruments when time starts at a finite date (rather than at date \(-\infty\)).

E.1 Derivation of the Phillips Curve and Reserves-Demand Equation

In this subsection, we derive the log-linearized Phillips curve (12) and reserves-demand equation (13). To derive the Phillips curve, we log-linearize firms’ first-order condition (C.12) around the unique steady state determined in Appendix D, and use the definition of the real wage \(w_t \equiv W_t/P_t\), to get

\[
\hat{P}_t^* = (1 - \beta\theta)E_t\left\{\sum_{k=0}^{+\infty}(\beta\theta)^k\left[\alpha_\phi \left(i_{t+k} - \tilde{w}_{t+k} + \hat{\tilde{P}}_{t+k} - \tilde{m}p_{t+k} + \hat{\tilde{\theta}}_t + \frac{\hat{\tilde{\theta}}_h}{\varepsilon - 1}\right)\right]\right\}, \tag{E.1}
\]

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where $\alpha_{\phi} \equiv \phi I^t / \phi I^t + (1 - \phi) I \in (0, 1]$, variables with hats denote log deviations from steady-state values, $i_t^e \equiv \hat{I}_t$, $i_t \equiv \hat{I}_t$, and $mp_{t+k}$ denotes the marginal productivity in period $t + k$ for a firm whose price was last set in period $t$. Log-linearizing the production function (C.9) gives

$$\hat{h}_t = \frac{f}{f^h} (\hat{g}_t - \hat{\varphi}_{2,t}) , \quad (E.2)$$

so that we can rewrite $\hat{m}p_{t+k}$ as

$$\hat{m}p_{t+k} = \hat{\varphi}_{2,t} + \frac{f''h}{f^h} \hat{h}_{t+k} = \hat{m}p_{t+k} + \frac{f''h}{f^h} (\hat{h}_{t+k} - \hat{h}_t)$$

$$= \hat{m}p_{t+k} + \frac{ff''}{(f^h)^2} (\hat{g}_t - \hat{g}_{t+k}) = \hat{m}p_{t+k} - \frac{\varepsilon ff''}{(f^h)^2} \left( \hat{P}_t^* - \hat{P}_{t+k} \right) , \quad (E.3)$$

where $mp_{t+k}$ denotes the average marginal productivity in period $t + k$. Using this result and following the same steps as in, e.g., Galí (2008, Chapter 3), we can rewrite (E.1) as

$$\pi_t = \beta \mathbb{E}_t \{ \pi_{t+1} \} + \frac{(1 - \theta)(1 - \beta \theta)}{\theta [1 - \varepsilon ff''/(f^h)^2]} \left[ \alpha_{\phi} (i_t^e - i_t) + \hat{w}_t - \hat{m}p_t - \hat{\varphi}_{3,t} - \hat{\varphi}_{4,t} \right] . \quad (E.4)$$

Now, log-linearizing the goods-market-clearing condition (C.17) gives

$$\tilde{c}_t + \tilde{g}_t = \hat{g}_t , \quad (E.5)$$

where $\tilde{c}_t \equiv (c/y)\hat{c}_t$ and $\tilde{g}_t \equiv (g/y)\hat{g}_t$. Log-linearizing the first-order condition (C.6), and using (E.5), gives

$$i_t^e - i_t = \alpha_{\ell} v' \hat{c}_t + \alpha_{\ell} \Gamma_{t} \hat{m}_t - \alpha_{\ell} v' \hat{g}_t + \alpha_{\ell} v' \hat{g}_t - \alpha_{\ell} \hat{\varphi}_{1,t} , \quad (E.6)$$

where $\alpha_{\ell} \equiv (I^t - I^e) / I^e \in (0, 1]$. Log-linearizing the first-order condition (C.5), and using (E.2) and (E.5), gives

$$\hat{w}_t = \left( - \frac{v''y}{u'} + \frac{v''h}{v'} \frac{f}{f^h} \right) \hat{g}_t + \frac{v''y}{u'} \hat{g}_t - \hat{\varphi}_{1,t} - \frac{v''h}{v'} \frac{f}{f^h} \hat{\varphi}_{2,t} . \quad (E.7)$$

Log-linearizing the constraint (C.10) holding with equality, and using (E.2) and (E.7), gives

$$\hat{\ell}_t = \left( - \frac{v''y}{u'} + \frac{v''h}{v'} \frac{f}{f^h} + \frac{f}{f^h} \right) \hat{g}_t + \frac{v''y}{u'} \hat{g}_t - \hat{\varphi}_{1,t} - \left( \frac{v''h}{v'} \frac{f}{f^h} + \frac{f}{f^h} \right) \hat{\varphi}_{2,t} - \hat{\varphi}_{3,t} . \quad (E.8)$$

Moreover, we have

$$\hat{m}p_t = \hat{\varphi}_{2,t} + \frac{ff''}{(f^h)^2} (\hat{g}_t - \hat{\varphi}_{2,t}) . \quad (E.9)$$

Using (E.6), (E.7), (E.8), and (E.9), we can then rewrite (E.4) as the Phillips curve

$$\pi_t = \beta \mathbb{E}_t \{ \pi_{t+1} \} + \kappa (\hat{g}_t - \delta_g \hat{m}_t - \delta_g \hat{g}_t - \delta_{\varphi} \hat{\varphi}_t) , \quad (E.8)$$

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where
\[\kappa \equiv \left[ -u''y / u' + v''h / v' f' / f - ff'' - \alpha_t \alpha_{\phi} u''y / u' + \alpha_t \alpha_{\phi} \Gamma_{t \ell \ell} \left( -u''y / u' + v''h / v' f' / f + f / f' \right) \right] \psi > 0,\]
\[\delta_m = -\alpha_t \alpha_{\phi} \left( \Gamma_{\ell m m} \right) \frac{\psi}{\kappa} > 0,\]
\[\delta_g = \left[ -u''y / u' \left[ 1 + \alpha_t \alpha_{\phi} \left( 1 + \frac{\Gamma_{t \ell \ell}}{\Gamma_{\ell}} \right) \right] \right] \frac{\psi}{\kappa} > 0,\]
\[\delta_{\varphi} = \left\{ \left[ 1 + \alpha_t \alpha_{\phi} \left( 1 + \frac{\Gamma_{t \ell \ell}}{\Gamma_{\ell}} \right) \right] \mathbb{1}_{\varphi_t = \varphi_{1, t}} + \left[ 1 + \frac{v''h / v' f' / f - ff'' + \alpha_t \alpha_{\phi} \Gamma_{t \ell \ell} \left( v''h / v' f' / f \right) + f / f' h \right] \mathbb{1}_{\varphi_t = \varphi_{2, t}} + \left( 1 + \alpha_t \alpha_{\phi} \Gamma_{t \ell \ell} \right) \mathbb{1}_{\varphi_t = \varphi_{3, t}} + \left( \frac{1}{\varepsilon - 1} \right) \mathbb{1}_{\varphi_t = \varphi_{4, t}} \right\} \frac{\psi}{\kappa} > 0,\]

where in turn
\[\psi = \frac{(1 - \theta) (1 - \beta \theta)}{\theta \left[ 1 - \frac{v''h / v' f' / f}{f / f'} \right]}.\]

Note that, to write this Phillips curve in a compact way, we have here considered a single supply shock \(\varphi_t \in \{ \varphi_{1, t}, \varphi_{2, t}, \varphi_{3, t}, \varphi_{4, t} \}\) and used indicator functions in the definition of \(\delta_{\varphi}\): for any \(k \in \{1, 2, 3, 4\}\), \(\mathbb{1}_{\varphi_t = \varphi_{k, t}}\) takes the value one if \(\varphi_t = \varphi_{k, t}\) and the value zero otherwise.

Equation (12) corresponds to this Phillips curve in which, for simplicity, the notations \(\hat{y}_t, \hat{m}_t, \hat{g}_t,\) and \(\hat{\varphi}_t\) are replaced by the notations \(y_t, m_t, g_t,\) and \(\varphi_t\) (as everywhere in the main text). This Phillips curve departs from the standard NK Phillips curve in two respects. First, the parameters \(\kappa, \kappa\delta_g,\) and \(\kappa\delta_{\varphi}\) now depend (positively) on \(\Gamma_{t \ell}\), as an increase in output \(\hat{y}_t\) (for given government expenditures and given supply shocks), a decrease in government expenditures \(\hat{g}_t\) (for a given output), and a negative supply shock \(\hat{\varphi}_t\) (for \(\varphi_t \in \{ \varphi_{1, t}, \varphi_{2, t}, \varphi_{3, t} \}\) and for a given output) all raise firms’ marginal cost of production also through the resulting increase in loans and banking costs. Second, and more importantly, a new term appears on the right-hand side, \(-\kappa \delta_m \hat{m}_t\), which reflects a cost channel of monetary policy: an increase in real reserve balances reduces firms’ marginal cost of production through the resulting decrease in banking costs. The parameter \(\kappa \delta_m\) thus depends (positively) on \(|\Gamma_{\ell m}|\).

To derive the reserves-demand equation (13), we log-linearize the first-order condition (C.7) and use (E.5) to get
\[i_t - i_t^m = -\alpha_m \frac{\Gamma_{\ell m \ell}}{\Gamma_m} \hat{i}_t - \alpha_m \frac{\Gamma_{m m}}{\Gamma_m} \hat{m}_t + \alpha_m \frac{u''y}{u'} \hat{y}_t - \alpha_m \frac{u''y}{u'} \hat{g}_t + \alpha_m \hat{\varphi}_{1, t}, \tag{E.10}\]

where \(i_t^m \equiv \hat{i}_t^m\) and \(\alpha_m \equiv (I^m - I) / I^m < 0\). Using (E.8), we can rewrite (E.10) as the reserves-demand equation
\[\hat{m}_t = \chi_g \hat{y}_t - \chi_i (i_t - i_t^m) - \chi_g \hat{g}_t - \chi_{\varphi} \hat{\varphi}_t,\]
where

\[ \chi_y \equiv \left[ \frac{u''y}{u'} - \frac{\Gamma_{\ell m}}{\Gamma_m} \left( -\frac{u''y}{u'} + \frac{v''h}{v'} f' + \frac{f}{f'h} \right) \right] \left( \frac{\Gamma_{mm}}{\Gamma_m} \right)^{-1} > 0, \]

\[ \chi_i \equiv \left( \alpha_m \frac{\Gamma_{mm}}{\Gamma_m} \right)^{-1} > 0, \]

\[ \chi_g \equiv \left( \frac{u''y}{u'} + \frac{\Gamma_{\ell m}}{\Gamma_m} \left( \frac{\Gamma_{mm}}{\Gamma_m} \right)^{-1} > 0, \]

\[ \chi_\varphi \equiv \left( \frac{v''h}{v'} f' + \frac{f}{f'h} \right) \left( \frac{\Gamma_{mm}}{\Gamma_m} \right)^{-1} \geq 0. \]

Here again, to write this reserves-demand equation in a compact way, we have considered a single supply shock \( \varphi_t \in \{ \varphi_{1,t}, \varphi_{2,t}, \varphi_{3,t}, \varphi_{4,t} \} \) and used indicator functions in the definition of \( \chi_\varphi \).

Equation (13) corresponds to this reserves-demand equation in which, for simplicity, the notations \( \hat{m}_t, \hat{y}_t, \hat{g}_t, \) and \( \hat{\varphi}_t \) are replaced by the notations \( m_t, y_t, g_t, \) and \( \varphi_t \) (as everywhere in the main text). This reserves-demand equation implies that the spread between the interest rates on bonds and on reserves depends positively on output, and negatively on real reserve balances, government expenditures, and supply shocks. The reason is that this spread represents the marginal opportunity cost of holding reserves (rather than bonds serving only as a store of value). It has to be equal to the marginal benefit of holding reserves, i.e. the marginal effect of reserves on banking costs, which depends positively on loans — and hence positively on output (for given government expenditures and given supply shocks), negatively on government expenditures (for a given output), and negatively on supply shocks (for \( \varphi_t \in \{ \varphi_{1,t}, \varphi_{2,t}, \varphi_{3,t} \} \) and for a given output) — and negatively on reserves. The parameters \( \chi_y, \chi_g, \) and \( \chi_\varphi \) thus depend positively on \( |\Gamma_{\ell m}| \), and the parameter \( \chi_i \) negatively on \( \Gamma_{mm} \).

E.2 Proof of the Double Inequality (14)

In this subsection, we prove the double inequality (14). To show that \( \chi_y < 1/\delta_m \), we define

\[ \Omega \equiv \frac{-\theta \left[ 1 - \frac{1}{1 + \frac{\delta_m}{\Gamma_m}} \right] \delta_m \kappa}{\alpha_m (1 - \theta) (1 - \beta \theta) \sigma \chi_i} > 0. \]
and we write

\[
\Omega \left( \frac{1}{\delta_m} - \chi_y \right) = \frac{u' \, \Gamma_{mm} m}{w'' y \, \Gamma_m} \left[ \alpha \alpha \phi \frac{\Gamma_{\ell \ell}}{\Gamma_{\ell}} \left( - \frac{u'' y}{u'} + \frac{v'' h}{v'} \frac{f}{f^\prime h} + \frac{f}{f^\prime h} \right) \right. \\
- (1 + \alpha \alpha \phi) \frac{w'' y}{u'} + \frac{v'' h}{v'} \frac{f}{f^\prime h} - \frac{f f''}{(f')^2} \\
\left. + \alpha \alpha \phi \phi \Gamma_{\ell m} \Gamma_{m} \left( \frac{u'}{u'' y} + \frac{v'' h}{v'} \frac{f}{f^\prime h} + \frac{f}{f^\prime h} \right) \right]
\]

Therefore, the whole expression is positive, which implies that \( \chi_y < 1/\delta_m \).

To show that \( \sigma = -u''(c)y/u'(c) < \chi_y \), we write

\[
\frac{1}{\sigma \chi_i} \left( \chi_y - \sigma \right) = -\alpha \frac{\Gamma_{mm} m}{\Gamma_m} - \alpha_m + \alpha_m \frac{u' \Gamma_{\ell \ell} m}{\Gamma_m} \left( - \frac{u'' y}{u'} + \frac{v'' h}{v'} \frac{f}{f^\prime h} + \frac{f}{f^\prime h} \right) \\
= -\alpha_m + \alpha_m \left( \frac{u'}{u'' y} + \frac{v'' h}{v'} \frac{f}{f^\prime h} + \frac{f}{f^\prime h} \right) - \alpha_m \frac{\Gamma_{\ell \ell} m + m \Gamma_{mm}}{\Gamma_m}.
\]

The last expression is the sum of three terms. The first two terms are positive. And so is the third one, given that

\[
\ell \Gamma_{\ell \ell} m + m \Gamma_{mm} = (1 - d) \ell \Gamma_{\ell \ell} m + d \ell \Gamma_{\ell \ell} m + m \Gamma_{mm} \\
= (1 - d) \ell \Gamma_{\ell \ell} m + d \ell \left[ \frac{v' b}{g_b g_m} + v' b g_{\ell m} \right] + m \left[ \frac{v' b}{g_m} \left( \frac{g_b}{m} \right)^2 + v' b g_{mm} \right] \\
= (1 - d) \ell \Gamma_{\ell \ell} m + v' b g_{\ell m} \left( d g_b + m g_{mm} \right) + v' b \left( d g_{\ell m} + m g_{mm} \right) \\
\leq 0,
\]

where the last equality comes from (C.25) and (C.27). Therefore, the whole expression is positive, which implies that \( \sigma < \chi_y \).

### E.3 Resolution of the Paradox of Flexibility

In this subsection, we formally show that our model with banks solves the paradox of flexibility. We proceed along the same lines as with the simple model in Subsection B.1.
The unique bounded solution of our model with banks is characterized by (8) and (9) with $Z_{t+k}$, \( \vartheta \), \( \delta_2 g_t + \delta_\varphi \varphi_t \), and \( \xi_j \) replaced by \( Z'_{t+k} \), \( \vartheta^* \), \( \delta_m M_t + \delta_2 g_t + \delta_\varphi \varphi_t \), and \( \xi^*_j \) for \( j \in \{1, 2\} \). Using the definition of \( Z^*_t \), and after some simple algebra, we can rewrite these two equations as

\[
\pi_t = - (1 - \rho) p_{t-1} + \frac{\kappa}{\beta (\omega_2 - \omega_1)} E_t \left\{ - \frac{1}{\sigma} \sum_{k=0}^{+\infty} \left( \omega_2^{-k-1} - \omega_1^{-k-1} \right) (r_{t+k}^m - r_{t+k}) \right\} + \sum_{k=0}^{+\infty} \left( \xi_1^M \omega_1^{-k-1} - \xi_2^M \omega_2^{-k-1} \right) M_{t+k} - \sum_{k=0}^{+\infty} \left( \xi_1^{g*} \omega_1^{-k-1} - \xi_2^{g*} \omega_2^{-k-1} \right) g_{t+k} + \sum_{k=0}^{+\infty} \left( \xi_1^{\varphi*} \omega_1^{-k-1} - \xi_2^{\varphi*} \omega_2^{-k-1} \right) \varphi_{t+k} \right\},
\]

(E.12)

\[
y_t = -\vartheta^* p_{t-1} + g_t + \frac{E_t}{\beta (\omega_2 - \omega_1)} \left\{ \frac{1}{\sigma} \sum_{k=0}^{+\infty} \left( \xi_1^* \omega_1^{-k-1} - \xi_2^* \omega_2^{-k-1} \right) (r_{t+k}^m - r_{t+k}) \right\} - \sum_{k=0}^{+\infty} \left( \xi_1^M \omega_1^{-k-1} - \xi_2^M \omega_2^{-k-1} \right) M_{t+k} + \sum_{k=0}^{+\infty} \left( \xi_1^{g*} \omega_1^{-k-1} - \xi_2^{g*} \omega_2^{-k-1} \right) g_{t+k} - \sum_{k=0}^{+\infty} \left( \xi_1^{\varphi*} \omega_1^{-k-1} - \xi_2^{\varphi*} \omega_2^{-k-1} \right) \varphi_{t+k} \right\},
\]

(E.13)

where

\[
\begin{align*}
\xi^*_j &\equiv \beta (\omega_j + \rho - 1) + \kappa \delta_m - 1, \\
\xi^M_j &\equiv \delta_m (\omega_j - 1) + \frac{1 - \delta_m \chi_y}{\sigma \chi_i}, \\
\xi^{g*}_j &\equiv (1 - \delta_y) (\omega_j - 1) + \frac{\delta_y \chi_y - \chi_y}{\sigma \chi_i}, \\
\xi^{\varphi*}_j &\equiv \delta_\varphi (\omega_j - 1) + \frac{\chi_\varphi - \delta_\varphi \chi_y}{\sigma \chi_i}
\end{align*}
\]

for \( j \in \{1, 2\} \).

The only parameter that depends on the degree of price stickiness \( \vartheta \) in the structural equations (1), (12), and (13) is the slope \( \kappa \) of the Phillips curve (12). We have \( \lim_{\vartheta \to 0} \kappa = +\infty \) and hence

\[
\lim_{\vartheta \to 0} \left[ \frac{-\beta \sigma P^* (X)}{\kappa (1 - \sigma \delta_m)} \right] = X (X - \omega^n_1)
\]

for any \( X \in \mathbb{R} \), where

\[
\omega^n_1 \equiv 1 + \frac{1 - \delta_m \chi_y}{(1 - \sigma \delta_m) \chi_i} > 1,
\]

where in turn the inequality follows from (14). Therefore, we get

\[
\lim_{\vartheta \to 0} \rho = 0, \quad \lim_{\vartheta \to 0} \omega_1 = \omega^n_1, \quad \lim_{\vartheta \to 0} \omega_2 = +\infty.
\]

(E.14)

Using (E.14) and

\[
(1 - \rho) (\omega_1 - 1) (\omega_2 - 1) = P^* (1) = \frac{(1 - \delta_m \chi_y) \kappa}{\beta \sigma \chi_i},
\]

we also get that

\[
\lim_{\vartheta \to 0} \frac{\kappa}{\omega_2} = \frac{\beta \sigma}{1 - \sigma \delta_m}.
\]

(E.15)

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Using (E.14) and (E.15), we can easily determine the limits of (E.12) and (E.13) as $\theta \to 0$:

$$
\lim_{\theta \to 0} \pi_t = -p_{t-1} + \frac{1}{1 - \sigma \delta_m} E_t \left\{ \sum_{k=0}^{+\infty} (\omega_t^n)^{k-1} \left\{ -\left( i_{t+k}^m - r_{t+k} \right) + (\omega_t^n - 1) M_{t+k} \right\} + \frac{\sigma}{1 - \sigma \delta_m} \left[ \sigma (1 - \delta_g) (\omega_t^n - 1) + \delta_g \chi_y - \chi_y \right] g_{t+k} + \left[ \sigma \delta \varphi (\omega_t^n - 1) + \frac{\chi_y - \delta \varphi \chi_y}{\chi_i} \right] \varphi_{t+k} \right\} \right\}
$$

$$
\lim_{\theta \to 0} y_t = \frac{\delta_m}{1 - \sigma \delta_m} E_t \left\{ \sum_{k=0}^{+\infty} (\omega_t^n)^{k-1} \left\{ i_{t+k}^m - r_{t+k} - (\omega_t^n - 1) M_{t+k} \right\} \right\}
+ \frac{1}{1 - \sigma \delta_m} \left[ \delta_m M_t + \left( \delta_g - \sigma \delta_m \right) g_t + \delta \varphi \varphi_t \right].
\tag{E.16}
$$

These limits are finite, unlike their counterparts in the basic NK model.

We now show that the right-hand sides of (E.16) and (E.17) coincide with the values taken by $\pi_t$ and $y_t$ when prices are perfectly flexible ($\theta = 0$). To determine these values, we first log-linearize the first-order condition of firms’ optimization problem under flexible prices (D.4), and use (E.2), to get

$$
\hat{w}_t = f f'' f'' f + \alpha_t \left( i_t^l - i_t \right) + \left[ 1 - \frac{f f''}{(f')^2} \right] \hat{\varphi}_{3,t} + \hat{\varphi}_{4,t} \frac{1}{\varepsilon - 1}.
\tag{E.18}
$$

Using (E.6), (E.7), (E.8), (E.18), considering a single supply shock $\varphi_t \in \{ \varphi_{1,t}, \varphi_{2,t}, \varphi_{3,t}, \varphi_{4,t} \}$, and replacing the notations $\hat{y}_t, \hat{m}_t, \hat{g}_t$, and $\hat{\varphi}_t$ by the notations $y_t, m_t, g_t$, and $\varphi_t$ (for simplicity and consistency with the main text), we then get

$$
y_t = \delta_m m_t + \delta_g g_t + \delta \varphi \varphi_t.
\tag{E.19}
$$

Finally, using the IS equation (1), the money-demand equation (13), the identity $m_t = M_t - p_t$, and the flexible-price equation (E.19), we get the following dynamic equation under flexible prices:

$$
p_t = (\omega_t^n)^{-1} E_t \left\{ p_{t+1} \right\} + \frac{\omega_t^n}{1 - \sigma \delta_m} \left\{ -\left( i_t^m - r_t \right) + \left( \frac{1 - \delta_m \chi_y}{\chi_i} - \sigma \delta_m \right) M_t + \sigma \delta_m E_t \left\{ M_{t+1} \right\} \right\}
+ \left[ \frac{\chi_y - \delta_g \chi_y}{\chi_i} + \sigma (1 - \delta_g) \right] g_t - \sigma (1 - \delta_g) E_t \left\{ g_{t+1} \right\}
+ \left[ \frac{\chi_y - \delta \varphi \chi_y}{\chi_i} - \sigma \delta \varphi \right] \varphi_t + \sigma \delta \varphi \varphi_t \left\{ \varphi_{t+1} \right\}.
\tag{E.20}
$$

Iterating this equation forward to $+\infty$ leads to the following value for the price level $p_t$ in our model with banks under flexible prices:

$$
p_t = \frac{1}{1 - \sigma \delta_m} E_t \left\{ \sum_{k=0}^{+\infty} (\omega_t^n)^{k-1} \left\{ -\left( i_{t+k}^m - r_{t+k} \right) + (\omega_t^n - 1) M_{t+k} \right\} \right\}
+ \frac{\sigma}{1 - \sigma \delta_m} \left[ \sigma (1 - \delta_g) (\omega_t^n - 1) + \delta_g \chi_y - \chi_y \right] g_{t+k} + \left[ \sigma \delta \varphi (\omega_t^n - 1) + \frac{\chi_y - \delta \varphi \chi_y}{\chi_i} \right] \varphi_{t+k} \right\} \right\}
+ \frac{\sigma}{1 - \sigma \delta_m} \left[ -\delta_m M_t + (1 - \delta_g) g_t - \delta \varphi \varphi_t \right].
\tag{E.20}
$$

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which implies in turn that the value of \(\pi_t \equiv p_t - p_{t-1}\) in our model with banks under flexible prices coincides with the right-hand side of (E.16). In turn, using (E.19), (E.20), and the identity \(m_t = M_t - p_t\), we get that the value of \(y_t\) in our model with banks under flexible prices coincides with the right-hand side of (E.17). Thus, our model with banks solves the paradox of flexibility: the limits of \(\pi_t\) and \(y_t\) as \(\theta \to 0\) are finite and coincide with the values of \(\pi_t\) and \(y_t\) when \(\theta = 0\).

E.4 Local-Equilibrium Determinacy With an Initial Date

In this subsection, we assume that time starts at date 0, rather than date \(-\infty\). We cannot assume that prices are sticky (\(\theta > 0\)) at date 0, since there is no date \(-1\). Instead, we assume that prices are flexible at date 0 and sticky afterwards. We show that, under this assumption, our model with banks still delivers local-equilibrium determinacy under exogenous monetary-policy instruments. Whether it delivers determinacy or not does not depend on the presence or the absence of exogenous shocks; so, for simplicity and without any loss in generality, we focus on the case in which there are no exogenous shocks (\(i_m^* = M_t = g_t = r_t = \varphi_t = 0\) for \(t \geq 0\)).

We proceed along the same lines as with the simple model in Subsection B.5. From date 1 onwards, the economy is characterized by the same equilibrium conditions as when time starts at date \(-\infty\), so that \(\pi_t\) and \(y_t\) for \(t \geq 1\) are given by (8) and (9) with \(Z_{t+k}, \vartheta, \delta_g g_t + \delta_\varphi \varphi_t\), and \(\xi_j\) replaced by \(Z^*_{1+k}, \vartheta^* \equiv (1-\rho)(1-\beta\rho)\alpha+\delta_m \rho, \delta_m M_t + \delta_g g_t + \delta_\varphi \varphi_t\), and \(\xi^*_j \equiv \beta(\omega_j + \rho - 1) + \kappa \delta_m - 1\) for \(j \in \{1,2\}\). Applying the operator \(E_0\) to the left- and right-hand sides of these two equations taken at date 1, and using \(M_1 = g_1 = \varphi_1 = Z^*_{1+k} = 0\) for all \(k \geq 0\), we get

\[
E_0 \{\pi_1\} = -(1-\rho) p_0, \quad (E.21)
\]
\[
E_0 \{y_1\} = -\vartheta^* p_0. \quad (E.22)
\]

At date 0, the IS equation (1) and the money-demand equation (13) still hold, and give

\[
y_0 = E_0 \{y_1\} - \frac{1}{\sigma} (i_0 - E_0 \{\pi_1\}), \quad (E.23)
\]
\[
p_0 = -\chi_g y_0 + \chi_t i_0. \quad (E.24)
\]

The Philips curve (12), however, no longer holds at date 0, both because there is no date \(-1\) and because prices are flexible at date 0. The following lemma indicates what becomes of the date-0 Phillips curve:

**Lemma 5 (Phillips Curve at Date 0 in the Model With Banks):** When prices are flexible at date 0 and sticky afterwards, the Phillips curve at date 0 is

\[
0 = \beta E_0 \{\pi_1\} + \kappa (y_0 + \delta_m p_0) \quad (E.25)
\]

in the model with banks in the absence of exogenous shocks.

**Proof:** See Subsection E.5. ■
This date-0 Phillips curve is identical to the equation that would be obtained by replacing \( \pi_0 \) by 0 in the Phillips curve (12) taken at date 0. The reason is essentially the same as in Subsection B.5. At any date \( t \geq 1 \), we have \( \pi_t = (1 - \theta)(p_t^* - p_{t-1}) \) and \( p_t = \theta p_{t-1} + (1 - \theta) p_t^* \), where \( p_t^* \) denotes the newly set price at date \( t \), so that \( \pi_t = \frac{(1-\theta)^2}{\theta} (p_t^* - p_t) \): inflation differs from zero if and only if the newly set price differs from the average price. The Phillips curve (12) at dates \( t \geq 1 \) can thus be rewritten as 

\[
(1 - \theta)^2 \frac{\theta}{\sigma} (p_t^* - p_t) = \beta E_t \{ \pi_{t+1} \} + \kappa (y_t + \delta_m p_t): \text{the newly set price departs from the average price by a factor proportional to expected future inflation and the current output gap. This equilibrium condition, written this way, still holds at date } 0; \text{ only, the newly set price cannot depart from the average price at that date, since all prices are set at that date, so that the equilibrium condition at date } 0 \text{ becomes (E.25).}
\]

The equilibrium conditions (E.21)-(E.25) are five linear equations in the five unknowns \( p_0, y_0, i_0, E_0 \{ \pi_1 \} \), and \( E_0 \{ y_1 \} \), without any forcing term. They imply 

\[
\Theta p_0 = 0,
\]

where 

\[
\Theta \equiv \left[ 1 + \beta (1 - \rho) + \frac{(1 - \sigma \delta_m) \kappa \alpha \phi}{\sigma} + \frac{\beta \chi_y}{\sigma \chi_i} \right] (1 - \rho) + \frac{(1 - \delta_m \chi_y) \kappa}{\sigma \chi_i}
\]

\[
= \left( 1 + \frac{\chi_y}{\sigma \chi_i} \right) \left( \frac{1 - \rho}{\rho} \right)
\]

\[
> 0,
\]

where the last equality is obtained using \( \mathcal{P}^*(\rho) = 0 \) (where, in turn, \( \mathcal{P}^*(X) \) is defined in the Appendix of the paper). Since \( \Theta \neq 0 \), we get \( p_0 = 0 \): the initial price level is uniquely pinned down. Using (E.21) and (E.25), we then obtain a unique solution for \( y_0 \) (\( y_0 = 0 \)). Using (8) and (9) recursively for \( t \geq 1 \), we also obtain a unique solution for \( \pi_t \) and \( y_t \) for \( t \geq 1 \) (\( \pi_t = y_t = 0 \)). We conclude that the model delivers local-equilibrium determinacy.

### E.5 Proof of Lemma 5

In this subsection, we prove Lemma 5. We proceed along the same lines as with the simple model in Subsection B.6.

Although all firms now choose their price at date 0, the first-order condition of firms’ optimization problem at date 0 is still given by (E.1):

\[
\hat{P}_0^* = (1 - \beta \theta) E_0 \left\{ \sum_{t=1}^{\infty} (\beta \theta)^{t-1} \left( \alpha \phi \left( i_t' - i_t \right) + \hat{w}_t + \hat{P}_t - \hat{m} p_{t|0} \right) \right\} = (1 - \beta \theta) \left( \alpha \phi \left( i_0' - i_0 \right) + \hat{w}_0 + \hat{P}_0 - \hat{m} p_{0|0} \right) + \beta \theta E_0 \left\{ \hat{P}_1^* \right\} \]

\[
+ \beta \theta (1 - \beta \theta) E_0 \left\{ \sum_{t=1}^{\infty} (\beta \theta)^{t-1} \left( \hat{m} p_{t|1} - \hat{m} p_{t|0} \right) \right\}.
\]

(E.26)
Equations (B.13)-(B.14) and (B.18)-(B.20) still hold in our model with banks, and enable us to rewrite the terms \( \hat{P}_0, \hat{P}_1, \hat{w}_0, \hat{m}_0, \hat{m}, \hat{p}_{z0} \) in (E.26) as functions of \( \hat{y}_0, \hat{P}_0 \), and \( \hat{P}_1 \). Moreover, we can also use (E.6), (E.8), the identity \( \hat{m}_0 = \hat{M}_0 - \hat{P}_0 \), and \( \hat{M}_0 = \hat{y}_0 = \hat{\varphi}_{1,0} = \hat{\varphi}_{2,0} = \hat{\varphi}_{3,0} = 0 \), to express the term \( i_0^* - i_0 \) in (E.26) as a function of \( \hat{y}_0 \) and \( \hat{P}_0 \):

\[
i_0^* - i_0 = \left[ -\alpha \frac{u''y}{u'} + \alpha \frac{\Gamma_{\ell M}}{\Gamma_{\ell}} \left( -\frac{u''y}{u'} + \frac{v''h}{v' + f'g} + \frac{f}{f'g} \right) \right] \hat{y}_0 - \alpha \frac{\Gamma_{\ell M}}{\Gamma_{\ell}} \hat{P}_0. \tag{E.27}
\]

Thus, using (B.13)-(B.14), (B.18)-(B.20), (E.27), and \( \pi_1 = \hat{P}_1 - \hat{P}_0 \), we can rewrite (E.26) as

\[
0 = \beta \mathbb{E}_0 \{\pi_1\} + \kappa \left( \hat{y}_0 + \delta_m \hat{P}_0 \right).
\]

Equation (E.25) corresponds to this equation in which, for simplicity and consistency with the main text, the notations \( \hat{y}_0 \) and \( \hat{P}_0 \) are replaced by the notations \( y_0 \) and \( p_0 \).

### Appendix F: Model With Banks – Basic-NK-Model Limit

In this appendix, which complements Section 4 in the paper, we show how to make the steady state and log-linearized reduced form of our model with banks converge towards the steady state and log-linearized reduced form of the basic NK model, and we show that the unique local equilibrium of our model with banks under exogenous monetary-policy instruments then converges towards a particular equilibrium (out of an infinity of equilibria) of the basic NK model under an exogenous policy rate, namely the equilibrium characterized by (10) and (11).

#### F.1 Convergence Towards the Basic NK Model and Selected Equilibrium

The results that we obtain in our model with banks hold for any given (dis)utility and production functions \( u, v, v^b, f, f^b \), any given values of the structural parameters \( \beta \in (0,1), \varepsilon > 0, \phi \in (0,1), \theta \in (0,1), \) and any given steady-state values of the IOR rate and government expenditures \( I^m \in [1,\beta^{-1}] \) and \( g \geq 0 \). In this subsection, we first show that the disutility function \( v^b \) and the steady-state value \( I^m \) can be chosen so as to make our model arbitrarily close, in terms of steady state and log-linearized reduced form, to the basic NK model characterized by the same (dis)utility and production functions \( u, v, f \), and the same values of the structural parameters \( \beta, \varepsilon, \theta \), and the same steady-state value of government expenditures \( g \). More specifically, we replace the disutility function \( v^b \) by \( \gamma v^b \) (and hence the banking-cost function \( \Gamma \) by \( \gamma \Gamma \)), where \( \gamma > 0 \) is a scale parameter, and we establish the following proposition:

**Proposition 2 (Convergence Towards the Basic NK Model):** As \((I^m, \gamma) \to (\beta^{-1}, 0)\) with \((\beta^{-1} - I^m)/\gamma \) bounded away from zero and infinity, the steady state and log-linearized reduced form of our model with banks converge towards the steady state and log-linearized reduced form of the basic NK model, i.e. \( h \to h^* \)

\[
(\sigma, \kappa, \delta_m, \varepsilon, \phi, \chi_i^{-1}, \chi_y^{-1}, \chi_g^{-1}, \chi_{\varphi}^{-1}) \to (\sigma^*, \kappa^*, 0, \delta_g^*, \delta_{\varphi}^*, 0, 0, 0),
\]

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where $\sigma^*, \kappa^*, \delta_g^*$, and $\delta_\varphi^*$ denote the parameters of the IS equation $y_t = \mathbb{E}_t\{y_{t+1}\} - (1/\sigma^*)(i_t - \mathbb{E}_t\{\pi_{t+1}\} - r_t) + g_t - \mathbb{E}_t\{g_{t+1}\}$ and the Phillips curve $\pi_t = \beta\mathbb{E}_t\{\pi_{t+1}\} + \kappa^*(y_t - \delta_g^*g_t - \delta_\varphi^*\varphi_t)$ of the basic NK model.

**Proof:** See Subsection F.2. ■

Proposition 2 can be interpreted as follows. Making the steady-state IOR rate $I^m$ go to the steady-state interest rate on bonds $I = \beta^{-1}$ asymptotically removes the steady-state opportunity cost of holding reserves. Making the banking-cost-scale parameter $\gamma$ go to zero asymptotically removes: (i) the steady-state marginal banking cost $\Gamma$, provided that the steady-state value of real reserve balances $m$ is bounded away from zero, and (ii) the steady-state marginal benefit of holding reserves $\Gamma_m$, even when $m$ is bounded from above. Imposing that $(\beta^{-1} - I^m)/\gamma$ be bounded away from zero and infinity ensures that the steady-state opportunity cost and marginal benefit of holding reserves go hand in hand to zero, so that $m$ is itself bounded away from zero and infinity. Asymptotically, given that all steady-state costs related to banking and reserve holding are removed, the steady-state employment level takes its frictionless value ($h = h^*$), the marginal cost of production becomes insensitive to the volume of loans ($\kappa = \kappa^*$, $\delta_g = \delta_g^*$, and $\delta_\varphi = \delta_\varphi^*$), the cost channel of monetary policy is shut down ($\delta_m = 0$), and the interest-rate spread becomes insensitive to reserves, output, government expenditures, and supply shocks ($\chi_i^{-1} = \chi_y\chi_i^{-1} = \chi_\gamma\chi_i^{-1} = \chi_\varphi\chi_i^{-1} = 0$).

We then show that, as $(I^m, \gamma) \to (\beta^{-1}, 0)$ with $(\beta^{-1} - I^m)/\gamma$ bounded away from zero and infinity, the unique local equilibrium of our model with banks under exogenous monetary-policy instruments, characterized by (E.12) and (E.13), converges towards the equilibrium of the basic NK model under an exogenous policy rate characterized by (10) and (11). To do so, we first use Proposition 2 to get that, as $(I^m, \gamma) \to (\beta^{-1}, 0)$ with $(\beta^{-1} - I^m)/\gamma$ bounded away from zero and infinity, we have $\mathcal{P}^*(X) \to (X - 1)\mathcal{P}_b(X)$ for any $X \in \mathbb{R}$, where $\mathcal{P}^*(X)$ is defined in the Appendix of the paper, and therefore $(\rho, \omega_1, \omega_2) \to (\rho_b, 1, \omega_b)$. Using this result, $\beta\rho_b\omega_b = 1$, $\rho_b + \omega_b = 1 + 1/\beta + \kappa/\sigma$, and $\mathcal{P}_b(\rho_b) = 0$, we then easily obtain that the limits of (E.12) and (E.13), as $(I^m, \gamma) \to (\beta^{-1}, 0)$ with $(\beta^{-1} - I^m)/\gamma$ bounded away from zero and infinity, are (10) and (11).

**F.2 Proof of Proposition 2**

In this subsection, we prove Proposition 2. The proof of steady-state convergence largely rests on reasonings similar to the ones conducted in Appendix D. With the introduction of parameter $\gamma$, Equation (D.5) becomes, at the steady state,

$$\gamma\Gamma_{\ell}[\mathcal{L}(h), m] = \mathcal{A}(h). \quad (F.1)$$
This equation implicitly and uniquely defines a function $\tilde{M}$ such that $m = \tilde{M}(h, \gamma)$. The function $\tilde{M}$ is strictly increasing in each of its two arguments ($\tilde{M}_h > 0$ and $\tilde{M}_\gamma > 0$). For any given $\gamma > 0$, the function $h \mapsto \tilde{M}(h, \gamma)$ is defined over $(\bar{h}, \bar{h})$, where $\bar{h} \in (\underline{h}, \underline{h}^*)$ is implicitly and uniquely defined by $\lim_{m \to +\infty} \gamma I_\ell [L(\bar{h}), m] = A(\bar{h})$. Note that $\bar{h}$ depends on $\gamma$ and satisfies

$$\lim_{\gamma \to 0} \bar{h} = \bar{h}^*. \quad (F.2)$$

Now, with the introduction of parameter $\gamma$, Equation (D.13) becomes

$$\bar{F}(h, \gamma) \equiv \frac{\gamma \Gamma_m \left[ L(h), \tilde{M}(h, \gamma) \right]}{u'[f(h) - g]} = \beta I^m - 1. \quad (F.3)$$

Lemma 4 implies that, for any given $\gamma > 0$,

$$\tilde{F}_h > 0 \quad \text{and} \quad \lim_{h_t \to h} \tilde{F}(h_t, \gamma) = 0. \quad (F.4)$$

We can rewrite $\bar{F}(h, \gamma)$ as $\bar{F}(h, \gamma) = \bar{F}_1(h, \gamma) F_2(h)/\phi$, where, for any given $\gamma > 0$, the function $h \mapsto \bar{F}_1(h, \gamma)$ is defined over $(\underline{h}, \bar{h})$ by

$$\bar{F}_1(h, \gamma) \equiv \frac{\Gamma_m \left[ L(h), \tilde{M}(h, \gamma) \right]}{\Gamma_\ell \left[ L(h), \tilde{M}(h, \gamma) \right]} = \frac{g^b_m \left[ L(h), \tilde{M}(h, \gamma) \right]}{g^b_\ell \left[ L(h), \tilde{M}(h, \gamma) \right]},$$

while $F_2$ is defined in Subsection D.2. We have

$$\left( g^b_\ell \right)^2 \bar{F}_{1,\gamma} = \left( g^b_\ell g^b_{mm} - g^b_m g^b_{\ell m} \right) \frac{\tilde{M}_\gamma}{\tilde{M}} = -g^b_{\ell m} \left( dLg^b_\ell + Mg^b_m \right) \frac{\tilde{M}_\gamma}{\tilde{M}} = -g^b g^b_{\ell m} \frac{\tilde{M}_\gamma}{\tilde{M}} > 0,$$

where the second equality is obtained by using (C.27), and the third equality by using (C.25). Therefore, we get $\bar{F}_{1,\gamma} > 0$ and hence, using $F_2 > 0$,

$$\bar{F}_\gamma > 0. \quad (F.5)$$

Using (F.2), (F.3), (F.4), and (F.5), we conclude that

$$\lim_{(I^m, \gamma) \to (\beta^{-1}, 0)} h = h^*. \quad (F.6)$$

As a consequence, the steady-state values of all endogenous variables converge, as $(I^m, \gamma) \to (\beta^{-1}, 0)$, towards their counterparts in the corresponding basic NK model — with the exception of the steady-state value of real reserves $m$, which does not exist in the basic NK model.

We now show that $m$ is bounded away from zero and infinity as $(I^m, \gamma) \to (\beta^{-1}, 0)$ when $(\beta^{-1} - I^m)/\gamma$ is bounded away from zero and infinity. Rewrite (F.3) as

$$\frac{-\Gamma_m \left[ L(h), m \right]}{u'[f(h) - g]} = \frac{1 - \beta I^m}{\gamma}. \quad (F.7)$$

Since the right-hand side of this equation is bounded away from zero, (C.22), (F.6), and (F.7) imply that $m$ is bounded from above. Moreover, (F.5) and $\bar{F} < 0$ imply that, for any given $h_t$,

$$\lim_{\gamma \to 0} \frac{-\bar{F}(h_t, \gamma)}{\gamma} = +\infty. \quad (F.8)$$

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Now, using (C.23) and (F.1), we get, for any given \( h_t \), \( \lim_{\gamma \to 0} \frac{-\tilde{F}(h_t, \gamma)}{\gamma} = \lim_{\gamma \to 0} \frac{-\Gamma_m \left[ \mathcal{L}(h_t), \tilde{M}(h_t, \gamma) \right]}{u'[f(h_t) - g]} = \lim_{m_i \to 0} \frac{-\Gamma_m \left[ \mathcal{L}(h_t), m_i \right]}{u'[f(h_t) - g]} \). (F.9)

Using (F.8) and (F.9), we then get, for any given \( h_t \),

\[
\lim_{m_i \to 0} \frac{-\Gamma_m \left[ \mathcal{L}(h_t), m_i \right]}{u'[f(h_t) - g]} = +\infty.
\]

Using this result, (F.6), (F.7), and the fact that the right-hand side of (F.7) is bounded from above, we conclude that \( m \) is bounded away from zero.

Finally, (F.6) and the boundedness of \( m \) away from zero and infinity imply that, as \((I^m, \gamma) \to (\beta^{-1}, 0)\): (i) the elasticities \( \Gamma_{\ell \ell}/\Gamma_\ell, \Gamma_{mm}m/\Gamma_m, \Gamma_{mn}/\Gamma_m \) and \( \Gamma_{nm}/\Gamma_\ell \) are themselves bounded away from zero and infinity; and (ii) the parameter \( \alpha_\ell \equiv (I^\ell - I)/I^\ell \), which can be rewritten as

\[
\alpha_\ell = \frac{\gamma \Gamma_\ell (\ell, m)}{u'[f(h) - g] + \gamma \Gamma_\ell (\ell, m)}
\]

by using the first-order condition (C.6) amended to take into account the introduction of parameter \( \gamma \), converges towards zero. Moreover, \( \alpha_m \equiv (I^m - I)/I^m \) also converges towards zero as \((I^m, \gamma) \to (\beta^{-1}, 0)\), and \( \alpha_\phi \) is bounded. Therefore, using the definitions of \( \sigma, \kappa, \delta_m, \delta_y, \delta_\varphi, \chi_y, \chi_x, \chi_g \), and \( \chi_\varphi \), we conclude that, as \((I^m, \gamma) \to (\beta^{-1}, 0)\) with \((\beta^{-1} - I^m)/\gamma \) bounded away from zero and infinity, we have

\[
\begin{align*}
\sigma & \to \sigma^* \equiv \frac{-u''y}{u'}, \\
\kappa & \to \kappa^* \equiv \frac{(1 - \theta)(1 - \beta \theta)}{\theta [1 - \frac{\epsilon f''}{(f')^2}]}, \\
\delta_y & \to \delta_y^* \equiv \frac{-u''y}{u'} + \frac{v''f}{\theta f'} \left[ 1 + \frac{v''f}{\theta f'} \right]^{-1}, \\
\delta_\varphi & \to \delta_\varphi^* \equiv \left[ 1 + \frac{v''f}{\theta f'} \right]^{-1} + \frac{v''f}{\theta f'} \left[ 1 + \frac{v''f}{\theta f'} \right]^{-1}
\end{align*}
\]

and \((\delta_m, \chi_x^{-1}, \chi_y \chi_x^{-1}, \chi_g \chi_x^{-1}, \chi_\varphi \chi_x^{-1}) \to (0, 0, 0, 0, 0)\), where all the functions in \( \sigma^*, \kappa^*, \delta_y^*, \) and \( \delta_\varphi^* \) are evaluated at \( h = h^* \) and \( c = f(h^*) - g \).

**Appendix G: Model With Banks – Extension With Liquid Bonds**

Our model with banks abstracts from government bonds and any role they may play in facilitating transactions. In reality, banks may hold government bonds (or other liquid assets), in addition to reserves, for liquidity management. Some regulatory constraints that give rise to a convenience yield for reserves – like the constraint on “high-quality liquid assets” imposed on...
US banks — can also be satisfied by holding government bonds. From this (regulatory) vantage point, bonds and reserves are perfect substitutes in satisfying liquidity needs. But government bonds are not as useful as reserves in satisfying the intra-day liquidity needs that arise from banking transactions, because bonds can either be sold for next-day settlement or used in repo transactions arranged to obtain liquidity; while reserves are readily available for any transaction.

Government bonds also provide a convenience yield to many non-bank entities (e.g. by serving as collateral or international reserve asset) and benefit from regulations (like restrictions on the asset portfolios of US money-market mutual funds). So, the observed returns on government bonds may reflect their convenience yield. If the returns are sufficiently attractive compared to the IOR rate, banks may hold government bonds to satisfy liquidity needs and regulatory constraints. If not, banks may hold mostly reserves for liquidity management.

In this appendix, which complements Section 5 of the paper, we show how our model with banks can be extended by adding government bonds to obtain an equilibrium in which the return on these bonds is below the IOR rate, but the features of equilibrium that we discussed in the main text still apply. Our model abstracts from non-bank financial institutions and foreign entities that may hold bonds. To formalize our main point, we will assume that workers get utility from government bonds (instead of modeling, say, a pension fund that holds bonds on workers’ behalf). We will show that bankers may use government bonds for liquidity management if the IOR rate is sufficiently low compared to the equilibrium return from holding liquid bonds; but bankers will only use reserves for liquidity management when the IOR rate is sufficiently high. Although we don’t explicitly model inside assets like federal-funds loans, we have in mind that our equilibrium with a relatively high IOR rate can also represent observed episodes in which banks don’t lend federal funds, and the federal-funds rate is below the IOR rate. Our main point is that financial institutions that don’t have direct access to the IOR rate may hold these assets in equilibrium, while banks hold reserves with a positive marginal convenience yield.

### G.1 Equilibrium Conditions Related to Households

Each household consists of workers and bankers, as described in Appendix C. Households get utility from consumption \(c_t\) and disutility from labor \(h_t\) for workers, \(h_b\) for bankers, as before. We now assume that workers also get utility from holding government bonds \(b^w\), and bankers may use bonds \(b^b\) as well as reserves \(m\) to produce loans \(\ell\). As before, we make a substitution for \(h_b\) in households’ primitive utility function and get the following reduced-form utility function:

\[
U_t = E_t \left\{ \sum_{k=0}^{\infty} \beta^k \left[ u(c_{t+k}) - \frac{v(h_{t+k})}{\varphi_{t,t+k}} \Gamma(\ell_{t+k}, m_{t+k} + \eta b^b_{t+k}) + z(b^w_{t+k}) \right] \right\},
\]

where \(\eta \in (0,1]\). The function \(z\), defined over the set of positive real numbers \(\mathbb{R}_{>0}\), is twice differentiable, strictly increasing \((z' > 0)\), and strictly concave \((z'' > 0)\); it also satisfies the
usual Inada conditions. Values of $\eta$ below unity may capture the fact that in reality reserves are more useful than government bonds for liquidity management because they provide immediate intra-day liquidity to banks. But allowing for $\eta < 1$ won’t play a major role in our analysis.

In the interest of realism (to make sure some reserves are always held in equilibrium), we also assume that the central bank imposes reserve requirements on banks. Since our model consolidates bankers and workers into households (thus, abstracting from deposits), we specify the reserve requirement as

$$m_t \geq \psi \ell_t,$$

where $\psi > 0$. The household budget constraint, expressed in real terms, is

$$c_t + b_t + b_t^b + b_t^w + \ell_t + m_t \leq \frac{I_{t-1}}{\Pi_t} b_{t-1} + \frac{I_{t-1}^b}{\Pi_t} \left( b_{t-1}^b + b_{t-1}^w \right) + \frac{I_{t-1}^m}{\Pi_t} m_{t-1} + w_t h_t + \tau_t,$$

where $I_t^b$ denotes the gross nominal interest rate on government bonds (while $b_t$ represents a private-sector bond in zero net supply, as we indicated in Appendix C). We let $\lambda_t$ and $\lambda_t^r$ denote the Lagrange multipliers on the period-$t$ budget constraint and reserve requirement (respectively). The optimality conditions are

$$\lambda_t = u'(c_t),$$

$$\lambda_t w_t = \frac{v'(h_t)}{\varphi_{1,t}},$$

$$\lambda_t = \beta I_t E_t \left\{ \frac{\lambda_{t+1}}{\Pi_{t+1}} \right\},$$

$$\lambda_t = z'(b_t^w) + \beta I_t^b E_t \left\{ \frac{\lambda_{t+1}}{\Pi_{t+1}} \right\},$$

$$\frac{\Gamma_t (\ell_t, m_t + \eta b_t^b)}{\varphi_{1,t}} + \lambda_t + \psi \lambda_t^r = \beta I_t^m E_t \left\{ \frac{\lambda_{t+1}}{\Pi_{t+1}} \right\},$$

$$\frac{\Gamma_m (\ell_t, m_t + \eta b_t^b)}{\varphi_{1,t}} + \lambda_t = \lambda_t^r + \beta I_t^m E_t \left\{ \frac{\lambda_{t+1}}{\Pi_{t+1}} \right\},$$

and

$$(m_t - \psi \ell_t) \lambda_t^r = 0.$$

We must also have

$$\eta \frac{\Gamma_m (\ell_t, m_t + \eta b_t^b)}{\varphi_{1,t}} + \lambda_t \geq \beta I_t^b E_t \left\{ \frac{\lambda_{t+1}}{\Pi_{t+1}} \right\}$$

and $b_t^b \geq 0$, with complementary slackness. Using (G.2), we can rewrite (G.3), (G.4), (G.5), and (G.6) in the following simpler forms, which will be useful in the rest of the appendix:

$$\frac{I_t^b}{I_t} = 1 - \frac{z'(b_t^w)}{\lambda_t},$$

$$\frac{I_t^b}{I_t} = 1 + \frac{\Gamma_t (\ell_t, m_t + \eta b_t^b)}{\varphi_{1,t} \lambda_t} + \psi \frac{\lambda_t^r}{\lambda_t},$$

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\[ \frac{I_m^m}{I_t} = 1 + \frac{\Gamma_m (\ell_t, m_t + \eta b^b_t)}{\varphi_{1,t} \lambda_t} - \frac{\lambda^r_t}{\lambda_t}, \quad (G.9) \]

and

\[ \frac{I_b^b}{I_t} \leq (1 - \eta) + \eta \frac{I_m^m}{I_t} + \eta \frac{\lambda^r_t}{\lambda_t}, \quad (G.10) \]

### G.2 Other Equilibrium Conditions

The remaining equilibrium conditions involve minor adjustments to our presentation in Subsections C.2-C.4 of Appendix C (for firms, the government, and market clearing), as we describe below. The equilibrium conditions associated with firms don’t change. For the government, we replace the consolidated budget constraint (C.14) by

\[ M_t + B_t = I_{t-1}^m M_{t-1} + I_{t-1}^b B_{t-1} + P_t g_t - T_t, \quad (G.11) \]

where \( B_t \) denotes the nominal stock of one-period public debt (held outside the central bank). A Ricardian fiscal policy adjusts the lump-sum tax \( T_t \) to stabilize the real public debt around a steady-state target \( b^* > 0 \). The market-clearing conditions of Appendix C still apply – in particular the condition (C.15), given that private-sector bonds are in zero net supply. In addition, we now have the market-clearing condition for government bonds:

\[ b^w_t + b^b_t = \frac{B_t}{P_t}, \quad (G.12) \]

### G.3 Equilibrium of Interest

The household optimality conditions above admit a solution with \( 1 = I_m^m < I_b^b \) that may represent the period before interest payment on reserves in the US (i.e. before 2008). If \( \eta \) is large enough, such a solution may have binding reserve requirements and \( b^b_t > 0 \). In this case, banks use government bonds – and if we extended our model, they could use inside assets like commercial paper – to manage the liquidity of their portfolios.

In this appendix, however, we are interested in a candidate equilibrium in which banks do not use government bonds \( (b^b_t = 0) \), the reserve requirement is not binding \( (\lambda^r_t = 0) \), and the IOR rate is above the government-bond yield \( (I_m^m > I_b^b) \). This candidate equilibrium may capture – admittedly, in a stark way – some features of US bank portfolios and T-bill returns in the aftermath of the financial crisis.

In this subsection, we show that this candidate equilibrium is indeed, under some parameter restrictions, an equilibrium of our model with liquid bonds, and that it coincides with the equilibrium of our model without liquid bonds (in the sense that all the endogenous variables that are common to both models, except the lump-sum tax \( T_t \), take the same equilibrium values). So, all the results that we obtain in our model without liquid bonds and that we discuss in Sections 4 and 5 of the main text also apply to our model with liquid bonds.
To do so, we proceed in four steps: first, we show that any equilibrium with \( b^b_t = \lambda^r_t = 0 \) in the model with liquid bonds necessarily coincides with the equilibrium of the model without liquid bonds; second, we state the parameter restrictions that we consider; third, we show the existence of our steady-state equilibrium of interest; and fourth, we show the existence of our dynamic equilibrium of interest.

In our first step, we note that in any equilibrium with \( b^b_t = \lambda^r_t = 0 \) (in particular in our candidate equilibrium of interest), the household optimality conditions (G.8) and (G.9) collapse to

\[
\frac{I^l_t}{I_t} = 1 + \frac{\Gamma_t(\ell_t, m_t)}{\varphi_{1,t}\lambda_t}
\]

and

\[
\frac{I^m_t}{I_t} = 1 + \frac{\Gamma_m(\ell_t, m_t)}{\varphi_{1,t}\lambda_t}.
\]

These conditions are identical to the optimality conditions (C.6) and (C.7) of our model without liquid bonds. Thus, any equilibrium with \( b^b_t = \lambda^r_t = 0 \) satisfies all the equilibrium conditions (listed in Appendix C) of our model without liquid bonds, except the government budget constraint (C.14), which is replaced by (G.11). Therefore, all the endogenous variables that are common to both models (with and without liquid bonds) take the same equilibrium values in both models, except the lump-sum tax \( T_t \) appearing in the government budget constraint. In essence, because banks do not hold government bonds (\( b^b_t = 0 \)) and face a non-binding reserve requirement (\( \lambda^r_t = 0 \)), they behave in exactly the same way as in our model without liquid bonds.

In our second step, we state the two restrictions that we impose on parameters. The first restriction is

\[
\psi < \overline{\psi}, \tag{G.13}
\]

where \( \overline{\psi} \) denotes the steady-state value of the reserves-to-loans ratio \( m_t/\ell_t \) in our model without liquid bonds. As we will see, this restriction will ensure that the reserve requirement (G.1) is not binding in our model with liquid bonds. Our second parameter restriction is

\[
\max \left\{ 1, \frac{1}{\beta} - \frac{z'(b^r)}{\beta \lambda} \right\} < I^m < \frac{1}{\beta}, \tag{G.14}
\]

where \( \overline{\lambda} \) denotes the upper bound of the steady-state values taken by the marginal utility of consumption \( \lambda_t \) as \( I^m \) varies from 1 to \( 1/\beta \) in our model without liquid bonds (this upper bound being reached for \( I^m = 1 \)). As we will see, that restriction will ensure that the interest rate on government bonds \( I^b_t \) is lower than the IOR rate \( I^m_t \) in our model with liquid bonds. In fact, that restriction will turn out to be sufficient but not necessary for \( I^b_t < I^m_t \); for simplicity, we relegate to the next subsection the statement of the (more complex) parameter restriction that is necessary and sufficient for that matter.

In our third step, we show that our model with liquid bonds, under the two parameter restrictions (G.13) and (G.14), has a steady-state equilibrium with \( I^b < I^m \) and \( b^b = \lambda^r = 0 \) that coincides
with the steady-state equilibrium of the model without liquid bonds (in the sense that all the endogenous variables that are common to both models, except the lump-sum tax $T_t$, take the same steady-state values). To that aim, consider a candidate steady-state equilibrium with $b^b = \lambda^r = 0$. In this candidate equilibrium, as follows from the first step above, all the endogenous variables that also appear in the model without liquid bonds, except the lump-sum tax $T_t$, take the same steady-state values as in that model. Using these values and $b^b = 0$, we then get residually the steady-state values of the other endogenous variables: (i) $b^w$ and $B$ from the market-clearing condition (G.12) and the steady-state target $B/P = b^*$(ii) $I^b$ from the first-order condition (G.7); and (iii) $T$ from the consolidated budget constraint of the government (G.11).

At this stage, all equality conditions for steady-state equilibrium are satisfied, and the steady-state value of all endogenous variables is pinned down. What remains to be shown is that: (i) the inequality conditions for steady-state equilibrium, i.e. the steady-state versions of (G.1) and (G.10), are satisfied as strict inequalities, implying that the candidate steady-state equilibrium is indeed a steady-state equilibrium; and (ii) this equilibrium has the property that $I^b < I^m$. We first establish this last inequality by using in turn the first-order condition (G.7) with $I = 1/\beta$ and $b^w = b^*$, the inequality $\lambda \leq \bar{X}$, and the parameter restriction (G.14), to get

$$I^b = \frac{1}{\beta} - \frac{z'(b^*)}{\beta \lambda} \leq \frac{1}{\beta} - \frac{z'(b^*)}{\beta \bar{X}} < I^m.$$  

In turn, the property $I^b < I^m$, together with $I^m < I$ and $\lambda^r = 0$, implies that the steady-state version of (G.10) is satisfied as a strict inequality:

$$\frac{I^b}{I} < (1 - \eta) + \eta \frac{I^m}{I} + \eta \frac{\lambda^r}{\bar{X}},$$

which implies in turn, through the complementary-slackness condition, that $b^b = 0$: banks hold only reserves because they pay more interest than government bonds ($I^b < I^m$) and are at least as liquid as government bonds ($\eta \leq 1$). Finally, the parameter restriction (G.13) straightforwardly implies that the steady-state version of (G.1) is satisfied as a strict inequality. We conclude that our model with liquid bonds does indeed have a steady-state equilibrium with $I^b < I^m$ and $b^b = \lambda^r = 0$ that coincides with the steady-state equilibrium of the model without liquid bonds.

In our fourth and last step, we proceed similarly to show that our model with liquid bonds, under the parameter restrictions (G.13) and (G.14), has a dynamic equilibrium with $I_t^b < I_t^m$ and $b_t^b = \lambda_t^r = 0$ that coincides with the dynamic equilibrium of the model without liquid bonds (in the sense that all the endogenous variables that are common to both models, except the lump-sum tax $T_t$, take the same equilibrium values). More specifically, we consider a candidate equilibrium with $b_t^b = \lambda_t^r = 0$. In this candidate equilibrium, as follows from the first step above, all the endogenous variables that also appear in the model without liquid bonds, except the lump-sum tax $T_t$, take the same equilibrium values as in that model. Using these values and $b_t^b = 0$, we then get residually the equilibrium values of the other endogenous variables (expressed
as log-deviations from their steady-state values, and denoted by letters with hats): (i) \( \hat{b}^m_t \) and \( \hat{B}_t \) from the log-linearized version of the market-clearing condition (G.12) and the fiscal-policy rule; (ii) \( \hat{I}_t \) from the log-linearized version of the first-order condition (G.7); and (iii) \( \hat{T}_t \) from the log-linearized version of the government’s consolidated budget constraint (G.11).

At this stage, all equality conditions for equilibrium are satisfied, and the equilibrium value of all endogenous variables is pinned down. What remains to be shown is that: (i) the inequality conditions for equilibrium, i.e. (G.1) and (G.10), are satisfied as strict inequalities, implying that the candidate equilibrium is indeed an equilibrium; and (ii) this equilibrium has the property that \( I^b_t < I^m_t \). Now, we have just shown that these three strict inequalities are satisfied at the steady state; therefore, by continuity, they are also satisfied in the neighborhood of this steady state, under the standard assumption that shocks are small enough. We conclude that our model with liquid bonds does indeed have a dynamic equilibrium with \( I^b_t < I^m_t \) and \( b^b_t = \lambda_t = 0 \) that coincides with the dynamic equilibrium of the model without liquid bonds.

### G.4 Relaxation of the Parameter Restriction (G.14)

To prove the existence of our equilibrium of interest in the previous subsection, we have used the parameter restriction (G.14), which involves the reduced-form parameter \( \bar{\lambda} \). This restriction, however, can be harmlessly relaxed to some extent, because our proof only rests on the weaker condition

\[
\max \left\{ 1, \frac{1}{\beta} - \frac{z'(b^*)}{\beta \lambda} \right\} < I^m < \frac{1}{\beta}, \tag{G.15}
\]

where the steady-state value \( \lambda \) depends on several parameters of the model – in particular \( \beta \) and \( I^m \), but not \( b^* \). To re-state (G.15) as a condition involving only parameters, we write \( \lambda \) as

\[
\lambda = \Lambda (\beta I^m),
\]

where the function \( \Lambda \) is defined by

\[
\Lambda(x) \equiv u' \{ f [F^{-1}(x - 1)] - g \}
\]

for \( x \in (-\infty, 1] \), where in turn the function \( F \) is defined in Subsection D.2. Given Lemma 4 in Subsection D.2, the function \( \Lambda \) is strictly decreasing \( (\Lambda' < 0) \), with \( \lim_{x \to -\infty} \Lambda(x) = +\infty \). Therefore, there exists a unique \( x^* \in (-\infty, 1) \) such that

\[
1 - \frac{z'(b^*)}{\Lambda(x^*)} = x^*.
\]

We can then re-state (G.15) as

\[
\max \left\{ 1, \frac{x^*}{\beta} \right\} < I^m < \frac{1}{\beta},
\]

where the reduced-form parameter \( x^* \) depends on several parameters of the model – in particular \( b^* \), but not \( \beta \) nor \( I^m \).
Appendix H: Model With Banks – Numerical Simulation

In this appendix, which complements Section 5 of the paper, we describe our non-linear simulation of the model with banks. Our goal is to illustrate that a large monetary expansion (as large as QE2, or even up to four times as large) can have fairly modest inflationary effects. To make this point, we first calibrate our model to a steady-state equilibrium that matches some features of the US economy in November 2010, leading up to QE2; then, we consider the effects of large monetary expansions.

H.1 Calibration

For our simulation, we consider iso-elastic functional forms:

\[ u(c_t) \equiv (1 - \tilde{\sigma})^{-1} c_t^{1 - \tilde{\sigma}}, \]

\[ v(h_t) \equiv V (1 + \eta)^{-1} (h_t)^{1 + \eta}, \]

\[ v_b(h_{bt}) \equiv V_b (1 + \eta)^{-1} (h_{bt})^{1 + \eta}, \]

\[ f(h_t) \equiv A (h_t)^{\alpha}, \]

\[ f_b(h_{bt}, m_t) \equiv A_b (h_{bt})^{1 - \varsigma} (m_t)\varsigma, \]

where \( \tilde{\sigma} > 0, V > 0, \eta \geq 0, V_b > 0, A > 0, 0 < \alpha < 1, A_b > 0, \) and \( 0 < \varsigma < 1. \) These specifications imply

\[ g_b(\ell_t, m_t) = A_b^{-\frac{1}{\varsigma}} (\ell_t)^{\frac{1}{1 - \varsigma}} (m_t)^{-\frac{\varsigma}{1 - \varsigma}}, \]

\[ \Gamma(\ell_t, m_t) = V_b (1 + \eta)^{-1} A_b^{-\frac{1 + \eta}{\varsigma}} (\ell_t)^{\frac{1 + \eta}{1 - \varsigma}} (m_t)^{-\frac{\varsigma}{1 - \varsigma}}. \]

We need to calibrate the parameters characterizing these functional forms (\( \tilde{\sigma}, V, \eta, V_b, A, \alpha, A_b, \varsigma \)), as well as the parameters \( \beta, \varepsilon, \phi, \theta, I^m, \) and \( g. \)

We have three degrees of freedom in our calibration, as we can freely pick units for output \( y_t \) and labor inputs (\( h_t \) and \( h_{bt} \)). So, without any loss in generality, we can set arbitrarily any three of the following four parameters: \( A, A_b, V, \) and \( V_b. \) We choose to normalize \( A, A_b, \) and \( V \) to one.

We set standard values for the parameters \( \tilde{\sigma}, \eta, \alpha, \phi, \varepsilon, \) and \( \theta \) that appear in standard models. The utility function is logarithmic in consumption (\( \tilde{\sigma} = 1 \)) and has a unitary Frisch elasticity of labor supply, for production workers as well as bankers (\( \eta = 1 \)). The elasticity of output with respect to the labor input is \( \alpha = 0.67. \) Firms borrow the entire wage bill (\( \phi = 1 \)), as in Christiano et al. (2005) and Ravenna and Walsh (2006). For the price-setting nexus, we set the elasticity of substitution across differentiated goods to 6 and the Calvo price-rigidity parameter to \( \theta = 0.67 \) (corresponding to “three-quarter price rigidity”). None of these values plays a major role in our simulation results.
We set the net IOR rate $I^n - 1$ to 25 basis points per annum (the value prevailing in November 2010 in the US). To set a value for $\beta$, we need to take a stand on the value of the unobservable interest rate $I$. Nagel (2016) estimates a liquidity premium of 10 basis points per annum on T-bills in November 2010 in the US. As a benchmark, we assume the same figure applies to our $I - I^n$ spread, making the net interest rate $I - 1$ equal to 35 basis points per annum. This assumption does matter for our results and we will discuss variations below. Since we have no inflation in the steady state, our target for $I$ then pins down the discount factor to $\beta = 1/I = 0.999$ (on a quarterly basis).

We set the remaining three parameters $g$, $\varsigma$, and $V_b$, so as to reach the following three steady-state targets: (i) the share of government purchases in output is $g/y = 0.3$; (ii) the net interest rate on bank loans $I_l - 1$ is 3.25% per annum (the prime loan rate in November 2010 in the US); and (iii) the ratio of bank reserves to loans is $m/\ell = 1/9$ (the ratio of total reserves to bank credit of all commercial banks in November 2010 in the US). Our simulation results, reported and discussed in the next subsection, are not sensitive to plausible variations in the values we pick for these targets.

To see how these targets pin down the parameters $g$, $\varsigma$, and $V_b$, we first rewrite firms’ first-order condition under flexible prices (C.13) as

$$w = \alpha A \left( \frac{\varepsilon - 1}{\varepsilon} \right) \left[ \phi \frac{I^\ell}{I} + (1 - \phi) \right]^{-1} h^{-(1-\alpha)}$$

(H.1)

in the steady state, and we use (C.3), (C.9), and (C.17) to rewrite households’ intra-temporal first-order condition (C.5) in the steady state as

$$w = VA^{\tilde{\sigma}} \left( 1 - \frac{g}{y} \right)^{\tilde{\sigma}} h^{\tilde{\sigma} + \alpha \tilde{\sigma}}.$$  

(H.2)

Equations (H.1) and (H.2) give the steady-state value of hours worked $h$ as a function of $g/y$, $I^\ell$, and already calibrated parameters:

$$h = \left\{ \alpha A^{1-\tilde{\sigma}} \left( \frac{\varepsilon - 1}{\varepsilon} \right) \left( 1 - \frac{g}{y} \right)^{-\tilde{\sigma}} \left[ \phi \frac{I^\ell}{I} + (1 - \phi) \right]^{-1} \right\}^{\frac{1}{\tilde{\sigma} + \alpha \tilde{\sigma} + (1-\alpha)}}.$$

Thus, the targets for $g/y$ and $I^\ell$ pin down $h$ and hence $g = (g/y)y = (g/y)Ah^\alpha$; we get $g = 0.28$.

Next, we rewrite households’ first-order conditions (C.6) and (C.7) in the steady state as

$$\beta I^\ell = 1 + \frac{V_b}{(1-\varsigma)\lambda} b^{\frac{1+\varsigma}{1-\varsigma}} \frac{\varsigma}{m^{\frac{1+\varsigma}{1-\varsigma}}}$$

(H.3)

and

$$\beta I^n = 1 - \frac{\varsigma V_b}{(1-\varsigma)\lambda} b^{\frac{1+\varsigma}{1-\varsigma}} \frac{\varsigma}{m^{\frac{1+\varsigma}{1-\varsigma}}}.$$  

(H.4)

Equations (H.3) and (H.4) give parameter $\varsigma$ as a function of $I^\ell$, $m/\ell$, and already calibrated parameters:

$$\varsigma = \left( \frac{m}{\ell} \right) \left( 1 - \frac{\beta I^n}{\beta I^\ell - 1} \right).$$

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Thus, the targets for $I^\ell$ and $m/\ell$ pin down $\varsigma$; we get $\varsigma = 0.0039$.

Finally, we can plug the value obtained for $h$ into either (H.1) or (H.2) to get the steady-state real wage $w$. Using the steady-state version of the borrowing constraint (C.10) holding with equality, we then get the steady-state value of real loans $\ell = \phi w h$, from which we get in turn the steady-state value of real reserves $m = (m/\ell)\ell$. The value that we have obtained for $h$ also gives us the steady-state value of consumption $c = y - g = (1 - g/y)Ah^\alpha$, from which we get in turn the steady-state value of the marginal utility of consumption $\lambda = c^{\tilde{\sigma}}$. By plugging these values of $\ell$, $m$, and $\lambda$, as well as the value that we have obtained for $\varsigma$, into either (H.3) or (H.4), we can recover the implied value of $V_b$; we get $V_b = 0.021$.

H.2 Simulations

We use Dynare for our non-linear simulations. Figure 2 in the text shows the results of four simulations. One simulation considers a monetary expansion like QE2, raising the balance-sheet size from $1$ trillion to $1.6$ trillion over the course of 3 quarters. The other three simulations raise the balance-sheet size by 2, 3, or 4 times as much, i.e. from $1$ trillion to $2.2$ trillion, $2.8$ trillion, or $3.4$ trillion. The monetary expansions that we consider are temporary: the balance-sheet size rises over 3 quarters, remains at its new value for 15 quarters, and goes back to its initial value over 3 quarters. Figure 2 shows that the “single QE2” expansion makes the spread $I_t - I^m_t$ fall from 10 to 6.2 basis points after a few quarters; it only increases inflation by 16 basis points (per annum) upon impact. Larger monetary expansions bring the spread closer to zero and still produce modest increases in inflation; a monetary expansion that is four times as large as QE2 only raises inflation by 30 basis points in our model.

Our simulations start with a small $I - I^m$ spread. This reflects a presumption that the already large level of reserve balances in the US prior to QE2 had “nearly satiated” the demand for real reserves (although our model does not have a finite satiation level of demand for real reserves). A large increase in the supply of reserves reduces the spread further, and this suffices to induce a large increase in demand for real reserves. Our results are not sensitive to the values we assume for most of our parameters (although we could make the impact effects on inflation even smaller if we raised the price rigidity parameter — say, to $\theta = 0.75$). Only two features really matter for the results.

First, the balance-sheet expansion is expected to be temporary. To see how his matters, note that in the extreme case of a permanent increase in nominal reserves, our model would imply a proportional price increase. This is because we assume that the central bank does not change $I^m$ and our representative-consumer setup pins down $I = 1/\beta$; so, the steady-state spread $I - I^m$ cannot shrink to raise the demand for real reserves if our monetary expansion is permanent. Our assumption that the unusual monetary expansion was not expected to last more than 5 years does not seem unreasonable to us, in light of commentary on how the financial crisis was not
expected to last as long as it did. At any rate, the inflationary effects of temporary monetary expansions that are expected to last longer than 5 years are also modest in our model. For example, if our “single QE2” expansion is expected to last 10 years (instead of 5), the impact effect on inflation only rises to 36 basis points (instead of 16).

The second assumption that matters for our low-inflation result is our small $I - I_m$ spread. If we set the steady-state spread to 20 basis points (instead of 10), the inflationary impact of our “single QE2” expansion is 33 basis points (instead of 16); and if we did set the spread to 50 basis points, our inflation number would rise to 80 basis points. Of course, smaller spreads than our value of 10 basis points would strengthen our claim: cutting the spread to 5 basis points reduces the inflationary impact to 8 basis points.

Appendix I: Model With Banks – Normal Times

In our model with banks, at the zero lower bound on nominal interest rates, the central bank has no possibility other than pegging the IOR rate, which is the interest rate it directly controls. Away from this lower bound, however, it has various possibilities. In this appendix, which complements Section 6 in the paper, we investigate the consequences of two possibilities motivated by alternative views about how the Federal Reserve may conduct monetary policy in the future.

The commentary on the Federal Reserve’s policy options (e.g., Bernanke, 2015; Dudley, 2018; Powell, 2017), which probably applies to other major central banks as well, identifies two main options that involve alternative plans for balance-sheet contraction. The first option is to shrink the balance sheet substantially, set an interbank-rate target depending on the state of the economy, and adjust endogenously the quantity of reserves to make the interbank rate hit this target. The policy-oriented discussions are not precise about the details of how the IOR rate may be set in this option. In particular, the IOR rate could be permanently set to zero (in net terms), as was the case in the US before the crisis. Alternatively, it could be set at a fixed spread from the interbank-rate target, as was the case in the euro area before the crisis. Some commentary refers to this first option as a “corridor system” (although a corridor system may also refer more specifically to the setting of a floor and a ceiling for the interbank rate). The second option is to keep the balance sheet large (or let it shrink slowly and predictably over time as central-bank assets mature), without actively managing the quantity of reserves, and set the interest rate on excess reserves (and, perhaps, the reverse-repo rate) depending on the state of the economy. Some commentary refers to this second option as a “floor system.”

We show that these two options can have substantially different implications for determinacy in the context of our model with banks. First, we consider a corridor system in which the spread between the IOR rate and the interbank rate is kept fixed (by endogenously adjusting the stock of bank reserves) and policy sets a state-contingent target for the interbank rate. We show that
under this corridor system, our model with banks is isomorphic to the basic NK model if we treat the interest rate in the IS equation as the interbank rate.\textsuperscript{10} As a consequence, our model under this corridor system has the same implications for determinacy as the basic NK model. In particular, if the interbank-rate target reacts only to current inflation, it needs to react more than one-to-one in order to ensure determinacy (the so-called “Taylor principle”). Second, we consider a floor system in which policy sets the size of the balance sheet exogenously and sets the IOR rate depending on the state of the economy. We show that if the IOR rate reacts only to current inflation, then we get determinacy for any positive response (or even a peg) under this floor system, in contrast to what we get under the corridor system above; and if the IOR rate also reacts to current output, determinacy conditions remain quite lax.

I.1 Corridor System: Fixed Spread and Interest-Rate Rule

The first option that we consider is a corridor system that maintains a fixed spread between the IOR rate and the interbank rate and sets an interbank-rate target depending on the state of the economy. We treat \( I_t \) as the interbank rate in our model, for the sake of comparability with the basic NK model in which the interest rate featuring in the IS equation is also treated as the interbank rate. It is easy to show, along the same lines as in Subsections D.1 and D.2, that our model has a unique zero-inflation steady state under such a corridor system.\textsuperscript{11} We log-linearize the model around this steady state and, for simplicity, keep the same notations as previously. Under such a corridor system, the reserves-demand equation (13) becomes

\[
m_t = \chi y_t - \chi g_t - \chi \varphi t,\]

so that the Phillips curve (12) can be rewritten as

\[
\pi_t = \beta \mathbb{E}_t \{ \pi_{t+1} \} + \kappa [(1 - \delta m \chi y) y_t - (\delta g - \delta m \chi g) g_t - (\delta \varphi - \delta m \chi \varphi) \varphi t]. \tag{I.1}
\]

The coefficient \( \kappa (1 - \delta m \chi y) \) of \( y_t \) in the right-hand side of this Phillips curve is positive, given the second inequality of (14). Moreover, the coefficient \( -\kappa (\delta g - \delta m \chi g) \) of \( g_t \) in the right-hand side of this Phillips curve is negative, since

\[
\frac{\Omega}{\delta_m} (\delta_g - \delta m \chi g) = \left[ 1 + \alpha_\ell \alpha_\phi \left( 1 + \frac{\Gamma_{\ell \ell}}{\Gamma_\ell} \right) \frac{\Gamma_{mm} m}{\Gamma_m} + \alpha_\ell \alpha_\phi \left( 1 + \frac{\Gamma_{\ell m}}{\Gamma_m} \right) \frac{\Gamma_{\ell m} m}{\Gamma_\ell} \right] > 0,
\]

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\[\text{\textsuperscript{10}Of course, both models may then have some limitations as they abstract from the convenience yield of interbank loans.}\]

\[\text{\textsuperscript{11}This steady state does not depend on the particular rule considered for the interbank-rate target} \ I^*_t. \text{ In particular, the employment level at this steady state is equal to} \ F(-S), \text{ where} \ S \equiv (I - I^m)/I \text{ does the constant value of the spread. For the steady state to exist, though, the rule has to prescribe the value} \ \beta^{-1} \text{ for} \ I^*_t \text{ at the steady state. Any given rule} \ I^*_t = R(\Theta_t) \text{ for the interbank-rate target} \ I^*_t, \text{ where} \ \Theta_t \text{ is a vector of variables observable at date} \ t, \text{ can be “implemented” by the rule} \ \mu_t = \beta R(\Theta_{t-1}) m_t/(\alpha_{t-1} m_{t-1}) \text{ for the policy instrument} \ \mu_t, \text{ in the spirit of Adão, Correia, and Teles (2011) and Loisel (2009).}\]
where the inequality is obtained using (C.28) and (E.11). Finally, the coefficient \(-\kappa(\delta_\varphi - \delta_m \chi_\varphi)\) of \(\varphi_t\) in the right-hand side of this Phillips curve is also negative, since

\[
\frac{\sigma\Omega}{\delta_m} (\delta_\varphi - \delta_m \chi_\varphi) = \left\{ - \left[ 1 + \alpha_\ell \alpha_\phi \left( 1 + \frac{\Gamma_\ell \ell}{\Gamma_m} \right) + \alpha_\ell \alpha_\phi \frac{\Gamma_{\ell m} \ell}{\Gamma_m} \left( 1 + \frac{\Gamma_{\ell m} \ell}{\Gamma_m} \right) \right] I_{\varphi_t = \varphi_{1,t}} + \left\{ - \left[ 1 + \frac{v''h}{v'} f' f h - \frac{f f''}{(f')^2} + \alpha_\ell \alpha_\phi \frac{\Gamma_\ell \ell}{\Gamma_m} \left( \frac{v''h}{v'} f' f h + \frac{f}{f h} \right) \right] I_{\varphi_t = \varphi_{2,t}} + \left\{ - \left( 1 + \alpha_\ell \alpha_\phi \frac{\Gamma_{\ell m} \ell}{\Gamma_m} \alpha_\ell \alpha_\phi \frac{\Gamma_{\ell m} \ell}{\Gamma_m} \right) I_{\varphi_t = \varphi_{3,t}} + \left[ \left( -1 + \frac{1}{\varepsilon - 1} \right) \frac{\Gamma_{\ell m} m}{\Gamma_m} \right] I_{\varphi_t = \varphi_{4,t}} \right\} \right\} I_{\varphi_t = \varphi_{3,t}} + \left\{ \left( 1 + \alpha_\ell \alpha_\phi \frac{\Gamma_{\ell m} \ell}{\Gamma_m} + \alpha_\ell \alpha_\phi \frac{\Gamma_{\ell m} \ell}{\Gamma_m} \right) I_{\varphi_t = \varphi_{3,t}} + \left( \frac{1}{\varepsilon - 1} \right) \frac{\Gamma_{\ell m} m}{\Gamma_m} \right\} I_{\varphi_t = \varphi_{4,t}} > 0,
\]

where again the inequality is obtained using (C.28) and (E.11). Therefore, the Phillips curve (I.1) of our model with banks under the corridor system is isomorphic to the Phillips curve (2) of the basic NK model. As a consequence, the reduced form of our model with banks under the corridor system, made of the IS equation (1), the Phillips curve (I.1), and any given rule for the interbank-rate target \(i_t\), is isomorphic to the reduced form of the basic NK model, made of the IS equation (1), the Phillips curve (2), and the same rule for \(i_t\). We conclude that our model with banks, under the corridor system, inherits all the implications of the basic NK model for equilibrium determinacy (and equilibrium dynamics) away from the zero lower bound on the nominal IOR rate. In particular, the Taylor principle applies: if the rule makes the interbank-rate target \(i_t\) react non-negatively to current inflation \(i_t = \nu_\pi \pi_t\) with \(\nu_\pi > 0\), then the reaction needs to be more than one-to-one \((\nu_\pi > 1)\) to ensure determinacy.

**I.2 Floor System: Exogenous Reserves and Taylor Rule for the IOR Rate**

The second option that we consider is a floor system under which the central bank sets the growth rate of nominal reserves \(\mu_t\) exogenously (around the value \(\mu = 1\), as previously) and sets the IOR rate \(I^m_t\) according to the Taylor rule

\[
I^m_t = R(\Pi_t, y_t),
\]

Woodford (2003, Chapter 4) obtains a similar isomorphism result (though under some parameter restrictions) in the context of the basic NK model with money entering the utility function in a non-separable way, when the central bank maintains a fixed spread between the interest rate on money and the interest rate on bonds.
where the function $R$, from $\mathbb{R}^2_{>0}$ to $\mathbb{R}_{\geq 0}$, is differentiable and non-decreasing in $\Pi_t$ and $y_t$ (i.e. $R_{\Pi} \geq 0$ and $R_y \geq 0$). Under this floor system, the set of steady states is still characterized by the dynamic flexible-price equation (D.12) with $\mu_{t+1} = \varphi_{1,t} = \varphi_{2,t+1} = \varphi_{2,t} = \varphi_{3,t} = \varphi_{4,t} = 1$, $g_{t+1} = g_t = g$, and $h_{t+1} = h_t$, but now with $I_t^m = R[1, f(h_t)]$ (instead of $I_t^m = I^m$). The resulting equation is

$$ F(h_t) \equiv \frac{\Gamma_m \left[ \mathcal{L}(h_t), \mathcal{M}(h_t) \right]}{\alpha' \left[ f(h_t) - g \right]} = \beta R[1, f(h_t)] - 1. $$

Given Lemma 4, a sufficient condition for existence of a steady state (which is also a necessary condition for existence and uniqueness of a steady state) is

$$ R \left[ 1, f \left( \frac{\Pi}{R_{\geq 0}} \right) \right] < \frac{1}{\beta}. $$

Log-linearizing the model around a steady state, we get the same IS equation (1), Phillips curve (12), and reserves-demand equation (13) as previously, plus now the Taylor rule

$$ i_t^m = \nu_\pi \pi_t + \nu_y y_t, \quad \text{(I.2)} $$

where $\nu_\pi \equiv R_{\Pi}/R \geq 0$ and $\nu_y \equiv (R_{y}y)/R \geq 0$, and where we have replaced the notation $\hat{y}_t$ by the notation $y_t$ (for simplicity and consistency with the other equations). Using (1), (12), (13), and (I.2), we easily get a dynamic equation in the price level $p_t$ whose characteristic polynomial is

$$ P^{**}(X) = X^3 - a_2 X^2 + a_1 X - a_0 $$

with

$$ a_2 = 2 + \frac{1}{\beta} + \frac{(1 - \sigma \delta_m) \kappa}{\beta \sigma} + \frac{\chi_y}{\sigma \chi_i} + \frac{\nu_y}{\sigma} > 3, $$

$$ a_1 = 1 + \frac{2}{\beta} + \frac{(1 - \sigma \delta_m) \kappa}{\beta \sigma} + \frac{1 + \beta}{\beta \sigma} \chi_y + \frac{(1 - \delta_m \chi_y) \kappa}{\beta \sigma \chi_i} + \frac{\kappa \nu_\pi}{\beta \sigma} + \frac{(1 + \beta - \delta_m \kappa) \nu_y}{\beta \sigma}, $$

$$ a_0 = \frac{1}{\beta} + \frac{\chi_y}{\beta \sigma \chi_i} + \frac{\kappa \nu_\pi}{\beta \sigma} + \frac{\nu_y}{\beta \sigma} > 0, $$

where the inequality $a_2 > 3$ follows from the double inequality (14). Given that there are two non-predetermined variables, there is local-equilibrium determinacy if and only if $P^{**}(X)$ has exactly one root inside the unit circle. This root has to be a real number (indeed, if it were a complex number, its conjugate would be another root inside the unit circle). We have $P^{**}(0) = -a_0 < 0$ and $P^{**}(1) = (1 - \delta_m \chi_y - \delta_m \chi_i \nu_y) \kappa / (\beta \sigma \chi_i)$. In the following, we consider two alternative cases in turn, depending on the sign of $P^{**}(1)$.

We first consider the case in which $P^{**}(1) > 0$, that is to say equivalently the case in which

$$ \nu_y < \zeta_1, \quad \text{(I.3)} $$

where

$$ \zeta_1 \equiv \frac{1 - \delta_m \chi_y}{\delta_m \chi_i} > 0, $$

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where in turn the last inequality follows from the second inequality of (14). In this case, $P^{**}(0)$ and $P^{**}(1)$ are of opposite signs, so that $P^{**}(X)$ has either one or three real roots inside $(0,1)$. Moreover, in this case, we have

$$a_1 = 1 + \frac{2}{\beta} + \frac{(1 - \sigma \delta_m) \kappa}{\beta \sigma} + \frac{(1 + \beta) \chi_y}{\beta \sigma \chi_i} + \frac{\kappa \nu_\pi}{\beta \sigma} + \frac{(1 + \beta) \nu_y}{\beta \sigma} + \frac{(\zeta_1 - \nu_\pi) \delta_m \kappa}{\beta \sigma},$$

$$> 1 + \frac{2}{\beta} + \frac{(1 - \sigma \delta_m) \kappa}{\beta \sigma} + \frac{(1 + \beta) \chi_y}{\beta \sigma \chi_i} + \frac{\kappa \nu_\pi}{\beta \sigma} + \frac{(1 + \beta) \nu_y}{\beta \sigma},$$

$$> 0,$$

where the first inequality comes from (I.3) and the second one from (14). In turn, $a_1 > 0$, together with $a_0 > 0$ and $a_2 > 0$, implies that $P^{**}(X) < 0$ for all $X < 0$, and hence that $P^{**}(X)$ has no negative real roots. So, $P^{**}(X)$ has at least one real root inside $(0,1)$, which we denote by $\rho$, and its other two roots, which we denote by $\omega_1$ and $\omega_2$ with $|\omega_1| \leq |\omega_2|$, are either (i) both real and inside $(0,1)$, or (ii) both real and higher than 1, or (iii) both complex and conjugates of each other. Now, since $\rho + \omega_1 + \omega_2 = a_2 > 3$, Case (i) is impossible, and in Case (iii) the common real part of $\omega_1$ and $\omega_2$ is higher than 1. Therefore, in the remaining two possible cases, namely Cases (ii) and (iii), $\omega_1$ and $\omega_2$ lie outside the unit circle. As a consequence, we get local-equilibrium determinacy.

We now turn to the alternative case in which $P^{**}(1) < 0$, that is to say equivalently the case in which Condition (I.3) is not met. In this case, $P^{**}(0)$ and $P^{**}(1)$ have the same sign, so that $P^{**}(X)$ has either zero or two real roots inside $(0,1)$. Therefore, a necessary condition for local-equilibrium determinacy is then that $P^{**}(-1)$ be of the opposite sign, i.e. $P^{**}(-1) > 0$, so that $P^{**}(X)$ can have either one or three real roots inside $(-1,0)$. This necessary condition for determinacy can be written as

$$[\delta_m \kappa - 2 (1 + \beta)] \nu_y > 4 (1 + \beta) \sigma + \frac{2 (1 + \beta) \chi_y}{\chi_i} + \frac{2 (1 - \sigma \delta_m) \kappa}{\chi_i} + \frac{(1 - \delta_m \chi_y) \kappa}{\chi_i} + 2 \kappa \nu_\pi.$$

The right-hand side of this inequality is positive, given the double inequality (14). Therefore, the necessary condition for determinacy can be equivalently rewritten as

$$\delta_m \kappa > 2 (1 + \beta) \quad \text{and} \quad \nu_y > \zeta_2 + \zeta_3 \nu_\pi,$$  \hspace{1cm} (I.4)

where

$$\zeta_2 = \frac{4 (1 + \beta) \sigma \chi_i + 2 (1 + \beta) \chi_y + 2 (1 - \sigma \delta_m) \kappa \chi_i + (1 - \delta_m \chi_y) \kappa}{[\delta_m \kappa - 2 (1 + \beta)] \chi_i} > \zeta_1,$$

$$\zeta_3 = \frac{2 \kappa}{\delta_m \kappa - 2 (1 + \beta)} > 0,$$

where in turn the last two inequalities follow from the first inequality of (I.4). We now show that Condition (I.4) is not only necessary, but also sufficient for local-equilibrium determinacy in that case. To that aim, assume that this condition is met. Then, $P^{**}(-1)$ and $P^{**}(0)$ are of opposite signs, so that $P^{**}(X)$ has either one or three real roots inside $(-1,0)$. Let $\rho$ denote
one root of $P^{**}(X)$ inside $(-1,0)$. The other two roots of $P^{**}(X)$, which we denote by $\omega_1$ and $\omega_2$ with $|\omega_1| \leq |\omega_2|$, can be either (i) both real and inside $(-1,0)$, or (ii) both real and inside $(0,1)$, or (iii) both real and outside $(-1,1)$, or (iv) both complex and conjugates of each other. Now, since $\rho + \omega_1 + \omega_2 = a_2 > 3$, Cases (i) and (ii) are impossible, and in Case (iv) the common real part of $\omega_1$ and $\omega_2$ is higher than 1. So, in the remaining two possible cases, namely Cases (iii) and (iv), $\omega_1$ and $\omega_2$ lie outside the unit circle. As a consequence, Condition (I.4) is, indeed, sufficient for local-equilibrium determinacy.

From the results obtained in the two alternative cases considered, we get that there is local-equilibrium determinacy if and only if either Condition (I.3) is met, or Condition (I.3) is not met and Condition (I.4) is met. Now, Conditions (I.3) and (I.4) are mutually exclusive, given that $\zeta_2 > \zeta_1$. Therefore, we conclude that there is local-equilibrium determinacy if and only if either Condition (I.3) or Condition (I.4) is met.

Thus, Condition (I.3) is a sufficient condition for determinacy in our model with banks under this floor system. If the IOR-rate rule (I.2) reacts only to inflation (i.e. $\nu_y = 0$), then Condition (I.3) is necessarily met, and determinacy obtains whatever the non-negative reaction to inflation (i.e. for any value of $\nu_\pi \geq 0$): the Taylor principle does not apply. Alternatively, if the IOR-rate rule (I.2) reacts also to output (i.e. $\nu_y > 0$), then, to get a sense of how lax or stringent Condition (I.3) is, we can consider the same calibration as in Subsection H.1, which corresponds to a large balance-sheet size (consistently with the commentary on the floor-system option for the Federal Reserve). Under this calibration, we get $\zeta_1 = 15.7$. This seems a comfortably high threshold value, in the sense that the coefficient on output in the Taylor rules encountered in the literature is typically one order of magnitude lower. Thus, we view Condition (I.3) as likely to be met, and therefore determinacy as likely to prevail, under such a floor system in our model with banks.

Appendix J: Comparison With CIA and MIU Models

In this appendix, which complements Section 4 in the paper, we compare our model with banks with two other models in turn: the cash-in-advance (CIA) model and the money-in-the-utility-function (MIU) model. The comparison is in terms of: (i) whether the model delivers local-equilibrium determinacy under exogenous monetary-policy instruments (to solve the forward-guidance and fiscal-multiplier puzzles), in particular as prices become perfectly flexible (to solve the paradox of flexibility); (ii) when it does, whether the roots of the characteristic polynomial are positive real numbers (to solve what we call the “reversal puzzle”); (iii) whether the log-linearized reduced form of the model can converge towards that of the basic New Keynesian (NK) model (to use the model as an equilibrium-selection device for the basic NK model).

The results are summarized in Table J.1 on the next page.
<table>
<thead>
<tr>
<th>Model</th>
<th>CIA Model</th>
<th>MIU Model</th>
<th>Model with Banks</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>About the model</strong></td>
<td>Includes a spread term in the Phillips curve, as firms have to keep the cash overnight.</td>
<td>Similar to Woodford’s (2003, Chapter 4) model. Standard assumptions on ( u(c,m) ), similar to our assumptions on ( f(h^b,m) ) in our model with banks.</td>
<td>See Appendix C</td>
</tr>
<tr>
<td><strong>Determinacy under exogenous monetary-policy instruments (3 limit puzzles)</strong></td>
<td>Determinacy not for all FFPVs. As ( \theta \rightarrow 0 ), not for all FFPVs either.</td>
<td>Determinacy for all FFPVs.</td>
<td>Determinacy for all FFPVs</td>
</tr>
<tr>
<td><strong>When determinacy: real positive roots vs. real negative or complex roots (“reversal puzzle”)</strong></td>
<td>2 roots. One in ((0,1)) for all FFPVs. When determinacy, depending on the FFPVs, the other root is either real negative or real positive.</td>
<td>3 roots. One in ((0,1)) for all FFPVs. Depending on the FFPVs, the other roots are either real positive or complex. In particular, for separable utility ( (u_{cm} = 0) ), real positive for ( mu_{mm}/u_m ) small enough, complex for ( mu_{mm}/u_m ) large enough.</td>
<td>3 roots. For all FFPVs, one root in ((0,1)) and the other roots real positive.</td>
</tr>
<tr>
<td><strong>Basic-NK-model limit (3 limit puzzles and paradox of toil)</strong></td>
<td>No convergence.</td>
<td>No convergence in the general case. In some specific cases, e.g. for separable utility ( (u_{cm} = 0) ), convergence as scale parameter of money utility goes to 0 at the same speed as ( I^m \rightarrow \beta^{-1} ). In those specific cases, same selected equilibrium as in our paper.</td>
<td>Convergence as scale parameter of ( v^b(h) ) goes to 0 at the same speed as ( I^m \rightarrow \beta^{-1} ).</td>
</tr>
</tbody>
</table>

**Note:** “FFPVs” stands for “Functional Forms and Parameter Values”.


Whenever possible, we use the same notations as in the paper. For simplicity and without any loss in generality (given our focus in this appendix), we abstract here from government-purchases, preference, and supply shocks (i.e. $\hat{g}_t = r_t = \hat{g}_t = 0$), and consider only monetary-policy shocks (i.e. IOR-rate shocks and money-supply shocks).

J.1 Cash-in-Advance Model

In this subsection, we consider the CIA model under exogenous monetary-policy instruments and do the following: (i) we derive the equilibrium conditions of this model; (ii) we show the existence and uniqueness of a steady state; (iii) we log-linearize the equilibrium conditions around this steady state; (iv) we show that the model may generate local-equilibrium indeterminacy (thus giving rise to the forward-guidance and fiscal-multiplier puzzles), in particular as prices are sufficiently flexible (thus giving rise to the paradox of flexibility); (v) we show that when the model delivers determinacy, one of the roots of the characteristic polynomial may be a negative real number (thus giving rise to what we call the “reversal puzzle”); and (vi) we highlight the fact that the log-linearized reduced form of the model cannot smoothly converge towards that of the basic NK model, and therefore cannot be used as an equilibrium-selection device for the basic NK model.

J.1.1 Equilibrium Conditions

In the CIA model, households choose $b_t$, $c_t$, $h_t$, and $m_t$ to maximize

$$U_t = \mathbb{E}_t \left\{ \sum_{k=0}^{\infty} \beta^k [u(c_{t+k}) - v(h_{t+k})] \right\}$$

subject to their budget constraint

$$b_t + m_t \leq \frac{I_{t-1} b_{t-1}}{\Pi_t} + \frac{I_m^{t-1}}{\Pi_t} (m_{t-1} - c_{t-1}) + w_t h_t + \tau_t$$

and their cash-in-advance constraint

$$c_t \leq m_t,$$

taking all prices as given. The first-order conditions of this maximization problem are

$$u'(c_t) = \beta I_t^m \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\Pi_{t+1}} \right\} + \tilde{\lambda}_t,$$

$$\lambda_t = \beta I_t^m \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\Pi_{t+1}} \right\} + \tilde{\lambda}_t,$$

$$\frac{1}{I_t} = \beta \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\lambda_t \Pi_{t+1}} \right\},$$

$$\lambda_t w_t = v'(h_t),$$

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where $\lambda_t$ and $\tilde{\lambda}_t$ denote the Lagrange multipliers associated with the budget and cash-in-advance constraints respectively. Firms are subject to Calvo’s (1983) constraints on the frequency at which they can change their prices. They receive cash from consumers and hoard it until the next period. Thus, in the specific case of perfectly flexible prices ($\theta = 0$), firm $i$ chooses $P_t(i)$ to maximize

$$
E_t \left\{ \frac{\beta I^m_t \lambda_{t+1}}{\lambda_t \Pi_{t+1}} \right\} \frac{P_t(i) y_t(i)}{P_t} - w_t h_t(i)
$$

subject to the production function

$$
y_t(i) = f[h_t(i)]
$$

and the demand schedule

$$
y_t(i) = \left[ \frac{P_t(i)}{P_t} \right]^{-\varepsilon} y_t.
$$

Using the Euler equation above, and the symmetry between firms, we can write the first-order condition of this flexible-price maximization problem as

$$
\left( \frac{\varepsilon}{\varepsilon - 1} \right) \frac{w_t}{f'(h_t)} = \frac{I^m_t}{I_t}.
$$

Finally, the bond-market-clearing condition is

$$
b_t = 0,
$$

the money-market-clearing condition is

$$
m_t = \frac{M_t}{P_t},
$$

and the goods-market-clearing condition is

$$
c_t + g = y_t.
$$

### J.1.2 Steady-State Existence and Uniqueness

We focus on steady states in which the cash-in-advance constraint is binding, i.e. $I > I^m$ and $c = m$ (as otherwise the model boils down to the basic NK model). As in our model with banks, we consider steady-state values of policy-instruments such that $I^m \geq 1$, $\mu = 1$, and $g \geq 0$. Since $\mu = 1$, the set of steady states is the same under sticky prices ($\theta > 0$) as under flexible prices ($\theta = 0$), so that we can use the first-order condition of firms’ optimization problem under flexible prices (above) to characterize this set. When all variables are constant over time (in particular $h \equiv h_t = h_{t+1}$), the equilibrium conditions above imply

$$
\frac{v'(h)}{f'(h)} = \left( \frac{\varepsilon - 1}{\varepsilon} \right) \beta I^m u'[f(h) - g].
$$

Under standard assumptions on $u$, $v$ and $h$, the left-hand side of this equation is increasing in $h$, from $0$ (as $h = 0$) to $+\infty$ (as $h \to +\infty$), while its right-hand side is decreasing in $h$, from $+\infty$ (as $h \to \overline{h}$, where $\overline{h}$ is defined by $f(\overline{h}) = g$) to $0$ (as $h \to +\infty$). Therefore, there exists a unique solution in $h$ to this equation, and hence a unique steady state of the model.
J.1.3 Log-Linearized System

Log-linearizing the equilibrium conditions around the unique steady state leads to the following equations:

\[
\hat{y}_t = E_t \{ \hat{y}_{t+1} \} - \frac{1}{\sigma} (i_t - E_t \{ \pi_{t+1} \}), \\
\pi_t = \beta E_t \{ \pi_{t+1} \} + \kappa [ \hat{y}_t + \delta_i (i_t - i^m_t) ], \\
\hat{y}_t = \gamma \hat{m}_t,
\]

where \( \hat{m}_t = \hat{M}_t - \hat{P}_t, \pi_t = \hat{P}_t - \hat{P}_{t-1} \), and all parameters are positive:

\[
\sigma \equiv -\frac{u''y}{u'} > 0, \\
\kappa \equiv \frac{(1 - \theta) (1 - \beta \theta)}{\theta [1 - z (f''(f'))^2]} > 0, \\
\delta_i \equiv \frac{u''y}{u'} + \frac{v''h}{v'} f' h - \frac{f f''}{(f')^2} > 0, \\
\gamma \equiv \frac{c}{y} \in (0, 1).
\]

J.1.4 Local-Equilibrium Determinacy

Using the log-linearized equations above, we get the following dynamic equation in \( \hat{P}_t \):

\[
E_t \left\{ L \mathcal{P}_{CIA} \left( L^{-1} \hat{P}_t \right) \right\} = \kappa \delta_i i^m_t - \gamma \sigma \kappa \delta_i E_t \{ \hat{M}_{t+1} \} + (\sigma \delta_i - 1) \gamma \kappa \hat{M}_t,
\]

where

\[
\mathcal{P}_{CIA}(X) \equiv [\beta + (1 - \gamma \sigma) \kappa \delta_i] X^2 - [1 + \beta + \gamma \kappa + (1 - \gamma \sigma) \kappa \delta_i] X + 1.
\]

Since the model has only one non-predetermined variable, we get local-equilibrium determinacy if and only if \( \mathcal{P}_{CIA}(X) \) has one root inside the unit circle and the other root outside. Now, since \( \mathcal{P}_{CIA}(0) = 1 > 0 \) and \( \mathcal{P}_{CIA}(1) = -\gamma \kappa < 0 \), \( \mathcal{P}_{CIA}(X) \) has one real root inside \((0, 1)\). Therefore, the other root of \( \mathcal{P}_{CIA}(X) \) is a real number, and it lies outside the unit circle if and only if \( \mathcal{P}_{CIA}(-1) > 0 \), that is to say if and only if

\[
2 [1 + \beta + (1 - \gamma \sigma) \kappa \delta_i] + \gamma \kappa > 0.
\]

This determinacy condition may be met or not met depending on the calibration. In particular, as prices become perfectly flexible, this condition becomes

\[
\tilde{\sigma} < 2 + \left( \frac{\eta + 1 - \alpha}{\alpha} \right) \gamma,
\]

where \( \alpha \equiv f' h / f, \tilde{\sigma} \equiv -u''c/u', \) and \( \eta \equiv v''h/v' \). As a benchmark calibration, let us set \((\tilde{\sigma}, \eta, \alpha)\) to \((1, 1, 0.67)\) as in Galí (2008, Chapter 3), and \( \gamma \) to 0.7. These are the values that we use in our Online Appendix – 58.
simple model (Section 3 of the main text, Subsections B.2 and B.3 of the online appendix) and in our model with banks (Section 5 of the main text, Subsections H.1, H.2, and I.2 of the online appendix). Under this benchmark calibration, Condition (J.1) is met; but it is no longer met if we depart from this calibration by setting, say, $\tilde{\sigma} = 4$. Thus, under this alternative calibration, the model does not solve the paradox of flexibility.

### J.1.5 Nature of the Roots

Let $\rho \in (0, 1)$ and $\omega$ denote the roots of $P_{CIA}(X)$. Since $P_{CIA}(0) > 0$, we have $\omega < 0$ if and only if $\lim_{X \to -\infty} P_{CIA}(X) < 0$, that is to say if and only if $\beta + (1 - \gamma \sigma) \kappa \delta_i < 0$. Therefore, we get local-equilibrium determinacy and $\omega < 0$ if and only if

$$-\left(1 + \frac{\gamma \kappa}{2}\right) < \beta + (1 - \gamma \sigma) \kappa \delta_i < 0.$$  

As prices become perfectly flexible, this condition becomes

$$-\left[\tilde{\sigma} + \left(\frac{\eta + 1 - \alpha}{\alpha}\right)\gamma \right] \leq 2 (1 - \tilde{\sigma}) \leq 0.$$  

This condition is met under our benchmark calibration described above; and it is also met if we depart from this calibration by setting, say, $\tilde{\sigma} = 2$ or $\tilde{\sigma} = 3$.

When we have local-equilibrium determinacy and $\omega < 0$ (as under our benchmark calibration), the effect of a given interest-rate change at date $T$ on $\pi_1$ is of a given sign for odd values of $T$, and of the opposite sign for even values of $T$ — a property of the model that we call the “reversal puzzle”. To see this, write the dynamic equation in $\hat{P}_t$ as

$$E_t \left\{ (1 - \rho L) \left( L^{-1} - \omega \right) \hat{P}_t \right\} = Z_{CIA,t},$$

where

$$Z_{CIA,t} \equiv \frac{1}{\beta + (1 - \gamma \sigma) \kappa \delta_i} \left[ \kappa \delta_i i_t^m - \gamma \sigma \kappa \delta_i E_t \left\{ \hat{M}_{t+1} \right\} + (\sigma \delta_i - 1) \gamma \kappa \hat{M}_t \right].$$

When there is local-equilibrium determinacy, we have $|\omega| > 1$, so that we can iterate this dynamic equation forward to $+\infty$ and get

$$(1 - \rho L) \hat{P}_t = E_t \left\{ Z_{CIA,t} \right\} = \frac{-1}{\omega} E_t \left\{ \frac{Z_{CIA,t}}{1 - (\omega L)^{-1}} \right\} = -E_t \left\{ \sum_{k=0}^{+\infty} \omega^{-k-1} Z_{CIA,t+k} \right\},$$

which implies

$$\pi_t = - (1 - \rho) \hat{P}_{t-1} - E_t \left\{ \sum_{k=0}^{+\infty} \omega^{-k-1} Z_{CIA,t+k} \right\}.$$  

Assume that the economy is at its steady state at date 0, and that the central bank announces at date 1 a given interest-rate change for some date $T \geq 1$ ($\hat{P}_0 = 0$, $\hat{M}_t = 0$ for $t \geq 1$, $i_t^m = i^* \neq 0$, and $i_t^m = 0$ for $t \geq 1$ and $t \neq T$). We then get

$$\pi_1 = \left[ \frac{-\kappa \delta_i}{\beta + (1 - \gamma \sigma) \kappa \delta_i} \right] \frac{i^*}{\omega^T}.$$  

Therefore, when $\omega < -1$, $\pi_1$ is of a given sign for odd values of $T$, and of the opposite sign for even values of $T$ (“reversal puzzle”).
J.1.6 Convergence Towards the Basic NK Model

As $I^m \to \beta^{-1}$, as long as $I^m \neq \beta^{-1}$, the cash-in-advance constraint is binding and the system of log-linearized equations does not converge towards the reduced form of the basic NK model: in fact, all the reduced-form parameters $\kappa$, $\delta_i$, and $\gamma$ keep constant as $I^m \to \beta^{-1}$, as long as $I^m \neq \beta^{-1}$. At the limit, when $I^m = \beta^{-1}$, the cash-in-advance constraint is slack and the model becomes exactly identical to the basic NK model. Therefore, we cannot use the CIA model to select an equilibrium of the basic NK model under an exogenous policy rate.

J.2 Money-in-the-Utility Model

In this subsection, we consider the MIU model under exogenous monetary-policy instruments and do the following: (i) we derive the equilibrium conditions of this model; (ii) we determine the necessary and sufficient condition for steady-state existence and uniqueness; (iii) we log-linearize the equilibrium conditions around this steady state; (iv) we show that the model always deliver local-equilibrium determinacy (thus solving the three limit puzzles); (v) we show that the characteristic polynomial of the log-linearized reduced form may have two complex (non-real) roots (thus giving rise to what we call the “reversal puzzle”); and (vi) we highlight the fact that except in particular cases, the model’s log-linearized reduced form cannot converge towards the basic NK model’s, and therefore cannot be used as an equilibrium-selection device for the basic NK model.

We use the same MIU model as in Woodford (2003, Chapter 4), except that he considers differentiated types of labor, while we consider for simplicity a single type of labor. This model includes, as a special case, the case of separability between consumption utility and money utility.

J.2.1 Equilibrium Conditions

We write households’ intertemporal utility function as

$$ U_t = E_t \left\{ \sum_{k=0}^{\infty} \beta^k \left[ u(c_{t+k}, m_{t+k}) - v(h_{t+k}) \right] \right\} $$

where the function $u$, defined over $\mathbb{R}_+^2$, is twice differentiable, strictly increasing ($u_c > 0$, $u_m > 0$), concave ($u_{cc} < 0$, $u_{mm} < 0$, $u_{cc} u_{mm} - (u_{cm})^2 \geq 0$), and such that $u_{cm} \geq 0$,

$$ \lim_{c \to 0} u_c(c, m) = +\infty, \quad (J.2) $$

$$ \lim_{c \to +\infty} u_c(c, m) = 0. \quad (J.3) $$

Households choose $b_t$, $c_t$, $h_t$, and $m_t$ to maximize their utility function subject to their budget constraint

$$ c_t + b_t + m_t \leq \frac{I_{t-1}}{\Pi_t} b_{t-1} + \frac{I^n_{t-1}}{\Pi_t} m_{t-1} + w_t h_t + \tau_t. \quad (J.4) $$

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taking all prices as given. The first-order conditions of this maximization problem are

\[ \lambda_t = u_c(c_t, m_t), \]

\[ \frac{1}{I_t} = \beta \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\lambda_t \Pi_{t+1}} \right\}, \]

\[ \lambda_t w_t = v'(h_t), \]

\[ u_m(c_t, m_t) = \lambda_t - \beta I_t^m \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\Pi_{t+1}} \right\}. \]

Using (J.5) and (J.6), we can rewrite the last condition as

\[ \frac{I_t^m}{I_t} = 1 - \frac{u_m(c_t, m_t)}{u_c(c_t, m_t)}. \]

Firms are subject to Calvo’s (1983) constraints on the frequency at which they can change their prices. In the limit case where prices are perfectly flexible (\( \theta = 0 \)), firm \( i \) chooses its price \( P_t(i) \) at date \( t \) to maximize its profit \( P_t(i) y_t(i) - w_t h_t(i) \) subject to the production function

\[ y_t(i) = f[h_t(i)] \]

and the demand schedule

\[ y_t(i) = \left[ \frac{P_t(i)}{P_t} \right]^{-\varepsilon} y_t. \]

The first-order condition of this (flexible-price) optimization problem implies

\[ P_t(i) = \left( \frac{\varepsilon}{\varepsilon - 1} \right) \frac{W_t}{f'(h_t(i))}. \]

In a symmetric (flexible-price) equilibrium, all firms set the same price:

\[ P_t = \left( \frac{\varepsilon}{\varepsilon - 1} \right) \frac{W_t}{f'(h_t)}. \]

Finally, the bond-market-clearing condition is

\[ b_t = 0, \]

the money-market-clearing condition is

\[ m_t = \frac{M_t}{P_t}, \]

and the goods-market-clearing condition is

\[ c_t + g = y_t. \]
### J.2.2 Steady-State Existence and Uniqueness

As in our model with banks, we consider steady-state values of policy-instruments such that \( I^m \geq 1, \mu = 1, \) and \( g \geq 0. \) Since \( \mu = 1, \) the set of steady states is the same under sticky prices \( (\theta > 0) \) as under flexible prices \( (\theta = 0), \) so that we can use the first-order condition of firms’ optimization problem under flexible prices (above) to characterize this set. We first use (J.5), (J.7), (J.9), (J.11), and (J.13) to get

\[
u_c [f (h) - g, m] = \left( \frac{1}{\varepsilon - 1} \right) \frac{v' (h)}{f' (h)}, \tag{J.14}\]

which implicitly and uniquely defines a function \( M \) such that

\[
m = M (h), \tag{J.15}\]

which is strictly increasing \( (M' > 0). \) This function is defined over \( (\underline{h}, +\infty), \) where \( \underline{h} \) is implicitly and uniquely defined by \( f(h) = g. \) We then use (J.6), (J.8), and (J.15) to get

\[
F (h) \equiv - \left( \frac{\varepsilon - 1}{\varepsilon} \right) f' (h) u_m [f (h) - g, M (h)] = \beta I^m - 1, \tag{J.16}\]

where the function \( F \) is defined over \( (\underline{h}, +\infty). \) If \( u_{cm} = 0, \) then \( z(h) \equiv u_m [f (h) - g, M (h)] \) is strictly decreasing in \( h. \) Alternatively, if \( u_{cm} > 0, \) then (J.14) implies that \( u_c [f (h) - g, M (h)] \) is strictly increasing in \( h, \) i.e. that

\[
u_{cc} [f (h) - g, M (h)] f' (h) + u_{cm} [f (h) - g, M (h)] M' (h) > 0,
\]

which implies in turn that

\[
z' (h) = u_{cm} f' (h) + u_{mm} M' (h) < \frac{f' (h)}{u_{cm}} (u_{cc} u_{mm} - u_{cm}^2) \leq 0,
\]

where the functions \( u_{cc}, u_{mm}, \) and \( u_{cm} \) are evaluated at \( [f (h) - g, M (h)], \) which implies in turn that \( z (h) \) is strictly decreasing in \( h. \) In both cases \( (u_{cm} = 0 \) and \( u_{cm} > 0), \) we thus get that \( z (h) \) is strictly decreasing in \( h, \) and therefore that the function \( F \) is strictly increasing \( (F' > 0), \) with

\[
\lim_{h_t \to \underline{h}} F (h_t) = -\infty, \tag{J.17}
\]

\[
\lim_{h_t \to +\infty} F (h_t) = 0.
\]

As a consequence, given (J.16), there is a unique steady state if and only if \( I^m < \beta^{-1}. \)

### J.2.3 Log-Linearized System

Log-linearizing the model around its unique steady state, we get the IS equation

\[
\hat{y}_t = E_t \{ \hat{y}_{t+1} \} - \frac{1}{\sigma} E_t \{ i_t - \pi_{t+1} \} - \xi E_t \{ \Delta \hat{m}_{t+1} \}, \tag{J.18}
\]

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where

\[ \sigma \equiv -\frac{u_{cc}y}{u_c} > 0, \]

\[ \xi \equiv \left( -\frac{u_{cc}y}{u_c} \right)^{-1} \frac{u_{cm}m}{u_c} \geq 0, \]

the Phillips curve

\[ \pi_t = \beta \mathbb{E}_t \{ \pi_{t+1} \} + \kappa (\hat{y}_t - \delta_m \hat{m}_t), \]  

(J.19)

where

\[ \kappa \equiv \frac{(1 - \theta)(1 - \beta \theta)}{\theta} \left[ \frac{-u_{cc}y}{u_c} + \frac{v''h}{v' f' h} \frac{f f''}{(f')^2} \right] > 0, \]

\[ \delta_m \equiv \left[ \frac{-u_{cc}y}{u_c} + \frac{v''h}{v' f' h} \frac{f f''}{(f')^2} \right]^{-1} \frac{u_{cm}m}{u_c} \geq 0, \]

and the money-demand equation

\[ \hat{m}_t = \chi_y \hat{y}_t - \chi_i (I_t - \hat{I}_t), \]  

(J.20)

where

\[ \chi_y \equiv \left( \frac{u_{cm}m}{u_c} - \frac{u_{mm}m}{u_m} \right)^{-1} \left( \frac{u_{cm}c}{u_m} - \frac{u_{cc}y}{u_c} \right) > 0, \]

\[ \chi_i \equiv \left( \frac{u_{cm}m}{u_c} - \frac{u_{mm}m}{u_m} \right)^{-1} \left( \frac{I_m}{I - \hat{I}_m} \right) > 0. \]

### J.2.4 Local-Equilibrium Determinacy

Using the IS equation (J.18), the Phillips curve (J.19), the money-demand equation (J.20), and the identity \( \hat{m}_t = \hat{M}_t - \hat{P}_t \), we get the following dynamic equation in \( \hat{P}_t \):

\[ \mathbb{E}_t \left\{ L \mathcal{P}_{MIU} (L^{-1}) \hat{P}_t \right\} = Z_{MIU,t} \equiv -\frac{\kappa}{\beta \sigma}i_t - \frac{(\xi - \delta_m) \kappa}{\beta} \mathbb{E}_t \{ \hat{M}_{t+1} \} + \left[ \frac{(\xi - \delta_m) \kappa}{\beta} + \frac{(1 - \delta_m X_y) \kappa}{\beta \sigma \chi_i} \right] \hat{M}_t, \]  

(J.21)

where

\[ \mathcal{P}_{MIU} (X) = X^3 - a_2 X^2 + a_1 X - a_0 \]

with

\[ a_2 \equiv 2 + \frac{1}{\beta} + \frac{\kappa}{\beta \sigma} + \frac{(\xi - \delta_m) \kappa}{\beta} + \frac{\chi_y}{\beta \sigma \chi_i} > 3, \]

\[ a_1 \equiv 1 + \frac{2}{\beta} + \frac{\kappa}{\beta \sigma} + \frac{(\xi - \delta_m) \kappa}{\beta} + \frac{(1 + \beta) \chi_y}{\beta \sigma \chi_i} + \frac{(1 - \delta_m X_y) \kappa}{\beta \sigma \chi_i} > 0, \]

\[ a_0 \equiv \frac{1}{\beta} + \frac{\chi_y}{\beta \sigma \chi_i} > 0. \]
The inequality $a_2 > 3$ comes from
\[
\xi - \delta_m = \left(\frac{-uy_c}{u_c}\right)^{-1} \frac{u_c}{u_c} - \left[\frac{-uy_c}{u_c} + \frac{f''}{f'} \frac{h}{h} - \frac{f''}{(f')^2}\right]^{-1} \frac{u_c}{u_c} X
\]
\[
= \left[\frac{-uy_c}{u_c} + \frac{f''}{f'} \frac{h}{h} - \frac{f''}{(f')^2}\right]^{-1} \frac{u_c}{u_c} X
\]
\[
\geq 0,
\]
where the inequality $a_1 > 0$ comes from (J.22) and
\[
1 - \delta_m \chi_g = 1 - \left[\frac{-uy_c}{u_c} + \frac{f''}{f'} \frac{h}{h} - \frac{f''}{(f')^2}\right]^{-1} \frac{u_c}{u_c} X
\]
\[
= \left[\frac{-uy_c}{u_c} + \frac{f''}{f'} \frac{h}{h} - \frac{f''}{(f')^2}\right]^{-1} \frac{u_c}{u_c} X
\]
\[
> 0.
\]
We have $P_{MIU}(0) = -a_0 < 0$ and
\[
P_{MIU}(1) = \frac{1 - \delta_m \chi_g}{\beta \sigma \chi_i} > 0,
\]
where the last inequality comes from (J.23). Therefore, $P_{MIU}(X)$ has either one or three real roots inside $(0, 1)$. Moreover, the inequalities $a_2 > 0$, $a_1 > 0$, and $a_0 > 0$ imply that $P_{MIU}(X) = X^3 - a_2 X^2 + a_1 X - a_0 < 0$ for all $X < 0$, so that $P_{MIU}(X)$ has no negative real roots. Therefore, $P_{MIU}(X)$ has at least one real root inside $(0, 1)$, which we denote by $\rho$, and its other two roots, which we denote by $\omega_1$ and $\omega_2$ with $|\omega_1| \leq |\omega_2|$, are either (i) both real and inside $(0, 1)$, or (ii) both real and higher than 1, or (iii) both complex and conjugates of each other. Now, we have $\rho + \omega_1 + \omega_2 = a_2 > 3$. Therefore, Case (i) is impossible, and in Case (iii) the common real part of $\omega_1$ and $\omega_2$ is higher than 1. As a consequence, in the remaining two possible cases, namely Cases (ii) and (iii), $\omega_1$ and $\omega_2$ lie outside the unit circle. Since the model has two non-predetermined variables, we therefore get local-equilibrium determinacy whatever the functional forms and parameter values considered. As a consequence, the model solves the three limit puzzles (i.e. the forward-guidance puzzle, the fiscal-multiplier puzzle, and the paradox of flexibility) for all functional forms and parameter values.

**J.2.5 Nature of the Roots**

Consider, for example, the separable and iso-elastic specification
\[
u(\chi_i, m_i) = \frac{\nu_i^{1-\sigma} - 1}{1-\sigma} + \frac{\nu_i^{1-\nu} - 1}{1-\nu},
\]
where $\sigma > 0$ and $\nu > 0$. Under this specification, $\sigma$, $\xi$, $\kappa$, $\delta_m$, and $\chi_g/\chi_i$ do not depend on $\nu$, but $\chi_i$ and $\chi_g$ do. Therefore, $a_2$ and $a_0$ do not depend on $\nu$, but $a_1$ does. Since $\lim_{\nu \to +\infty} \chi_i = 0$, the model solves the three limit puzzles (i.e. the forward-guidance puzzle, the fiscal-multiplier puzzle, and the paradox of flexibility) for all functional forms and parameter values.
we have \( \lim_{\nu \to +\infty} a_1 = +\infty \). As a consequence, for sufficiently large values of \( \nu \), \( \mathcal{P}_{MIU}(X) = X^3 - a_2X^2 + a_1X - a_0 \) is positive for all \( X \geq 1 \), so that Case (ii) is impossible and \( \omega_1 \) and \( \omega_2 \) are complex numbers. Moreover, since \( \lim_{\nu \to 0} x_i = +\infty \), we have

\[
\lim_{\nu \to 0} \mathcal{P}_{MIU} \left( 1 + \frac{\chi_y}{\sigma x_i} \right) = - \left( 1 + \frac{\chi_y}{\sigma x_i} \right) \frac{\chi_i}{\beta \sigma^2 x_i} < 0.
\]

Therefore, for sufficiently low values of \( \nu \), we have \( \mathcal{P}_{MIU}[1 + \chi_y/(\sigma x_i)] < 0 \), which, together with \( \mathcal{P}_{MIU}(1) > 0 \), implies that \( \omega_1 \) and \( \omega_2 \) are positive real numbers.

By continuity, there also exist non-separable specifications of \( u \) that can make \( \omega_1 \) and \( \omega_2 \) complex, for instance the iso-elastic specification

\[
u_U(t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma} + m_t^{1-\nu} - \frac{1}{1-\nu} + \epsilon \delta \nu - m_t^{1-\delta},
\]

where \( \delta \in (0,1) \) and \( \epsilon > 0 \) is sufficiently small.

When \( \omega_1 \) and \( \omega_2 \) are complex, the sign of the effect of a given interest-rate change at date \( T \) on \( \pi_1 \) changes an infinity of times as \( T \) moves from 1 to \( +\infty \) — which is another form of the “reversal puzzle” described in Subsubsection J.1.5. To see this, write the dynamic equation in \( \hat{P}_t \) as

\[
E_t \left\{ (L^{-1} - \omega_1) (L^{-1} - \omega_2) \left( 1 - \rho L \right) \hat{P}_t \right\} = Z_{MIU,t},
\]

and iterate this dynamic equation forward to \( +\infty \) to get

\[
\hat{P}_t - \rho \hat{P}_{t-1} = E_t \left\{ Z_{MIU,t} \left( L^{-1} - \omega_1 \right) \left( L^{-1} - \omega_2 \right) \right\} = \frac{E_t}{\omega_2 - \omega_1} \left\{ \sum_{k=0}^{+\infty} \left( \omega_1^{-k-1} - \omega_2^{-k-1} \right) Z_{MIU,t+k} \right\},
\]

which implies in turn that

\[
\pi_t = - (1 - \rho) \hat{P}_{t-1} + \frac{E_t}{\omega_2 - \omega_1} \left\{ \sum_{k=0}^{+\infty} \left( \omega_1^{-k-1} - \omega_2^{-k-1} \right) Z_{MIU,t+k} \right\}.
\]

Assume that the economy is at its steady state at date 0, and that the central bank announces at date 1 a given interest-rate change for some date \( T \geq 1 \) (\( \hat{P}_0 = 0, \hat{M}_t = 0 \) for \( t \geq 1 \), \( i_t^m = i^* \neq 0 \), and \( i_t^m = 0 \) for \( t \geq 1 \) and \( t \neq T \)). Then we get

\[
\pi_1 = \frac{- \left( \omega_1^{-T} - \omega_2^{-T} \right) \kappa i^*}{\beta \sigma (\omega_2 - \omega_1)}.
\]

When \( \omega_1 \) and \( \omega_2 \) are complex, we can write them as \( \omega_1 = re^{i\varphi} \) and \( \omega_2 = re^{-i\varphi} \), where \( r > 0 \) and \( \varphi \in (0,2\pi) \), so that we can rewrite \( \pi_1 \) as

\[
\pi_1 = \frac{- \kappa r^{-T-1} \sin (T \varphi) i^*}{\beta \sigma \sin (\varphi)},
\]

which implies that the sign of \( \pi_1 \) changes an infinity of times as the date \( T \) of the interest-rate change moves from 1 to \( +\infty \) (“reversal puzzle”).
J.2.6 Convergence Towards the Basic NK Model

For some specifications of $u$, one can make the system converge towards the reduced form of the basic New Keynesian model, and we then select the same equilibrium under an exogenous policy rate as in the main text. For separable specifications of $u$, in particular, one can introduce a multiplicative scale parameter into the money-utility function and make this parameter go to zero at the same speed as $I^m$ goes to $\beta^{-1}$. For constant-elasticity-of-substitution specifications $u = [(1 - a)c^z + am^n]^{1/z}$, one can make the weight parameter $a$ go to zero at the same speed as $I^m$ goes to $\beta^{-1}$. But in the general case, however, it is not possible to make the MIU model converge towards the basic NK model.

Appendix K: Comparison With Discounting Models

In this appendix, which complements Section 1 in the paper, we generalize Cochrane’s (2016) comments on Gabaix (2016) to highlight four properties of the “discounting models” proposed in the literature to solve the forward-guidance puzzle, and we show how our model with banks is different.\footnote{None of these four properties is related to the fiscal-multiplier puzzle or the paradox of toil. Exploring the implications of discounting models for this puzzle and that paradox would be interesting, but is beyond the scope of this online appendix.} First, these models do not solve the paradox of flexibility.\footnote{They may attenuate this paradox, though, as Angeletos and Lian (2016) show in the context of their discounting model.} Second, they require a sufficiently large departure from the basic NK model to solve the forward-guidance puzzle. Third, they cannot solve this puzzle without generating a negative long-term relationship between the inflation rate and the interest rate on bonds; by contrast, our model with banks generates the standard Fisher effect, i.e. a one-to-one long-term relationship between these two variables.\footnote{Gabaix (2016), however, introduces price indexation and “inflation guidance” into his benchmark discounting model and shows that the resulting model can both solve the forward-guidance puzzle and make inflation respond positively to the nominal interest rate in the long term.} And fourth, these models cannot solve the forward-guidance puzzle without having non-standard implications for equilibrium determinacy in “normal times,” i.e. away from the zero lower bound on nominal interest rates. By contrast, our model with banks does not necessarily question the implications of the basic NK model about how monetary policy works during normal times; for example, under a corridor system, our model with banks inherits the familiar conditions for equilibrium determinacy.

By “discounting model,” we mean more specifically in this appendix any model whose reduced form, in the absence of shocks other than interest-rate shocks, is made of an IS equation and a Phillips curve of type

\begin{align*}
\hat{y}_t &= \xi_1 \mathbb{E}_t \{\hat{y}_{t+1}\} - \frac{\xi_2}{\sigma} \mathbb{E}_t \{i_t - \pi_{t+1}\}, \tag{K.1} \\
\pi_t &= \beta \xi_3 (\theta) \mathbb{E}_t \{\pi_{t+1}\} + \kappa (\theta) \{\hat{y}_t - \xi_4 (\theta) \mathbb{E}_t \{\hat{y}_{t+1}\}\}. \tag{K.2}
\end{align*}

By “discounting model,” we mean more specifically in this appendix any model whose reduced form, in the absence of shocks other than interest-rate shocks, is made of an IS equation and a Phillips curve of type

\begin{align*}
\hat{y}_t &= \xi_1 \mathbb{E}_t \{\hat{y}_{t+1}\} - \frac{\xi_2}{\sigma} \mathbb{E}_t \{i_t - \pi_{t+1}\}, \tag{K.1} \\
\pi_t &= \beta \xi_3 (\theta) \mathbb{E}_t \{\pi_{t+1}\} + \kappa (\theta) \{\hat{y}_t - \xi_4 (\theta) \mathbb{E}_t \{\hat{y}_{t+1}\}\}. \tag{K.2}
\end{align*}
where \( \beta \in (0, 1), \sigma > 0, \xi_1 > 0, \xi_2 > 0 \), and, for all \( \theta \in (0, 1), \xi_3(\theta) \geq 0, \xi_4(\theta) \in ]0, 1) \), and \( \kappa(\theta) > 0 \), with \( \lim_{\theta \to 0} \xi_3(\theta) < +\infty \) and \( \lim_{\theta \to 0} \kappa(\theta) = +\infty \). This class of reduced forms nests the reduced form of the basic NK model as a special case in which \( \xi_1 = \xi_2 = \xi_3(\theta) = 1 \) and \( \xi_4(\theta) = 0 \). More generally, this class allows the coefficients of \( \mathbb{E}_t\{\hat{y}_{t+1}\} \) and \( \mathbb{E}_t\{\pi_{t+1}\} \) to be smaller (“positive discounting”) or larger (“negative discounting”) than in the basic NK model, and also allows for a \( \mathbb{E}_t\{\hat{y}_{t+1}\} \) term in the Phillips curve. In particular, this class encompasses the reduced forms of three models that have been shown to be able to solve the forward-guidance puzzle: (i) Gabaix’s (2016) benchmark model, in which \( (\xi_1, \xi_3(\theta)) \in ]0, 1) \) and \( \xi_4(\theta) = 0 \); (ii) Angeletos and Lian’s (2016) model, in which \( (\xi_1, \xi_2, \xi_3(\theta), \xi_4(\theta)) \in ]0, 1) \); and (iii) Bilbiie’s (2017) model with myopic firms, in which \( (\xi_1, \xi_2) \in ]0, 1) \) and \( \xi_3(\theta) = \xi_4(\theta) = 0 \). In addition, it also encompasses the reduced forms of: (iv) Bilbiie’s (2017) model with non-myopic firms, in which \( (\xi_1, \xi_2) \in ]0, 1) \), \( \xi_3(\theta) = 1 \), and \( \xi_4(\theta) = 0 \); (v) McKay, Nakamura, and Steinsson’s (2017) model, in which also \( (\xi_1, \xi_2) \in ]0, 1) \), \( \xi_3(\theta) = 1 \), and \( \xi_4(\theta) = 0 \); (vi) Ravn and Sterk’s (2017) model with risk-neutral equity investors, in which \( \xi_3(\theta) = 1 \) and \( \xi_4(\theta) \in ]0, 1) \); and (vii) Woodford’s (2018) model with exponentially distributed planning horizons and no learning, in which \( \xi_1 = \xi_2 = \xi_3(\theta) \in ]0, 1) \) and \( \xi_4(\theta) = 0.17 \).

**K.1 Paradox of Flexibility**

Unlike our model with banks, discounting models do not solve the paradox of flexibility:

**Proposition 3 (Paradox of Flexibility in Discounting Models):** In models whose reduced form is made of an IS equation of type \( (K.1) \) and a Phillips curve of type \( (K.2) \), when the interest rate is set exogenously from date \( 1 \) to date \( T \geq 2 \) and the economy is at its steady state at date \( T+1 \), the responses of \( |\pi_1| \) and \( |\hat{y}_1| \) to an interest-rate change at date \( T \) go to infinity as \( \theta \to 0 \).

**Proof:** Assume that \( i_t = 0 \) for \( 1 \leq t \leq T - 1 \), \( i_T = i^* \neq 0 \), and \( \hat{y}_{T+1} = \pi_{T+1} = 0 \). We start with the case in which \( \xi_3(\theta) > 0 \) or \( \xi_4(\theta) > 0 \). In this case, the system made of the IS equation \( (K.1) \) and the Phillips curve \( (K.2) \) can be rewritten as

\[
\mathbb{E}_t \left\{ \begin{bmatrix} \hat{y}_{t+1} \\ \pi_{t+1} \end{bmatrix} \right\} = C \begin{bmatrix} \hat{y}_t \\ \pi_t \end{bmatrix} + D i_t \tag{K.3}
\]

with

\[
C \equiv \frac{1}{\varphi(\theta)} \begin{bmatrix} \kappa(\theta) \xi_2 + \beta \sigma \xi_3(\theta) - \xi_2 \xi_1 & -\xi_2 \\ \kappa(\theta) \sigma [\xi_4(\theta) - \xi_1] & \sigma \xi_4 \end{bmatrix} \quad \text{and} \quad D \equiv \frac{\xi_2}{\varphi(\theta)} \begin{bmatrix} \beta \xi_3(\theta) \\ \kappa(\theta) \xi_4(\theta) \end{bmatrix},
\]

\( \varphi(\theta) = \kappa(\theta) + \beta \sigma \xi_3(\theta) + \sigma \xi_4(\theta) \).

---

\( ^{16} \)We focus on discrete-time discounting models for the sake of comparability with our model with banks, but we have no reason to expect that continuous-time discounting models behave differently. Indeed, Michaillat and Saez (2018) show that their continuous-time discounting model has the same four properties as the ones listed above.

\( ^{17} \)However, it does not encompass the reduced forms of McKay, Nakamura, and Steinsson’s (2016) and Del Negro, Giannoni, and Patterson’s (2015) models, which involve some discounting too but are more complex.
where \( \varphi(\theta) \equiv \beta \sigma \xi_1 \xi_3(\theta) + \kappa(\theta) \xi_2 \xi_4(\theta) > 0 \). The characteristic polynomial of \( C \) is

\[
C(X) \equiv X^2 - \frac{\sigma \xi_1 + \kappa(\theta) \xi_2 + \beta \sigma \xi_3(\theta)}{\varphi(\theta)} X + \frac{\sigma}{\varphi(\theta)}
\]

Since \( C(0) \neq 0 \), \( C \) is invertible. Iterating the dynamic equation (K.3) forward to date \( T \), and using the terminal condition \( \hat{y}_{T+1} = \pi_{T+1} = 0 \) and the invertibility of \( C \), we get

\[
\begin{bmatrix}
\hat{y}_1 \\
\pi_1 
\end{bmatrix} = -C^{-T} \textbf{D} \textbf{i}^*.
\]

For any \( X \in \mathbb{R} \), we have

\[
\lim_{\theta \to 0} \frac{\varphi(\theta) C(X)}{\kappa(\theta) \xi_2} = \left[ \lim_{\theta \to 0} \frac{\xi_4(\theta)}{\xi_2(\theta)} \right] X^2 - X.
\]

One root of the polynomial on the right-hand side of this equation is zero. Therefore, one root of \( C(X) \) converges towards zero as \( \theta \to 0 \), which implies in turn that \( \lim_{\theta \to 0} ||C^{-1}|| = +\infty \). Using the fact that \( ||\textbf{D}|| \) is bounded away from zero as \( \theta \to 0 \), we then get \( \lim_{\theta \to 0} |\hat{y}_1| = \lim_{\theta \to 0} |\pi_1| = +\infty \).

In the alternative case in which \( \xi_3(\theta) = \xi_4(\theta) = 0 \), the system made of the IS equation (K.1) and the Phillips curve (K.2) implies the following dynamic equation in inflation:

\[
\begin{bmatrix}
\xi_1 + \frac{\kappa(\theta) \xi_2}{\sigma} \\
\end{bmatrix} E_t \{ \hat{y}_{t+1} \} = \pi_t + \frac{\kappa(\theta) \xi_2}{\sigma} i_t.
\]

Iterating this dynamic equation forward to date \( T \), and using the terminal condition \( \pi_{T+1} = 0 \), we get

\[
\pi_1 = -\left[ \xi_1 + \frac{\kappa(\theta) \xi_2}{\sigma} \right]^{T-1} \frac{\kappa(\theta) \xi_2 i^*}{\sigma},
\]

so that \( \lim_{\theta \to 0} |\pi_1| = +\infty \). Using the Phillips curve (K.2) with \( \xi_3(\theta) = \xi_4(\theta) = 0 \), we then get \( \lim_{\theta \to 0} |\hat{y}_1| = +\infty \). 

This proposition results from two properties of discounting models: (i) these models generate indeterminacy under an exogenous policy rate when prices are sufficiently flexible, as their dynamic system then has one stable eigenvalue not matched by any predetermined variable, and (ii) this eigenvalue converges to zero as prices become more and more flexible. Indeed, this stable eigenvalue magnifies the effects of future conditions (at date \( T \)) on initial outcomes (at date \( 1 \)), and these effects grow explosively as this eigenvalue goes to zero – thus giving rise to the paradox of flexibility.

In turn, indeterminacy under sufficiently flexible prices follows, by continuity, from indeterminacy under perfectly flexible prices. Under perfectly flexible prices, the Phillips curve (K.2) collapses to the dynamic equation \( \hat{y}_t = [\lim_{\theta \to 0} \xi_4(\theta)] E_t \{ \hat{y}_{t+1} \} \), which pins down \( \hat{y}_t \) uniquely if \( \lim_{\theta \to 0} \xi_4(\theta) \neq 1 \). Under an exogenous policy rate \( i_t \), the IS equation (K.1) then pins down expected future inflation \( E_t \{ \pi_{t+1} \} \), but not current inflation \( \pi_t \). Thus, discounting the basic
NK model may deliver determinacy under an exogenous policy rate for some degrees of price stickiness, but cannot do it for sufficiently small degrees of price stickiness.

In our model with banks, by contrast, the interest rate pegged at the zero lower bound (ZLB) is the IOR rate, not the interest rate on bonds. Under an exogenous IOR rate and exogenous reserves, the interest rate on bonds evolves according to a shadow Wicksellian rule, as we explain in the main text. This shadow Wicksellian rule ensures determinacy whatever the degree of price stickiness, and in particular under perfectly flexible prices — thus solving the paradox of flexibility.

K.2 Distance From the Basic NK Model

Unlike our model with banks, discounting models require a discrete departure from the basic NK model to deliver determinacy under an exogenous policy rate and, therefore, to solve the forward-guidance puzzle. Indeed, for a sufficiently small departure, i.e. for \((\xi_1, \xi_2, \xi_3(\theta), \xi_4(\theta))\) sufficiently close to \((1, 1, 1, 0)\), indeterminacy obtains, by continuity with the basic NK model. Thus, a sufficiently high degree of bounded rationality (in Gabaix, 2016), information frictions (in Angeletos and Lian, 2016), or market incompleteness (in Bilbiie, 2017), is needed to solve the forward-guidance puzzle.

By contrast, our model with banks solves the forward-guidance puzzle even for an arbitrarily small departure from the basic NK model, i.e. even for arbitrarily small banking costs and convenience yield of bank reserves. Again, the key element at the source of this result is that the interest rate pegged at the ZLB is the IOR rate, not the interest rate on bonds. Under an exogenous IOR rate and exogenous reserves, the interest rate on bonds evolves according to a shadow Wicksellian rule, as we explain in the main text. The smaller the departure from the basic NK model, the smaller the coefficients of output and the price level in this shadow rule. If these coefficients were exactly zero, then indeterminacy would ensue; but as long as the coefficient of the price level is positive, however small it is, determinacy prevails.

K.3 Fisher Effect

Because it generates the standard IS equation (1), our model with banks trivially implies the standard Fisher effect, i.e. a one-to-one long-term relationship between the inflation rate and the interest rate on bonds. Thus, a permanent rise in the nominal-reserves-growth rate will raise the inflation rate and the interest rate on bonds by the same amount in the long term.

By contrast, discounting models cannot deliver determinacy under an exogenous policy rate without making the inflation rate respond negatively to the interest rate in the long term; therefore, they cannot both solve the forward-guidance puzzle and imply a long-term relationship consistent in sign (let alone in size) with the standard Fisher effect. This result is stated in the Online Appendix — 69.
following proposition:

**Proposition 4 (No Fisher Effect in Discounting Models):** In models whose reduced form is made of an IS equation of type (K.1) and a Phillips curve of type (K.2), if setting exogenously the policy rate delivers local-equilibrium determinacy, then a permanent increase in the policy rate leads to a permanent decrease in the inflation rate.

**Proof:** Consider first the case in which \( \xi_3(\theta) > 0 \) or \( \xi_4(\theta) > 0 \). In this case, under a permanent peg \( i_t = i^* \), the system made of the IS equation (K.1) and the Phillips curve (K.2) can be rewritten as (K.3) with \( i_t = i^* \). If the peg ensures local-equilibrium determinacy, then \( C(X) \), the characteristic polynomial of \( C \) (derived in Subsection K.1), must have no root inside the unit circle, because the system has two non-predetermined variables. In particular, \( C(X) \) must have no root inside the real-number interval \([0, 1]\), which requires that \( C(0) C(1) > 0 \), i.e. equivalently

\[
\sigma \left(1 - \xi_1\right) \left[1 - \beta \xi_3(\theta)\right] - \kappa(\theta) \xi_2 \left[1 - \xi_4(\theta)\right] > 0. \tag{K.5}
\]

In the unique local equilibrium, the (constant) inflation rate is easily obtained as

\[
\pi_t = \pi^* \equiv \frac{-\kappa(\theta) \xi_2 \left[1 - \xi_4(\theta)\right] i^*}{\sigma \left(1 - \xi_1\right) \left[1 - \beta \xi_3(\theta)\right] - \kappa(\theta) \xi_2 \left[1 - \xi_4(\theta)\right]}.
\]

Given (K.5), \( \pi^* \) is negatively related to \( i^* \).

In the alternative case in which \( \xi_3(\theta) = \xi_4(\theta) = 0 \), under a permanent peg \( i_t = i^* \), the system made of the IS equation (K.1) and the Phillips curve (K.2) implies the dynamic equation (K.4) with \( i_t = i^* \). Therefore, for the peg to ensure determinacy, we need

\[
\sigma \left(1 - \xi_1\right) - \kappa(\theta) \xi_2 > 0. \tag{K.6}
\]

In the unique local equilibrium, the (constant) inflation rate is easily obtained as

\[
\pi_t = \pi^* \equiv \frac{-\kappa(\theta) \xi_2 i^*}{\sigma \left(1 - \xi_1\right) - \kappa(\theta) \xi_2}.
\]

Given (K.6), \( \pi^* \) is negatively related to \( i^* \). ■

This proof is simple, but mechanical. In what follows, we offer an interpretation of Proposition 4 that involves a shadow interest-rate rule and the Taylor principle. The question (negatively) answered by Proposition 4 is whether the system made of the modified IS equation (K.1), the modified Phillips curve (K.2), and the permanent peg \( i_t = i^* \) can have a unique stationary solution and make inflation, in this unique stationary solution, depend positively on \( i^* \). This question will receive exactly the same answer if that system is replaced by the system made of the standard IS equation (1), the modified Phillips curve (K.2), and the shadow interest-rate rule

\[
i_t = \xi_2 i^* + \sigma \left(1 - \xi_1\right) E_t \{\hat{y}_{t+1}\} + (1 - \xi_2) E_t \{\pi_{t+1}\}. \tag{K.7}
\]

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Indeed, the two systems have exactly the same implications for local-equilibrium determinacy and the dynamics of inflation and output (they differ only in terms of the implied dynamics for $i_t$). So consider the latter system. The Taylor principle (as defined by Woodford, 2003, Chapter 4) states that a necessary condition for local-equilibrium determinacy is that the modified Phillips curve (K.2) and the shadow interest-rate rule (K.7) should make the interest rate react more than one-to-one to the inflation rate in the long term, that is to say

$$\zeta \equiv \frac{\sigma (1 - \xi_1) [1 - \beta \xi_3 (\theta)]}{\kappa (\theta) [1 - \xi_4 (\theta)]} + (1 - \xi_2) > 1,$$  \hspace{1cm} (K.8)

which is equivalent to (K.5). In the unique local equilibrium, the (constant) interest rate $i$ and the (constant) inflation rate $\pi$ are therefore linked to each other by the relationship $i = \xi_2 i^* + \zeta \pi$, where $\zeta > 1$. Now, the standard IS equation (1) implies that they should be equal to each other: $i = \pi$. As a consequence, we get

$$\pi = \frac{-\xi_2 i^*}{\zeta - 1}.$$

Thus, the necessary condition for local-equilibrium determinacy (K.8) imposed by the Taylor principle requires that $\pi$ be negatively related to $i^*$.

In our model with banks, the previous reasoning does not apply. Despite the standard nature of its IS equation (which trivially implies the Fisher effect), our model with banks delivers determinacy under an exogenous policy rate essentially because the policy rate pegged is the IOR rate, not the interest rate on bonds as in the basic NK model and in discounting models. Under an exogenous IOR rate and exogenous reserves, the interest rate on bonds evolves according to a shadow Wicksellian rule that always ensures determinacy.

K.4 Normal Times

Discounting models have only one interest rate, which is the policy instrument. To solve the forward-guidance puzzle, they need to deliver determinacy when this interest rate is pegged at the ZLB. The parameter restrictions that deliver determinacy under a peg at the ZLB, however, also deliver determinacy under a peg at a higher interest rate; and, by continuity, policy rules that make the interest rate react sufficiently weakly to inflation will also deliver determinacy. Thus, discounting models do not support the conventional wisdom that interest-rate rules have to be active during normal times.

By contrast, our model with banks has two interest rates. At the ZLB, the central bank has no possibility other than pegging the IOR rate, which is the interest rate it directly controls. Away from the ZLB, however, it has various possibilities. One of these possibilities is to operate under a corridor system that maintains a fixed spread between the IOR rate and the interbank rate and sets an interbank-rate target depending on the state of the economy. In Subsection I.1, we have shown that the reduced form of our model with banks under such a corridor system is isomorphic to the reduced form of the basic NK model, and therefore that our model then
inherits all the standard implications of the basic NK model for equilibrium determinacy and dynamics.

References


speech at Lehman College, New York, New York, April 18th.


