New Principles For Stabilization Policy

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Abstract: In a broad class of discrete-time rational-expectations models, I consider stabilization-policy rules making the policy instrument react with coefficient $\phi \in \mathbb{R}$ to a (past, current, or expected future) generic variable at time horizon $h \in \mathbb{Z}$, possibly among other variables. Using two complex-analysis theorems, I establish analytically some simple, easily interpretable, necessary or sufficient conditions on ϕ and h for these rules to ensure local-equilibrium determinacy. These conditions lead to new, general principles for stabilization policy in terms of whether, and how strongly or weakly, to react to any variable, at any horizon, in any model. Building on these conditions, I characterize circumstances under which the long-run Taylor principle is (not) necessary, (not) sufficient, or irrelevant for determinacy. I also provide the first hard guidelines for finding rules with robust determinacy properties across alternative models.

Keywords: stabilization policy, policy-instrument rule, local-equilibrium determinacy, Taylor principle, backward- and forward-looking policy, Rouché's theorem, Erdős-Turán theorem.

JEL codes: E32, E52.

1 Introduction

Dynamic rational-expectations models are widely used in macroeconomics. It is well known that these models can have "sunspot equilibria" in which the economy fluctuates around a steady state because of self-fulfilling expectations. Such fluctuations may notably explain the relatively large macroeconomic volatility in the 1960s and 1970s in the US, as argued by Clarida et al. (2000) and Lubik and Schorfheide (2004). Since these fluctuations are typically detrimental to welfare, a natural goal for stabilization policy is

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to eliminate these equilibria by ensuring "local-equilibrium determinacy" (i.e. existence and uniqueness of a stationary solution to the locally log-linearized model).

A large number of papers have thus studied, in various specific contexts, the conditions under which a policy-instrument rule ensures determinacy; that is, in discrete time, the inequality conditions on the coefficients of the rule for the resulting dynamic system to satisfy Blanchard and Kahn's (1980) determinacy conditions. Probably the best known result along these lines is about the so-called "Taylor principle" for monetary policy. Since Taylor (1993), monetary policy is commonly modeled by a simple interest-rate rule; in its simplest version, the Taylor principle states that the rule should make the interest rate react more than one-for-one to the inflation rate (when it reacts only to the inflation rate). This principle has been found to be necessary, if not also sufficient, for determinacy in some simple prominent models and for different inflation horizons in the rule (see, e.g., Woodford, 2003, Chapter 4).

These determinacy conditions, however, have so far been studied − analytically or numerically − only on a model-by-model, rule-by-rule basis; no general determinacy condition has yet been established. Some patterns emerge from this literature, but no clear-cut result stands out; the Taylor principle, in particular, is a good guide for determinacy in many monetary-policy models, but a poor one in others (see, e.g., Benhabib et al., 2001, Bilbiie, 2008). For monetary policy as for other stabilization policies, we still lack a general picture and understanding of determinacy outcomes depending on the model, the variables in the rule, and the coefficients and time horizons of these variables. We lack them essentially because the literature has been able to derive determinacy conditions analytically only in simple models and for simple rules with short horizons. The main difficulty in getting more general analytical results is that Blanchard and Kahn's (1980) determinacy conditions are about the roots of the characteristic polynomial of the dynamic system; and these roots depend on (the coefficients and horizons of) the policy-instrument rule in a complicated way.

In this paper, I use two complex-analysis theorems to overcome this difficulty and establish analytically some general, necessary or sufficient conditions for determinacy in dynamic rational-expectations models. These conditions depend on the coefficients and horizons of the policy-instrument rule in a simple and easily interpretable way. They lead to new principles for stabilization policy in terms of whether, and how strongly or weakly, to react to any variable, at any horizon, in any model.

More specifically, I consider a broad class of (locally log-linearized) discrete-time infinitehorizon rational-expectations models. For simplicity, I first focus on (locally log-linearized) rules that make the policy instrument react to a single variable (or single linear combination of variables) with coefficient $\phi \in \mathbb{R}$. The time horizon of this variable is $h \in \mathbb{Z}$: the policy instrument reacts to the $|h|$-period-lagged variable (when $h \leq -1$), the current variable (when $h = 0$), or the current expectation of the h-period-ahead variable (when $h \geq 1$). A negative horizon, making the rule backward-looking, may be due to "inside lags" (as are called recognition, decision, and implementation lags, which delay the reaction of policy to the state of the economy). A positive horizon, making the rule forward-looking, captures a reaction to forecasts or expectations (e.g., for monetary policy, the central bank's inflation forecasts, or market- or survey-based measures of inflation expectations).

The determinacy status of the dynamic system composed of the model and the rule can be either "determinacy" (unique stationary solution), or "multiplicity" (infinity of stationary solutions), or "explosiveness" (no stationary solution). I characterize this determinacy status as a function of the coefficient ϕ and the horizon h in the rule, for a sufficiently small or large $|\phi|$ and any h, as well as for a sufficiently large $|h|$ and any ϕ.

I show that there exists a positive threshold $\underline{\phi}$ such that for any $|\phi| < \underline{\phi}$, the determinacy status is independent of h and is the same as under a policy-instrument peg ($\phi = 0$). Intuitively, for $|\phi|$ sufficiently small, the structural equations "dominate" the rule in the system's dynamics: the rule does not change the system's dynamics enough, relatively to a peg, to affect the determinacy status. There also exist a higher threshold $\bar{\phi}$ and a horizon $h^* \in \mathbb{Z}$ such that for any $|\phi| > \bar{\phi}$, there is explosiveness if $h \leq h^* - 1$, determinacy if $h = h^*$, and multiplicity if $h \geq h^* + 1$. Intuitively, for $|\phi|$ sufficiently large, it is conversely the rule that dominates the structural equations in the system's dynamics: a sufficiently large weight $|\phi|$ on outcomes before (resp. after) horizon h^* favors exploding (resp. imploding) paths and leads to explosiveness (resp. multiplicity).

For $|\phi| \in (\underline{\phi}, \bar{\phi})$, the structural equations do not completely dominate the rule in the system's dynamics (since $|\phi| > \underline{\phi}$), nor does the rule completely dominate the structural equations (since $|\phi| < \bar{\phi}$). As $|h| \to +\infty$, the roots of the system's characteristic polynomial distribute themselves between inside and outside the unit circle $\mathcal{C}$ of the complex plane *in proportion of* the share of $\mathcal{C}$ on which the structural equations dominate the rule and the share of $\mathcal{C}$ on which the rule dominates the structural equations; so, the number of roots outside $\mathcal{C}$ increases less than one-for-one with $|h|$. As $h \to -\infty$ (resp. as $h \to +\infty$), the number of non-predetermined variables remains constant (resp. increases one-for-one with h), so we eventually get more (resp. fewer) roots outside $\mathcal{C}$ than non-predetermined variables, and hence explosiveness (resp. multiplicity), for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$. Determinacy may still obtain for arbitrarily large $|h|$'s, but *only if* $|\phi|$ is arbitrarily close to $\underline{\phi}$, i.e. only if the portion of $\mathcal{C}$ on which the rule dominates the structural equations is arbitrarily small.

I address the question of whether the set of "determinacy horizons" (i.e. the set of horizons $h \in \mathbb{Z}$ such that determinacy obtains for at least one value of $\phi \in \mathbb{R}$) is bounded or not, below or above. This question matters for the desirability of backward- or forward-looking stabilization policy. The answer depends on whether the model delivers determinacy, multiplicity, or explosiveness under a policy-instrument peg ($\phi = 0$). In the monetary-policy literature, with the interest rate as the policy instrument, one comes across the three kinds of models, as I discuss below and as I illustrate in the main text.

For models that deliver determinacy under a peg, the set of determinacy horizons is unbounded below and above, since determinacy obtains for $|\phi| < \underline{\phi}$ at any horizon. For models that deliver multiplicity (resp. explosiveness) under a peg, the set of determinacy horizons is bounded above (resp. below). The reason is that large positive (resp. negative) horizons h do not much "*perturb*" the imploding (resp. exploding) equilibrium paths obtained under a peg, as the reaction of the policy instrument prescribed by the rule on these paths decreases exponentially with $|h|$; so, these horizons *preserve* the determinacy status obtained under a peg if this status is multiplicity (resp. explosiveness). For these models, the set of determinacy horizons can also be bounded both below and above; I establish necessary or sufficient conditions for this outcome to obtain.

I derive the implications of all these results for the validity of the Taylor principle as a necessary or sufficient condition for determinacy. I consider Woodford's (2001, 2003) version of the Taylor principle, also called the long-run Taylor principle, which has a broader scope than the simpler version of the Taylor principle described above. I provide a formal, general definition of this principle, which applies to any stabilization-policy model and any variable in the rule, and which comes down to an inequality of type $\phi > \phi_W$, where the positive threshold ϕ_W depends on the model and the variable in the rule, but not on the horizon h of this variable. I characterize circumstances under which this principle is (alternatively) irrelevant, not necessary, not sufficient, sufficient, or locally necessary and sufficient for determinacy. Key to understand these results, as well as the contrasting results in the monetary-policy literature about the Taylor principle as a guide for determinacy, is whether $\phi_W = \underline{\phi}$, or $\phi_W = \bar{\phi}$, or $\phi_W \in (\underline{\phi}, \bar{\phi})$; whether $h = h^*$

or $h \neq h^*$; and whether h belongs to the set of determinacy horizons or not.

I illustrate all these results numerically with several simple monetary-policy examples. The models and calibrations are entirely borrowed from the literature. Even though the models are not quantitative, it is worth noting that $\underline{\phi}$ and $\bar{\phi}$ can be, in these examples, of the same order of magnitude as standard values of ϕ in the literature. In addition, the highest determinacy horizon, when it exists, is not higher than 2 periods; the lowest one, when it exists, is not lower than -1 period (the period being typically a quarter).

Finally, I extend the results to rules involving several variables with different horizons and coefficients, one of which is a variable with horizon h and coefficient ϕ; and to inertial rules, i.e. rules involving some past values of the policy instrument in addition to a variable with horizon h and coefficient ϕ. The extended results are similar to the benchmark results, and can be interpreted in the same way. They imply notably that for models delivering multiplicity under a peg, the set of determinacy horizons can always be made unbounded below and above with a "superinertial rule," i.e. a rule that would make the policy instrument explode over time if the variables set by the private sector were taken out of the rule.

A few remarks may serve to put my contribution in the context of the literature. The paper is, to my knowledge, the first to establish general determinacy conditions about the coefficients and horizons of policy-instrument rules. In particular, the concepts of $\underline{\phi}$, $\bar{\phi}$, and h^* are new. The literature has derived determinacy conditions analytically only in simple models and for simple rules with short horizons (so that the degree of the characteristic polynomial of the dynamic system is typically not higher than 3). Early examples of such contributions include Benhabib et al. (2001), Bullard and Mitra (2002), Carlstrom and Fuerst (2002), and Woodford (2003, Chapter 4), for horizons between -1 and 1.

The two complex-analysis theorems that I use to establish my general results are those of Rouché (1862) and Erdős and Turán (1950). One of these theorems is not new to economics: Bhattarai et al. (2014) use (another version of) Rouché's theorem to derive a sufficient condition for determinacy in a monetary-policy model with partial price indexation and habit formation in consumption. Their numerical simulations on a large grid of parameter values indicate that this sufficient condition is only slightly stronger than the long-run Taylor principle, which is necessary for determinacy in the context of their model and their interest-rate rule. There are, in substance, three key differences between their sufficient condition for determinacy and my sufficient conditions for determinacy, or multiplicity, or explosiveness.

The first difference is that Bhattarai et al. (2014) are after the weakest possible sufficient condition for determinacy in the context of their model and their rule; to that aim, they use a stronger version of Rouché's theorem, established by Glicksberg (1976); the analytical condition that they get depends on each coefficient of the rule in a complicated and opaque way (even though this condition is numerically found to be close to the simple long-run Taylor principle). By contrast, I am after some simple and easily interpretable sufficient conditions; I get them using the standard version of Rouché's theorem, and applying it differently; these analytical conditions depend on the coefficient ϕ of the policy-instrument rule in a simple and transparent way, through the thresholds $\underline{\phi}$ and $\bar{\phi}$. The other side of the coin, however, is that my sufficient conditions are not the weakest possible, and they are about one coefficient at a time (taking the other coefficients as given, if the rule has several coefficients).

The second difference is that my conditions characterize the determinacy status for sufficiently small or large coefficients at any horizon, as well as for any coefficient at horizons sufficiently distant in the future and in the past – while Bhattarai et al. (2014), like the rest of the literature, focus on a few specific short horizons (namely 0 and 1). The third difference, finally, is that I establish these sufficient conditions for a generic rule in a generic model, in order to derive general principles for stabilization policy (in terms of whether, and how strongly or weakly, to react to any variable, at any horizon, in any model), while Bhattarai et al. (2014) focus on a specific model and a specific rule. In particular, while the long-run Taylor principle is necessary (and numerically almost sufficient) for determinacy in their setup, I identify and analyze circumstances under which it is not necessary, or not sufficient, or irrelevant for determinacy.

Some of the results I establish are conditional on whether the model delivers multiplicity, determinacy, or explosiveness under a policy-instrument peg. One comes across the three types of models in the monetary-policy literature. Standard New Keynesian models typically deliver *multiplicity* under an interest-rate peg; this property is emphasized by Cochrane (2011); Giannoni and Woodford (2002) and Woodford (2003, Chapter 8) call it the "Sargent-Wallace property," after Sargent and Wallace (1975). Older models often deliver *explosiveness* under an interest-rate peg; this property is emphasized by Cochrane (2011), who calls these models "Old Keynesian." More recently, models have been developed that can deliver *determinacy* under an interest-rate peg (and, as a result, can solve some New Keynesian puzzles and paradoxes at the zero lower bound). Examples include the heterogenous-agents models of Acharya and Dogra (2020) and Bilbiie (2008, 2021), and the bounded-rationality model of Gabaix (2020).

My results about positive determinacy horizons offer an explanation for the propensity of forward-looking interest-rate rules to generate multiplicity in New Keynesian models, as found in, e.g., Levin et al. (2003). Existing results on this front are mostly numerical and sparsely distributed across calibrated models and rules; my analytical results generalize them to a broad class of models and a broad class of rules (making the policy instrument react to any expected future variable). Woodford (1994) and Bernanke and Woodford (1997) were the first to warn against forward-looking rules on multiplicity grounds.

My results about negative determinacy horizons matter in the presence of inside lags. Benhabib (2004) analyzes the implications of inside lags for determinacy in a simple monetary-policy model (analytically in continuous time, numerically in discrete time); he argues that the lag structure may be long in discrete time – e.g., up to sixty periods if inflation changes twice daily and the interest rate is set according to inflation lagged thirty days. In Loisel (2022b), I investigate the ability of stabilization policy to ensure determinacy and to control the anticipation and convergence rates in the presence of inside or outside lags; the approach I take there (starting from a targeted characteristic polynomial and deriving a corresponding, arbitrarily complex policy-instrument rule) is radically different from the one I am taking here, and does not lead to any simple "principle" for stabilization policy.

Benhabib et al. (2001, 2003) derive determinacy conditions analytically for backward- and forward-looking interest-rate rules in a simple monetary-policy model, depending on whether prices are flexible or sticky and on how money enters preferences and technology. Their backward- and forward-looking rules differ from mine; in particular, their backward-looking rules amount to inertial rules, as Benhabib et al. (2003) note. Like Benhabib (2004), Benhabib et al. (2001, 2003) conduct most of their analysis in continuous time. The mathematical tools that I use are helpful for establishing general analytical determinacy conditions in discrete time, but not in continuous time.[1]

As I discuss in the main text, my general results provide the first hard guidelines for finding rules with robust determinacy properties across alternative models. The literature on the robustness of interest-rate rules across alternative monetary-policy models,

[1]The determinacy status depends on the number of characteristic roots inside the unit circle of the complex plane (in discrete time), or on the number of characteristic roots in the left half-plane (in continuous time). Rouché's theorem, which I use, can directly characterize the former number, not the latter.

which includes notably Levin et al. (1999, 2003), Levin and Williams (2003), Taylor and Williams (2011), and Wieland et al. (2012, 2016), exclusively rests on numerical simulations. Relatedly, my results shed light on the poor performance of superinertial rules in Old Keynesian models (Rudebusch and Svensson, 1999, and Levin and Williams, 2003), and on the degree of superinertia of "robustly optimal rules" (Woodford, 2003, Chapter 8, and Giannoni and Woodford, 2002, 2003, 2005).

Most of the literature on policy-instrument rules is about monetary policy. My results apply more generally to any stabilization policy. In particular, fiscal policy also raises indeterminacy issues, as first shown by Schmitt-Grohé and Uribe (1997).

I establish not only determinacy conditions, but also multiplicity conditions and explosiveness conditions. Clarida et al. (2000) and Lubik and Schorfheide (2004) have famously argued that US macroeconomic volatility before 1979 may be due to multiplicity. Beaudry et al. (2017, 2020) argue that recent US macroeconomic data are consistent with explosiveness (and convergence to a limit cycle).

Two limitations of my work are worth mentioning. First, like most of the related literature, I take the case of local-equilibrium multiplicity seriously. Some authors argue that an equilibrium-selection criterion should be used in this case, like e.g. the minimal-state-variable criterion of McCallum (1983). From this alternative point of view, the distinction between determinacy and multiplicity may not matter anymore; but my results about explosiveness vs. either determinacy or multiplicity should still be of interest. Second, and again like most of the related literature (but unlike, e.g., Benhabib et al., 2001, 2002, 2003), I restrict attention to local equilibria, or their non-existence, in locally log-linearized models. This focus may not be that much of a limitation if non-local equilibria can be ruled out with the type of escape clause considered in, e.g., Benhabib et al. (2002); whether they can or cannot is, however, subject to debate (see, e.g., Cochrane, 2011).

The rest of the paper is organized as follows. Section 2 illustrates some of the main results of the paper in the basic New Keynesian model, with a rule making the interest rate react to inflation or output. Section 3 generalizes the analysis to a broad class of models and to rules involving any single variable. Section 4 extends the results to rules involving several variables and to inertial rules, and discusses the scope of application of the results as well as their implications for the robustness of rules across alternative models. I then conclude and provide a technical appendix.

2 A basic New Keynesian illustration

In this illustrative section, I derive some of the main results of the paper in a specific, simple and well known context: the basic New Keynesian model, with a rule making the interest rate react to inflation or output. The analysis is a special case, in terms of model and rule, of the more general analysis conducted in the next sections.

2.1 Determinacy status under Rule 1

I refer the reader to Woodford (2003) and Galí (2015) for a detailed presentation of the basic New Keynesian model. In this model, at each date $t \in \mathbb{Z}$, the private sector sets inflation π_t and output y_t according to the following (locally log-linearized) IS equation and Phillips curve:

$$y_t = \mathbb{E}_t\{y_{t+1}\} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t\{\pi_{t+1}\}\right), \tag{1}$$

$$\pi_t = \beta\mathbb{E}_t\{\pi_{t+1}\} + \kappa y_t, \tag{2}$$

where $\mathbb{E}_t\{.\}$ denotes the date-t rational-expectations operator, and $\sigma > 0$, $\beta \in (0,1)$, and $\kappa > 0$ are three parameters.[2] I abstract from exogenous shocks in these structural equations, as they are irrelevant for determinacy issues. The policymaker is a central bank setting the short-term nominal interest rate i_t. I start with the case in which the central bank reacts to the past, current, or expected future inflation rate; i.e., I consider the following (locally log-linearized) interest-rate rule:

$$i_t = \phi\mathbb{E}_t\left\{\pi_{t+h}\right\}, \tag{Rule 1}$$

where $(\phi, h) \in \mathbb{R} \times \mathbb{Z}$ (with $\mathbb{E}_t\left\{\pi_{t+h}\right\} = \pi_{t+h}$ when $h \leq 0$). I call ϕ and h the coefficient and horizon of inflation in the rule – or, with slight abuse of language, the coefficient and horizon of the rule.

Using the Phillips curve (2) and Rule 1 to replace y_t, y_{t+1}, and i_t in the IS equation (1), I get the dynamic equation

$$\beta\mathbb{E}_t\left\{\pi_{t+2}\right\} - \left(1 + \beta + \frac{\kappa}{\sigma}\right)\mathbb{E}_t\left\{\pi_{t+1}\right\} + \pi_t + \frac{\phi\kappa}{\sigma}\mathbb{E}_t\left\{\pi_{t+h}\right\} = 0.$$

Using the lag operator L, I rewrite this dynamic equation as

$$\mathbb{E}_t\left\{Q(L)\pi_{t+2}\right\} + \frac{\phi\kappa}{\sigma}\mathbb{E}_t\left\{\pi_{t+h}\right\} = 0, \tag{3}$$

[2] I use the same notations for the parameters as in Galí (2015). Woodford (2003) uses the same notations for β and κ, but replaces $1/\sigma$ by σ.

where $Q(z) := \beta - (1+\beta+\kappa/\sigma)z + z^2 \in \mathbb{R}[z]$.[3] Let ν denote the number of non-predetermined variables of this dynamic equation.[4] Let $P(z)$ denote the *reciprocal* polynomial of this dynamic equation's characteristic polynomial, $\mathcal{C}$ the circle of radius 1 centered at the origin of the complex plane, and p the number of roots of $P(z)$ inside $\mathcal{C}$.[5] As follows from Blanchard and Kahn (1980), the dynamic equation has an infinity of stationary solutions if $p < \nu$, a unique stationary solution if $p = \nu$, and no stationary solution if $p > \nu$. I say that the "determinacy status" $S(\phi, h)$ of the system composed of the structural equations (1)-(2) and Rule 1 is equal to M (for "multiplicity") in the first case, D (for "determinacy") in the second case, and E (for "explosiveness") in the third case.

Under an interest-rate peg ($\phi = 0$), we have $\nu = 2$ non-predetermined variables, and $P(z) = Q(z)$. Since $Q(0) = \beta > 0$, $Q(1) = -\kappa/\sigma < 0$, and $\lim_{z\in\mathbb{R}, z\to+\infty} Q(z) = +\infty$, $Q(z)$ has one root in $(0,1)$ and another in $(1,+\infty)$. With $p = 1$ roots inside $\mathcal{C}$ for $\nu = 2$ non-predetermined variables, thus, the dynamic equation has an infinity of stationary solutions: $S(0,h) = M$ for any $h \in \mathbb{Z}$.

When the interest rate is not pegged ($\phi \neq 0$), we generically have $\nu = \max(2,h)$ non-predetermined variables, and

$$P(z) = Q(z)z^{\max(0,h-2)} + \frac{\phi\kappa}{\sigma}z^{\max(0,2-h)}.$$

This result is "generic" in the sense of holding for all $(\phi, h) \in (\mathbb{R}\setminus\{0\}) \times \mathbb{Z}$ except $(\phi,h) = (-\beta\sigma/\kappa, 2)$. If $(\phi,h) = (-\beta\sigma/\kappa, 2)$, then the coefficient of $\mathbb{E}_t\{\pi_{t+2}\}$ in the dynamic equation is 0, and we get $\nu = 1$ instead of $\nu = 2$. I study such zero-measure cases in detail in Loisel (2009); I ignore them in the present paper.

I determine the determinacy status $S(\phi,h)$ for $|\phi|$ sufficiently small or large and any h, as well as for $|h|$ sufficiently large and any ϕ. I obtain the following results:[6]

Proposition 1 (Determinacy status in the basic New Keynesian model under

[3] Throughout the paper, $\mathbb{R}[z]$ denotes the set of polynomials in z with real-number coefficients. Similarly, $\mathbb{C}[z]$ denotes the set of polynomials in z with complex coefficients.

[4] Throughout the paper, the non-predetermined variables of a dynamic equation (or a dynamic system) are defined, following Blanchard and Kahn (1980), as the non-predetermined elements of the vector $\mathbf{Z}_t$ when the dynamic equation (or the dynamic system) is written in a first-order form of type $\mathbb{E}_t\{\mathbf{Z}_{t+1}\} = \mathbf{M}\mathbf{Z}_t$ (abstracting from exogenous shocks), where $\mathbf{M}$ is a square matrix.

[5] For any $\tilde{P}(z) \in \mathbb{R}[z]$ of degree d, the reciprocal polynomial of $\tilde{P}(z)$ is $z^d\tilde{P}(z^{-1})$. I work with the reciprocal polynomial of the characteristic polynomial, rather than with the characteristic polynomial itself, as the former is more convenient to use than the latter in conjunction with the lag operator.

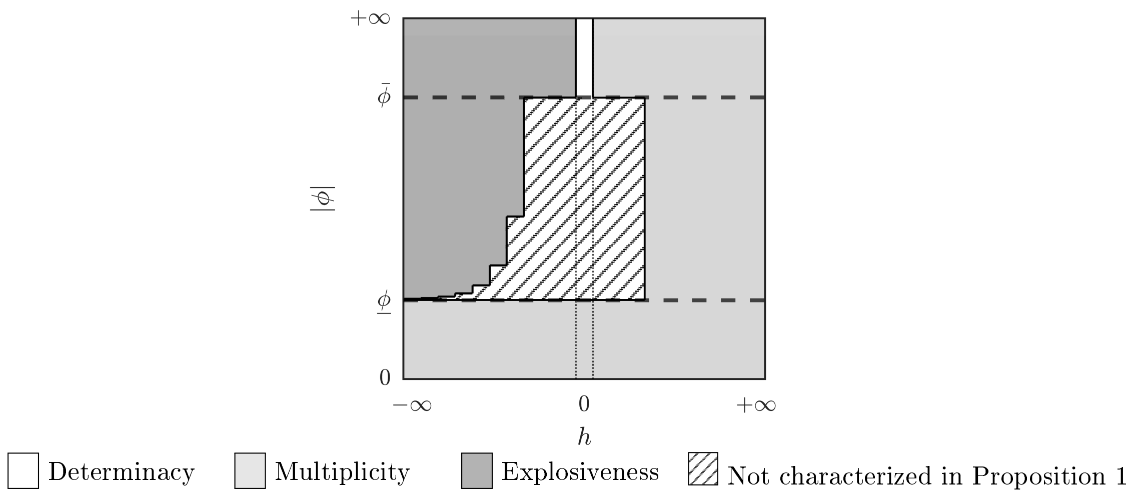
[6] In this proposition and in the rest of the paper, I use the shortcut "$\forall\,|\phi|$ ..." for "$\forall\phi \in \mathbb{R}$ such that $|\phi|$ ...".

Rule 1): *Consider the basic New Keynesian model (1)-(2) with the rule* $i_t = \phi \mathbb{E}_t \{\pi_{t+h}\}$, *where* $(\phi, h) \in \mathbb{R} \times \mathbb{Z}$. *Let* $\underline{\phi} := (\sigma/\kappa) \min_{z \in \mathcal{C}} |Q(z)|$ *and* $\bar{\phi} := (\sigma/\kappa) \max_{z \in \mathcal{C}} |Q(z)|$. *Then:*
(a) $\forall |\phi| < \underline{\phi}$, $\forall h \in \mathbb{Z}$, $S(\phi, h) = M$;
(b) $\forall |\phi| > \bar{\phi}$, *(i)* $\forall h \leq -1$, $S(\phi, h) = E$, *(ii)* $S(\phi, 0) = D$, *(iii)* $\forall h \geq 1$, $S(\phi, h) = M$;
(c) $\exists \bar{h} \in \mathbb{Z}$, $\forall |\phi| \in (\underline{\phi}, \bar{\phi})$, $\forall h \geq \bar{h}$, $S(\phi, h) = M$;
(d) $\exists \underline{h} : (\underline{\phi}, \bar{\phi}) \to \mathbb{Z}$, *(i)* $\forall |\phi| \in (\underline{\phi}, \bar{\phi})$, $\forall h \leq \underline{h}(|\phi|)$, $S(\phi, h) = E$, *(ii)* $\forall \varepsilon \in (0, \bar{\phi} - \underline{\phi})$, $\underline{h}$ *is bounded on* $(\underline{\phi} + \varepsilon, \bar{\phi})$.

Proof: See Subsection 2.2 and the Appendix. ■

This proposition may look a bit cryptic at first sight. To enable the reader to grasp it at a glance, I represent it diagrammatically in Figure 1. This figure shows the determinacy status $S(\phi, h)$ in the pseudo half-plane $(h, |\phi|) \in \mathbb{Z} \times \mathbb{R}_+$, according to Proposition 1.

Figure 1: Determinacy status for the basic New Keynesian model and Rule 1



In the next two subsections, to convey the intuition behind Proposition 1, I prove Points (a)-(b) and I provide an outline of the proof of Points (c)-(d). The analytical determination of $\underline{\phi}$ and $\bar{\phi}$ (equal to 1 and $1 + 2(1+\beta)\sigma/\kappa$ respectively) can be found in Online Appendix A.1.

2.2 Proof and intuition for Points (a)-(b) of Proposition 1

Points (a)-(b) of Proposition 1 are about the determinacy status $S(\phi, h)$ for a sufficiently small or large absolute value of the coefficient ϕ. To prove these points, I use the theorem of Rouché (1862). I refer the reader to Henrici (1988, Theorem 4.10b, Page 280) or

Marden (1966, Page 2) for a general and modern statement of this theorem. Because I will apply it only to polynomials, I only need the following, more restrictive version of the theorem, where the term "Jordan curve" refers to a non-self-intersecting closed curve in the complex plane, and where the subscripts "b" and "s" stand respectively for "big" and "small":

Theorem 1 (Rouché, 1862): *Let $\mathcal{J}$ be a Jordan curve, $P_b(z) \in \mathbb{C}[z]$, and $P_s(z) \in \mathbb{C}[z]$. If $\forall z \in \mathcal{J}$, $|P_b(z)| > |P_s(z)|$, then $P_b(z) + P_s(z)$ and $P_b(z)$ have the same number of roots inside $\mathcal{J}$ (counting multiplicity).*

Proof: See Henrici (1988, Page 280) or Marden (1966, Page 3). ■

To determine $S(\phi, h)$ for $|\phi|$ sufficiently small, I apply Rouché's theorem to $\mathcal{J} = \mathcal{C}$, $P_b(z) = Q(z) z^{\max(0,h-2)}$, and $P_s(z) = (\phi\kappa/\sigma) z^{\max(0,2-h)}$ (with, thus, $P_b(z)+P_s(z) = P(z)$). For any $|\phi| < \underline{\phi} := (\sigma/\kappa) \min_{\tilde{z} \in \mathcal{C}} |Q(\tilde{z})|$ and any $z \in \mathcal{C}$, we have

$$\left|Q(z) z^{\max(0,h-2)}\right| = |Q(z)| \geq \min_{\tilde{z} \in \mathcal{C}} |Q(\tilde{z})| = \frac{\underline{\phi}\kappa}{\sigma} > \frac{|\phi|\,\kappa}{\sigma} = \left|\frac{\phi\kappa}{\sigma} z^{\max(0,2-h)}\right|.$$

So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{C}$ as $Q(z) z^{\max(0,h-2)}$. The latter polynomial has exactly $\max(1, h-1)$ roots inside $\mathcal{C}$, since $Q(z)$ has exactly one root inside $\mathcal{C}$. Therefore, $p = \max(1, h-1) < \max(2, h) = \nu$, and we get $S(\phi, h) = M$ for any $h \in \mathbb{Z}$.

Intuitively, the polynomial $P(z)$, which characterizes the system's dynamics, is the sum of two terms: one coming from the rule and proportional to ϕ, the other coming from the structural equations and independent of ϕ. For $|\phi| < \underline{\phi}$, the latter term is larger in modulus than the former term for any $z \in \mathcal{C}$: in this sense, the structural equations *dominate* the rule in the system's dynamics. So, the rule does not change the system's dynamics enough, relatively to an interest-rate peg, to affect the determinacy status; and this status remains the same as under an interest-rate peg – i.e., multiplicity.

To determine $S(\phi, h)$ for $|\phi|$ sufficiently large, I switch $P_b(z)$ and $P_s(z)$: i.e., I apply Rouché's theorem to $\mathcal{J} = \mathcal{C}$, $P_b(z) = (\phi\kappa/\sigma) z^{\max(0,2-h)}$, and $P_s(z) = Q(z) z^{\max(0,h-2)}$. For any $|\phi| > \bar{\phi} := (\sigma/\kappa) \max_{\tilde{z} \in \mathcal{C}} |Q(\tilde{z})|$ and any $z \in \mathcal{C}$, we have

$$\left|\frac{\phi\kappa}{\sigma} z^{\max(0,2-h)}\right| = \frac{|\phi|\,\kappa}{\sigma} > \frac{\bar{\phi}\kappa}{\sigma} = \max_{\tilde{z} \in \mathcal{C}} |Q(\tilde{z})| \geq |Q(z)| = \left|Q(z) z^{\max(0,h-2)}\right|.$$

So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{C}$ as $(\phi\kappa/\sigma) z^{\max(0,2-h)}$. The latter polynomial has exactly $\max(0, 2-h)$ roots inside $\mathcal{C}$; so,

$p = \max(0, 2-h)$. Since $\nu = \max(2,h)$, we get: (i) if $h \leq -1$, then $p > \nu$ and $S(\phi,h) = E$; (ii) if $h = 0$, then $p = \nu$ and $S(\phi,h) = D$; and (iii) if $h \geq 1$, then $p < \nu$ and $S(\phi,h) = M$.

Intuitively, for $|\phi| > \bar{\phi}$, it is conversely the rule that dominates the structural equations in the system's dynamics; as a result, the determinacy status depends only on the horizon h in the rule. A large weight $|\phi|$ on past inflation ($h \leq -1$) favors exploding paths and leads to explosiveness; a large weight $|\phi|$ on expected future inflation ($h \geq 1$) favors imploding paths and leads to multiplicity; and a large weight $|\phi|$ on current inflation ($h = 0$) strikes the right balance between exploding and imploding paths, and leads to determinacy.

2.3 Proof outline and intuition for Points (c)-(d) of Prop. 1

Point (c) of Proposition 1 is about the existence of $\bar{h} \in \mathbb{Z}$ such that $S(\phi,h) = M$ for any $|\phi| \in (\underline{\phi}, \bar{\phi})$ and any $h \geq \bar{h}$. In my proof (in the Appendix), I do not seek to find the smallest integer $\bar{h}$ of that kind; I postpone this question to Subsection 2.6. Let z_o denote the root of $Q(z)$ in $(1,+\infty)$, with the subscript "o" standing for "outside $\mathcal{C}$." Consider a Jordan curve $\mathcal{J}_o$ surrounding z_o and not intersecting nor surrounding $\mathcal{C}$. I apply Rouché's theorem to $\mathcal{J} = \mathcal{J}_o$, $P_b(z) = Q(z)z^{h-2}$, and $P_s(z) = \phi\kappa/\sigma$. I obtain that for h sufficiently large, $P(z)$ has exactly one root inside $\mathcal{J}_o$, and hence at least one root outside $\mathcal{C}$, which implies $p < \nu$ and $S(\phi,h) = M$.

The intuition for this result is the following. Under an interest-rate peg ($\phi = 0$), we have a multiplicity of equilibrium paths that converge over time to zero at rate z_o^{-1}. When the interest rate is not pegged ($\phi \neq 0$), these paths are no longer equilibrium paths: they do not satisfy the dynamic equation (3) because of the term $(\phi\kappa/\sigma)\mathbb{E}_t\{\pi_{t+h}\}$ in this equation. When h is large, however, they are "*close to satisfying*" the dynamic equation, as the term $(\phi\kappa/\sigma)\mathbb{E}_t\{\pi_{t+h}\}$ is, on these paths, proportional to z_o^{-h} and hence close to zero. As a result, by continuity, there are neighboring paths that do satisfy the dynamic equation; i.e., there are equilibrium paths that converge over time to zero at a rate close to z_o^{-1}. As $h \to +\infty$, these equilibrium paths uniformly converge to those under an interest-rate peg, as the rate at which they converge over time to zero converges to z_o^{-1} (as can be readily checked by considering an arbitrarily small Jordan curve $\mathcal{J}_o$ around z_o in the reasoning above). In this sense, arbitrarily large horizons in the rule *preserve* all the local equilibria existing under an interest-rate peg.

Point (d) of Proposition 1 is about the determinacy status for $|\phi| \in (\underline{\phi}, \bar{\phi})$ and $-h$

sufficiently large. To prove this point in the Appendix, I consider a given $|\phi| \in (\underline{\phi}, \bar{\phi})$ and I proceed in four steps. In the first step, I show that all but one root of $P(z)$ converge uniformly to $\mathcal{C}$ as $h \to -\infty$. I get this result by applying Rouché's theorem twice: once to a circle approaching $\mathcal{C}$ from inside, and another time to a circle approaching $\mathcal{C}$ from outside. In the second step, I show that the roots of $P(z)$ uniformly converging to $\mathcal{C}$ as $h \to -\infty$ converge in distribution to the *uniform distribution* on $\mathcal{C}$. This result is a direct consequence of the second complex-analysis theorem that I use in the paper: the theorem of Erdős and Turán (1950), which I state in the Appendix.

In the third step, I consider an arc $\mathcal{A}$ of $\mathcal{C}$ on which the rule *dominates* the structural equations: $\forall z \in \mathcal{A}$, $|\phi| > |Q(z)|\,\sigma/\kappa$. I use again Rouché's theorem to show that as $h \to -\infty$, any root of $P(z)$ close to $\mathcal{A}$ lies inside $\mathcal{C}$. Given the result of the second step, therefore, the share of roots of $P(z)$ inside $\mathcal{C}$ is bounded below by $\ell(\mathcal{A})/\ell(\mathcal{C})$ as $h \to -\infty$, where $\ell(.)$ denotes the standard length operator (i.e., the Lebesgue measure on $\mathcal{C}$). As a consequence, as $h \to -\infty$, the number of roots of $P(z)$ inside $\mathcal{C}$ grows unboundedly ($p \to +\infty$) and eventually exceeds the constant number of non-predetermined variables ($\nu = 2$), leading to explosiveness. In the fourth step, finally, I use the fact that if $|\phi|$ is bounded away from $\underline{\phi}$, then $\ell(\mathcal{A})$ is bounded away from zero, and the function $\underline{h}(.)$ mentioned in Point (d) of Proposition 1 can be chosen bounded.

In Point (d), thus, like in Point (b), a *sufficiently large weight* $|\phi|$ on *sufficiently ancient outcomes* favors exploding paths and leads to explosiveness. The two points suggest some *substitutability* between "sufficiently large weight" and "sufficiently ancient outcomes." In Point (b), the weight $|\phi|$ is higher than $\bar{\phi}$ (i.e. the rule dominates the structural equations on the entire circle $\mathcal{C}$), and even the most recent outcomes are enough to generate explosiveness. In Point (d), conversely, for very ancient outcomes, even a weight $|\phi|$ hardly higher than $\underline{\phi}$ (i.e. even a very small portion of $\mathcal{C}$ on which the rule dominates the structural equations) is enough to generate explosiveness.

2.4 Determinacy status under Rule 2

I now replace inflation with output in the rule; i.e., I consider the rule

$$i_t = \phi \mathbb{E}_t \{y_{t+h}\}. \tag{Rule 2}$$

Using the Phillips curve (2) and Rule 2 to replace y_t, y_{t+1}, and i_t in the IS equation (1), I get the following dynamic equation:

$$\beta \mathbb{E}_t \{\pi_{t+2}\} - \left(1 + \beta + \frac{\kappa}{\sigma}\right) \mathbb{E}_t \{\pi_{t+1}\} + \pi_t + \frac{\phi}{\sigma} \left(\mathbb{E}_t \{\pi_{t+h}\} - \beta \mathbb{E}_t \{\pi_{t+h+1}\}\right) = 0.$$

So, when the interest rate is not pegged ($\phi \neq 0$), there are now $\nu = \max(2, h+1)$ non-predetermined variables, and the reciprocal polynomial of the characteristic polynomial is

$$P(z) = Q(z) z^{\max(0,h-1)} + \frac{\phi}{\sigma} (z - \beta) z^{\max(0,1-h)}.$$

It is easy to conduct the same analysis as in the previous subsections, replacing $Q(z) z^{\max(0,h-2)}$ by $Q(z) z^{\max(0,h-1)}$, $(\phi\kappa/\sigma) z^{\max(0,2-h)}$ by $(\phi/\sigma)(z-\beta) z^{\max(0,1-h)}$, $\underline{\phi} := (\sigma/\kappa) \min_{z \in \mathcal{C}} |Q(z)|$ by $\underline{\phi} := \sigma \min_{z \in \mathcal{C}} |Q(z)/(z-\beta)|$, and $\bar{\phi} := (\sigma/\kappa) \max_{z \in \mathcal{C}} |Q(z)|$ by $\bar{\phi} := \sigma \max_{z \in \mathcal{C}} |Q(z)/(z-\beta)|$. Since $\beta \in (0,1)$, the results are unchanged: Points (a)-(d) of Proposition 1 still hold with the new thresholds $\underline{\phi}$ and $\bar{\phi}$.

Proposition 2 (Determinacy status in the basic New Keynesian model under Rule 2): *Consider the basic New Keynesian model (1)-(2) with the rule $i_t = \phi \mathbb{E}_t \{y_{t+h}\}$, where $(\phi, h) \in \mathbb{R} \times \mathbb{Z}$. Let $\underline{\phi} := \sigma \min_{z \in \mathcal{C}} |Q(z)/(z-\beta)|$ and $\bar{\phi} := \sigma \max_{z \in \mathcal{C}} |Q(z)/(z-\beta)|$. Then, Points (a)-(d) of Proposition 1 still hold.*

Proposition 2 can be interpreted in exactly the same way as Proposition 1. It can also be represented in exactly the same diagrammatic form: Figure 1 shows the determinacy status in the basic New Keynesian model not only under Rule 1, but also under Rule 2. The analytical expressions of the thresholds $\underline{\phi}$ and $\bar{\phi}$ under Rule 2 are determined in Online Appendix A.2.

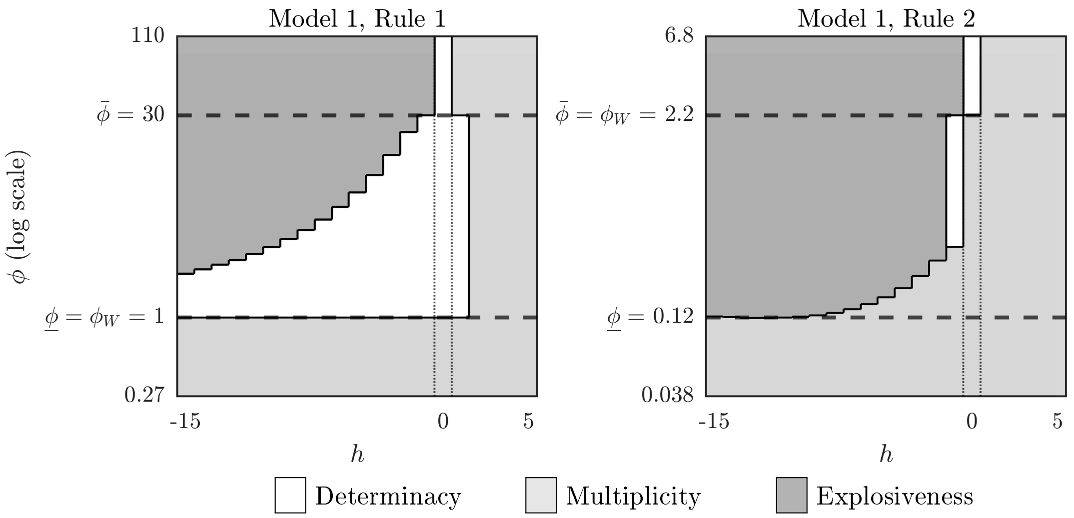
2.5 Numerical example

In order to illustrate Proposition 1-2 and the next propositions numerically, I consider Woodford's (2003, Chapter 4) calibration of the basic New Keynesian model: $(\beta, \kappa, \sigma) = (0.99, 0.022, 0.16)$, the period being one quarter. I call "Model 1" the resulting calibrated model, as it is the first of several calibrated models that I will consider in the paper.

The results obtained for Model 1 and Rules 1-2 are presented in Figure 2. This figure represents the determinacy status $S(\phi, h)$ in the pseudo half-plane $(h, \phi) \in \mathbb{Z} \times \mathbb{R}_+$ with a log scale for ϕ. I focus on positive values of ϕ for consistency with the theoretical and empirical literatures. The coefficient threshold ϕ_W featuring in the figure will be introduced and commented upon in the next subsection.

Even though the basic New Keynesian model is clearly not a quantitative model, a few features of this figure are worth emphasizing. First, the horizon threshold at and above which the rule can no longer deliver determinacy (i.e., the lowest integer $\bar{h}$ in Point (c)

Figure 2: Determinacy status for Model 1 and Rules 1-2 with $\phi > 0$



of Propositions 1-2) is two quarters for Rule 1, and one quarter for Rule 2. So, in this numerical example, monetary policy should hardly be forward-looking (under Rule 1), or not be forward-looking at all (under Rule 2), in order to ensure determinacy. In the next subsection, I will determine analytically this horizon threshold for Rule 1, and I will argue that it is typically low.

Second, Rule 2 can no longer deliver determinacy for a horizon equal to or lower than minus two quarters. So, in this numerical example, a central bank that would react to output with a delay of two or more quarters, say because of data-publication lags, would necessarily be behind the curve and fail to ensure determinacy, no matter how strongly or weakly it reacts to output. In Subsection 2.7, I will derive analytically a necessary and sufficient condition for existence of such a horizon threshold at and below which Rule 2 can no longer deliver determinacy.

Third, compared to standard values of ϕ in the literature (often between 0.5 and 2), the lower coefficient threshold $\underline{\phi}$ is of the same order of magnitude or one order of magnitude smaller, while the upper coefficient threshold $\bar{\phi}$ is of the same order of magnitude or one order of magnitude larger.

In addition, I have also considered Galí's (2015, Chapter 3) calibration of the basic New Keynesian model: $(\beta, \kappa, \sigma) = (0.99, 0.125, 1)$, the period being again one quarter. Most of the results under this alternative calibration are qualitatively and quantitatively similar. In particular, determinacy can again be obtained only for $h \leq 1$ under Rule 1, and only for $h \in \{-1, 0\}$ under Rule 2. The only notable difference is that $\underline{\phi}$ and $\bar{\phi}$ for Rule 2 are roughly multiplied by a factor of 6 (essentially because of the difference in the value of σ

between the two calibrations).

2.6 Determinacy horizons and Taylor principle under Rule 1

I now characterize more precisely the set of horizons for which Rules 1 and 2 can deliver determinacy, i.e. the set $\mathbb{H}_D := \{h \in \mathbb{Z} | \exists \phi \in \mathbb{R}, S(\phi, h) = D\}$, which I call the set of "determinacy horizons." This set is an important element to consider when assessing the desirability of forward- or backward-looking monetary policy. I also examine the validity of the Taylor principle as a condition for determinacy. I do that for Rule 1 in this subsection, and for Rule 2 in the next subsection.

Before stating the results, I need to specify what I mean by "Taylor principle." There are several versions of this principle in the literature. The simplest and narrowest version is that the rule should make the interest rate react more than one-for-one to the inflation rate, when it reacts only to the inflation rate. Another, more general version, sometimes called the long-run Taylor principle, was proposed by Woodford (2001, 2003) and is used in, e.g., Galí (2015, Chapter 4). For the sake of generality, I adopt the latter version of the Taylor principle – which, in the basic New Keynesian model under Rule 1, amounts anyway to the former version.

I will provide a formal, general definition of the long-run Taylor principle in Section 3. In the current section, I only need to state this principle in the specific context of the basic New Keynesian model, under Rule 1 or 2. In this context, loosely speaking, the long-run Taylor principle states that if the inflation rate were permanently higher by one percentage point, then the system composed of the Phillips curve (2) and Rule 1 or 2 should make the interest rate permanently higher by more than one percentage point. Under Rule 1, this principle straightforwardly translates into $\phi > \phi_W := 1$, where the subscript W stands for "Woodford." Under Rule 2, this principle amounts to $\phi > \phi_W := \kappa/(1-\beta)$, since the Phillips curve (2) implies that a permanent increase in inflation of one percentage point leads to a permanent increase in output of $(1-\beta)/\kappa$ percentage points.

I can now state the results for Rule 1 as follows:

Proposition 3 (Determinacy horizons and Taylor principle in the basic New Keynesian model under Rule 1): *Consider the basic New Keynesian model (1)-(2) with the rule* $i_t = \phi \mathbb{E}_t \{\pi_{t+h}\}$, *where* $(\phi, h) \in \mathbb{R} \times \mathbb{Z}$. *Then* $\phi_W = \underline{\phi}$ *and:*
(a) $\mathbb{H}_D = \{h \in \mathbb{Z} | h < 1 + (1-\beta)\sigma/\kappa\}$;

(b) $\forall h \in \mathbb{Z} \setminus \mathbb{H}_D$*, the Taylor principle is irrelevant for* D*;*

(c) $\forall h \in \mathbb{H}_D$*, the Taylor principle is locally necessary and sufficient for* D*;*

(d) $\forall h \in \mathbb{H}_D$*, the Taylor principle is sufficient for* D *if and only if* $h = 0$*.*

Proof: See Online Appendix A.3.[7] ■

Point (a) of this proposition characterizes the set $\mathbb{H}_D$. It gives the analytical expression of the smallest integer $\bar{h}$ in Point (c) of Proposition 1. In standard calibrations of the basic New Keynesian model, the horizon threshold $1 + (1 - \beta)\sigma/\kappa$ is typically between 1 and 2, because β is typically set to 0.99 (on a quarterly basis). In the two calibrations considered in the previous subsection, in particular, this threshold takes the values 1.07 and 1.08, which are much closer to 1 than to 2. So, in the basic New Keynesian model, a central bank reacting to inflation should hardly be forward-looking, if at all, in order to ensure determinacy.

Point (a) of Proposition 3 also says that a central bank reacting to inflation can be arbitrarily backward-looking and still ensure determinacy; but, as we know from Proposition 1 and Figure 1, the set of ϕ values leading to determinacy gradually shrinks to the empty set as $h \to -\infty$.

Points (b)-(d) of Proposition 3 are about the Taylor principle as a condition for determinacy. Point (b) trivially follows from the definition of $\mathbb{H}_D$. Point (d) of Proposition 3 essentially follows from Point (b) of Proposition 1. Point (c) states a local result, i.e. a result holding for ϕ in the neighborhood of $\phi_W = 1$. As ϕ crosses ϕ_W from below, one root of $P(z)$ crosses $\mathcal{C}$ at point 1. When $h < 1 + (1 - \beta)\sigma/\kappa$, the root goes from outside to inside $\mathcal{C}$, so the determinacy status moves from multiplicity to determinacy (reflecting the fact that increasing the weight on outcomes sufficiently distant in the past favors exploding paths). Alternatively, when $h > 1 + (1 - \beta)\sigma/\kappa$, the root goes from inside to outside $\mathcal{C}$, so the determinacy status remains multiplicity (reflecting the fact that increasing the weight on outcomes sufficiently distant in the future favors imploding paths). All these results about the Taylor principle are illustrated in the left panel of Figure 2, where I have featured ϕ_W.

[7] The paper derives results in two different contexts in turn: (i) the basic New Keynesian model under Rule 1 or 2; and (ii) a generic model under a generic rule. The proofs of all these results are at least briefly discussed in the main text. I relegate to an Online Appendix the complete proofs of some results for (i), in particular Proposition 3, as well as the complete proofs of the results for (ii). The reason is that these proofs are essentially variants or generalizations of some key proofs that are in the main text and in the Appendix of the paper (most notably the proof of Proposition 1); they rest on similar arguments and use the same theorems as the latter proofs. I make clear the connection between these proofs in the main text.

2.7 Determinacy horizons and Taylor principle under Rule 2

I now turn to Rule 2: I characterize again, this time partially, the set of determinacy horizons $\mathbb{H}_D$, and I study again the validity of the Taylor principle as a condition for determinacy. As discussed in the previous subsection, the Taylor principle under Rule 2 is $\phi > \phi_W := \kappa/(1-\beta)$. Let me define

$$\eta := \left[(1-\beta)^2 + \left(1+\beta+\frac{\kappa}{\sigma}\right)^2\right]\beta - (1+\beta)\left(1+\beta^2\right)\left(1+\beta+\frac{\kappa}{\sigma}\right).$$

I can then state the results for Rule 2 as follows:

Proposition 4 (Determinacy horizons and Taylor principle in the basic New Keynesian model under Rule 2): *Consider the basic New Keynesian model (1)-(2) with the rule* $i_t = \phi\mathbb{E}_t\{y_{t+h}\}$, *where* $(\phi, h) \in \mathbb{R} \times \mathbb{Z}$. *Then:*

(a) $\mathbb{H}_D$ *is bounded above;*

(b) $\mathbb{H}_D$ *is bounded below if and only if* $|\eta - 4\beta^2| < 4\beta\left(1+\beta^2\right)$*;*

(c) $\forall h \in \mathbb{Z} \setminus \mathbb{H}_D$, *the Taylor principle is irrelevant for D;*

(d) $\forall h \in \mathbb{H}_D$, *the Taylor principle is sufficient for D if and only if* $h = 0$*;*

(e) if (and only if) $\kappa/\sigma \geq (1+\beta)(1-\beta)^2/\beta$, *then: (i)* $\phi_W = \bar{\phi}$, *(ii)* $\forall h \leq -1$, *the Taylor principle is sufficient for E, (iii) for* $h = 0$, *it is sufficient for D, (iv)* $\forall h \geq 1$, *it is sufficient for M;*

(f) if (and only if) $\eta - 4\beta^2 < -4\beta\left(1+\beta^2\right)$, *then: (i)* $\phi_W = \underline{\phi}$, *(ii)* $\forall h \in \mathbb{Z}$, *the Taylor principle is locally necessary and sufficient for D if and only if* $h < (1-\beta)(1+\sigma/\kappa)$.

Proof: see Online Appendix A.4. ■

Points (a) and (c)-(e) of this proposition straightforwardly or essentially follow from Proposition 2 and the definition of $\mathbb{H}_D$. Note that Point (e) of Proposition 4 has no counterpart in Proposition 3; the reason is that under Rule 1, unlike under Rule 2, we necessarily have $\phi_W = \underline{\phi}$ and hence $\phi_W \neq \bar{\phi}$ (as stated in Proposition 3).

Point (b) of Proposition 4 states the necessary and sufficient condition on the structural parameters for $\mathbb{H}_D$ to be bounded below. If this condition is met, then a central bank reacting to output with a sufficiently long delay will necessarily generate either multiplicity or explosiveness, no matter how strongly or weakly it reacts to output. The condition stated in this point is, in fact, necessary and sufficient for $\text{argmin}_{z\in\mathcal{C}} |Q(z)/(z-\beta)| \subset \mathcal{C} \setminus \{-1, 1\}$, i.e. for the minimum in the definition of $\underline{\phi}$ to be obtained for some non-real complex numbers, whose number is necessarily even. So, as $|\phi|$ crosses $\underline{\phi}$ from below, an *even* (possibly zero) number of roots of $P(z)$ cross $\mathcal{C}$, either from inside to outside, or vice-versa. To move from multiplicity to determinacy, however, we would need exactly *one* root of $P(z)$ to go from outside to inside $\mathcal{C}$. So, the determinacy status either remains multiplicity, or jumps directly from multiplicity to explosiveness. For $-h$ sufficiently large, if no $|\phi|$ in the neighborhood of $\underline{\phi}$ can ensure determinacy, then more generally no $|\phi| \in \mathbb{R}_+$ can ensure determinacy. Note that Point (b) of Proposition 4 has no counterpart in Proposition 3; the reason is that under Rule 1, unlike under Rule 2, the minimum in the definition of $\underline{\phi}$ is necessarily obtained for $z = 1$: $\text{argmin}_{z \in \mathcal{C}} |Q(z)| = \{1\} \not\subset \mathcal{C} \backslash \{-1, 1\}$.

The conditions stated in Point (b) and (e) of Proposition 4 are met in Model 1, as apparent in the right panel of Figure 2 (where I have featured ϕ_W). They are also met under the other calibration considered in Subsection 2.5.

Point (f) of Proposition 4 is similar to Point (c) of Proposition 3. Under the condition stated in this point, one root of $P(z)$ crosses $\mathcal{C}$ at point 1 as ϕ crosses ϕ_W from below. When $h < (1-\beta)(1+\sigma/\kappa)$, the root goes from outside to inside $\mathcal{C}$, so the determinacy status moves from multiplicity to determinacy (reflecting the fact that increasing the weight on outcomes sufficiently distant in the past favors exploding paths). Alternatively, when $h > (1-\beta)(1+\sigma/\kappa)$, the root goes from inside to outside $\mathcal{C}$, so the determinacy status remains multiplicity (reflecting the fact that increasing the weight on outcomes sufficiently distant in the future favors imploding paths). Note that the validity of the Taylor principle as a condition for determinacy across different horizons depends crucially on whether $\phi_W = \bar{\phi}$, as in Point (e), or $\phi_W = \underline{\phi}$, as in Point (f). I will come back to this point in the more general analysis of the next section, and I will use it to shed light on some contrasting results in the monetary-policy literature about the Taylor principle as a guide for determinacy.

3 General determinacy analysis

In this section, I generalize the results of the previous section to a broad class of dynamic rational-expectations models, and to rules making the policy instrument react to any variable (or linear combination of variables) at horizon h with coefficient ϕ.

3.1 Model and rule

At each date $t \in \mathbb{Z}$, the private sector sets an n-dimension vector of endogenous variables $\mathbf{X}_t$ according to the following (locally log-linearized) structural equations:

$$\mathbb{E}_t \left\{ \mathbf{\Delta} \left(L^{-1} \right) \left[\mathbf{A} \left(L \right) \mathbf{X}_t + L^{-\gamma} \mathbf{B} \left(L \right) i_t \right] \right\} = \mathbf{0}, \tag{4}$$

where again i_t denotes the policy instrument at date t, L the lag operator, and $\mathbb{E}_t\{.\}$ the date-t rational-expectations operator. I abstract again from exogenous shocks, as they are irrelevant for determinacy issues. These structural equations are parameterized by $n \in \mathbb{N} \backslash \{0\}$, $\gamma \in \mathbb{N}$, $\mathbf{A}(z) \in \mathbb{R}^{n \times n}[z]$, $\mathbf{B}(z) \in \mathbb{R}^{n \times 1}[z]$, $\mathbf{\Delta}(z) = \text{diag}\left(z^{\delta_1}, ..., z^{\delta_n}\right) \in \mathbb{R}^{n \times n}[z]$, and $(\delta_1, ..., \delta_n) \in \mathbb{N}^n$.[8]

I make two non-restrictive assumptions on $\mathbf{A}(z)$ and $\mathbf{B}(z)$. First, I assume that $\det[\mathbf{A}(0)] \neq 0$; this assumption is made without any loss in generality because any system of independent structural equations of type (4) that does not satisfy this assumption can be equivalently rewritten as a system of type (4) that does. Second, I assume that $\mathbf{B}(z) \neq \mathbf{0}$; this assumption is needed simply for the policy instrument to have some effect on the endogenous variables set by the private sector.

The class of models of type (4) is broad enough to include, arguably, most existing dynamic stochastic general-equilibrium (DSGE) models. The structural equations (1)-(2) of the basic New Keynesian model, in particular, can straightforwardly be written in a form of type (4) with $n = 2$, $\gamma = 0$,

$$\mathbf{X}_t = \begin{bmatrix} y_t \\ \pi_t \end{bmatrix}, \ \mathbf{\Delta}(z) = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix}, \ \mathbf{A}(z) = \begin{bmatrix} 1 - z & \frac{1}{\sigma} \\ \kappa z & \beta - z \end{bmatrix}, \text{ and } \mathbf{B}(z) = \begin{bmatrix} -\frac{z}{\sigma} \\ 0 \end{bmatrix}.$$

This system satisfies the two assumptions made above: $\det[\mathbf{A}(0)] = \beta \neq 0$, and $\mathbf{B}(z) \neq \mathbf{0}$.

The policymaker follows the (log-linearized) policy-instrument rule

$$i_t = \phi \mathbb{E}_t \left\{ v_{t+h} \right\}, \tag{5}$$

where $\phi \in \mathbb{R}$ and $h \in \mathbb{Z}$ (with again $\mathbb{E}_t \left\{ v_{t+h} \right\} = v_{t+h}$ when $h \leq 0$), and where v_t can be any linear combination of current and past endogenous variables:

$$v_t := \mathbf{V}(L)\mathbf{X}_t, \tag{6}$$

[8] Throughout the paper, letters in bold denote vectors and matrices that have potentially more than one element. $\mathbf{0}$ denotes a vector or a matrix whose elements are all equal to zero and whose dimensions depend on the specific context in which it is used. For any $(n_1, n_2) \in (\mathbb{N} \setminus \{0\})^2$, $\mathbb{R}^{n_1 \times n_2}[z]$ denotes the set of polynomials in z whose coefficients are $n_1 \times n_2$ matrices with real-number elements.

with $\mathbf{V}(z) \in \mathbb{R}^{1\times n}[z]$. I make the following non-restrictive assumption on $\mathbf{V}(z)$:

$$W(z) := \det \begin{bmatrix} \mathbf{A}(z) & \mathbf{B}(z) \\ \mathbf{V}(z) & 0 \end{bmatrix} \neq 0.$$

If this assumption were not satisfied, then v_t could be expressed as a linear combination of (a backward-looking version of) the structural equations, and would therefore be exogenous. In the basic New Keynesian model, for instance, we have $W(z) = [(\beta - z)V_1(z) - \kappa z V_2(z)]z/\sigma$, where $V_1(z)$ and $V_2(z)$ denote the two elements of $\mathbf{V}(z)$; so, imposing $W(z) \neq 0$ amounts to ruling out variables v_t of type $v_t = \tilde{V}(L)(\pi_{t-1} - \beta\pi_t - \kappa y_{t-1})$ with $\tilde{V}(z) \in \mathbb{R}[z]$, i.e. variables v_t that are "proportional" to a backward-looking version of the Phillips curve (2). Such variables are exogenous because they can be rewritten, using the Phillips curve (2), as the sum of past expectation errors: $v_t = -\beta\tilde{V}(L)(\pi_t - \mathbb{E}_{t-1}\{\pi_t\})$. For a horizon h higher than the degree of $\tilde{V}(z)$, in particular, the term $\mathbb{E}_t\{v_{t+h}\}$ in the rule (5) would simply be zero.

3.2 Determinacy status

As in Section 2, let ν denote the number of non-predetermined variables of the system (4)-(5), and $P(z)$ the reciprocal polynomial of the characteristic polynomial of this system. In addition, let $\omega \in \mathbb{N}$ denote the multiplicity of 0 as a root of $W(z)$ (with $\omega = 0$ if $W(0) \neq 0$). I start by establishing a useful preliminary result:

Lemma 1 (Expression of ν and $P(z)$): *If $\phi = 0$, then $\nu = \delta := \sum_{j=1}^{n} \delta_j$ and $P(z) = Q(z) := \det[\mathbf{A}(z)]$. If $\phi \neq 0$, then, except possibly for a zero-measure set of ϕ values, $\nu = \delta + \max(0, h - m)$ and*

$$P(z) = Q(z)z^{\max(0,h-m)} + \phi R(z) z^{\max(0,m-h)}, \tag{7}$$

where $m := \omega - \gamma$ and $R(z) := -z^{-\omega}W(z)$.

Proof: See Online Appendix A.5. ■

This lemma generalizes similar preliminary results obtained in Section 2: in the specific context of the basic New Keynesian model, we had $Q(z) = \beta - (1 + \beta + \kappa/\sigma)z + z^2$ and $\delta = 2$, with $m = 2$ and $R(z) = \kappa/\sigma$ for Rule 1, and $m = 1$ and $R(z) = (z - \beta)/\sigma$ for Rule 2. The "zero-measure set of ϕ values" mentioned in the lemma refers again to the possibility of reducing ν below $\delta + \max(0, h - m)$ with carefully designed policy-instrument rules as in Loisel (2009), a possibility that I ignore in the present paper.

The polynomial $Q(z)$ depends on the model (4), not on the rule (5). It is the reciprocal polynomial of the characteristic polynomial *under a policy-instrument peg*, i.e. under the policy-instrument rule $i_t = 0$. The polynomial $R(z)$ depends on the model (4) and the variable v_t in the rule (5), not on the coefficient ϕ nor on the horizon h of the rule (5). It is the reciprocal polynomial of the characteristic polynomial *under the "targeting rule"* $v_t = 0$.

Let $q_{\mathcal{C}} := \#\{z \in \mathcal{C} | Q(z) = 0\}$ and $r_{\mathcal{C}} := \#\{z \in \mathcal{C} | R(z) = 0\}$ denote the number of roots of $Q(z)$ and $R(z)$ *exactly* on $\mathcal{C}$ (counting multiplicity). I distinguish between the "regular case" in which $q_{\mathcal{C}} = r_{\mathcal{C}} = 0$ (as in Section 2), and the "non-regular cases" in which $q_{\mathcal{C}} \geq 1$ or $r_{\mathcal{C}} \geq 1$. I focus on the regular case in the present paper, i.e. I assume

$$q_{\mathcal{C}} = r_{\mathcal{C}} = 0. \tag{8}$$

I study non-regular cases in a (longer) working-paper version (Loisel, 2022a).[9]

I distinguish between three kinds of models, depending on their *determinacy status under a peg*, which I denote by S_{peg} ($S_{peg} := S(0,h)$ for any $h \in \mathbb{Z}$). Let $p := \#\{z \in \mathbb{C} | P(z) = 0, |z| < 1\}$ and $q := \#\{z \in \mathbb{C} | Q(z) = 0, |z| < 1\}$ denote the number of roots of $P(z)$ and $Q(z)$ inside $\mathcal{C}$ (counting multiplicity). Under a peg ($\phi = 0$), we have $p = q$ and $\nu = \delta$ (as follows from Lemma 1); so, Blanchard and Kahn's (1980) *root-counting* condition for determinacy, $p = \nu$, is simply $q = \delta$. I assume here, and everywhere else in the paper, that Blanchard and Kahn's (1980) *no-decoupling* condition is met (it is straightforward to check that it is met in all the specific examples I consider in the paper).[10] As a result, the determinacy status under a peg is multiplicity for models with $q \leq \delta - 1$ ($S_{peg} = M$), determinacy for models with $q = \delta$ ($S_{peg} = D$), and explosiveness for models with $q \geq \delta + 1$ ($S_{peg} = E$).

Using Lemma 1, it is easy to conduct the same analysis as in Section 2, and thus to generalize Propositions 1-2 to the class of models (4) and the class of rules (5). I obtain the following proposition, where $r := \#\{z \in \mathbb{C} | R(z) = 0, |z| < 1\}$ denotes the number of

[9] As I elaborate in the working-paper version, non-regular cases arise under some specific circumstances. The case $q_{\mathcal{C}} \geq 1$, for instance, typically arises when the variable in the rule appears only in first difference in the structural equations (e.g. the price level in the rule and the inflation rate in the structural equations). The case $r_{\mathcal{C}} \geq 1$ can arise when, conversely, the variable in the rule is the first difference of a variable in the structural equations (e.g. the output growth rate in the rule and the output level in the structural equations). As I will illustrate below, the case $r_{\mathcal{C}} \geq 1$ can also arise when the variable in the rule is the output level and the long-run Phillips curve is vertical.

[10] The "no-decoupling condition" requires that the system should not be "decoupled" in the sense of Sims (2007). It is formulated as a matrix-rank condition in Blanchard and Kahn (1980, Page 1308), and is often called the "rank condition" in the literature. Sims' (2007) bare-bones example of a system meeting the root-counting condition but not the no-decoupling condition is $x_t = 1.1x_{t-1} + \varepsilon_t$ and $\mathbb{E}_t\{y_{t+1}\} = 0.9y_t + \nu_t$.

roots of $R(z)$ inside $\mathcal{C}$ (counting multiplicity):

Proposition 5 (Determinacy status): *Consider a model of type (4) and a variable v_t of type (6). Let $\underline{\phi} := \min_{z\in\mathcal{C}} |Q(z)/R(z)|$, $\bar{\phi} := \max_{z\in\mathcal{C}} |Q(z)/R(z)|$, and $h^* := m + r - \delta$. Then, under the rule $i_t = \phi\mathbb{E}_t\{v_{t+h}\}$ with $(\phi, h) \in \mathbb{R}\times\mathbb{Z}$:*

(a) $\forall\, |\phi| < \underline{\phi}$, $\forall h \in \mathbb{Z}$, $S(\phi, h) = S_{peg}$;

(b) $\forall\, |\phi| > \bar{\phi}$, (i) $\forall h \leq h^* - 1$, $S(\phi, h) = E$, (ii) $S(\phi, h^*) = D$, (iii) $\forall h \geq h^* + 1$, $S(\phi, h) = M$;

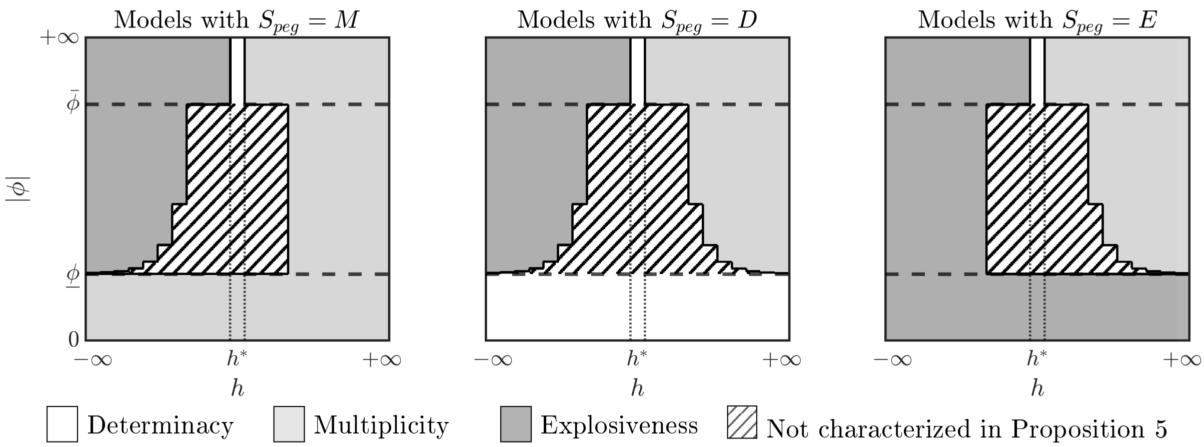
(c) $\exists \bar{h} : (\underline{\phi}, \bar{\phi}) \to \mathbb{Z}$, (i) $\forall\, |\phi| \in (\underline{\phi}, \bar{\phi})$, $\forall h \geq \bar{h}(|\phi|)$, $S(\phi, h) = M$, (ii) $\forall \varepsilon \in (0, \bar{\phi} - \underline{\phi})$, $\bar{h}$ is bounded on $(\underline{\phi} + \varepsilon, \bar{\phi})$, (iii) if $S_{peg} = M$, then $\bar{h}$ is bounded on $(\underline{\phi}, \bar{\phi})$;

(d) $\exists \underline{h} : (\underline{\phi}, \bar{\phi}) \to \mathbb{Z}$, (i) $\forall\, |\phi| \in (\underline{\phi}, \bar{\phi})$, $\forall h \leq \underline{h}(|\phi|)$, $S(\phi, h) = E$, (ii) $\forall \varepsilon \in (0, \bar{\phi} - \underline{\phi})$, $\underline{h}$ is bounded on $(\underline{\phi} + \varepsilon, \bar{\phi})$, (iii) if $S_{peg} = E$, then $\underline{h}$ is bounded on $(\underline{\phi}, \bar{\phi})$.

Proof: See Online Appendix A.6. ■

Like Propositions 1-2, this proposition may look a bit cryptic at first sight. Like Propositions 1-2, however, it can be represented in a simple diagrammatic form: Figure 3 shows the determinacy status $S(\phi, h)$ in the pseudo half-plane $(h, |\phi|) \in \mathbb{Z} \times \mathbb{R}_+$, according to Proposition 5.

Figure 3: Determinacy status for models of type (4) and rules of type (5)



The intuitions behind Proposition 5 are identical or similar to those behind Propositions 1-2. In Point (a), as $|\phi| < \underline{\phi}$, the structural equations *dominate* the rule in the system's dynamics, and the determinacy status remains the same as under a peg. Compared to Point (a) of Propositions 1-2, the novelty is that the determinacy status under a peg, S_{peg}, can now be not only M, but also D and E. In Point (b), as $|\phi| > \bar{\phi}$, the rule *dominates* the structural equations in the system's dynamics and makes the determinacy status depend only on h. Compared to Point (b) of Propositions 1-2, the novelty is that the pivotal horizon h^* can now be different from zero.

Points (c)-(i), (c)-(ii), (d)-(i), and (d)-(ii) of Proposition 5 generalize Point (d) of Propositions 1-2. For $|\phi| \in (\underline{\phi}, \bar{\phi})$, as $|h| \to +\infty$, the roots of $P(z)$ distribute themselves between inside and outside $\mathcal{C}$ *in proportion of* the share of $\mathcal{C}$ on which the rule dominates the structural equations and the share of $\mathcal{C}$ on which the structural equations dominate the rule; so, the number of inside roots increases less than one-for-one with $|h|$. As $h \to -\infty$ (resp. as $h \to +\infty$), the number of non-predetermined variables remains constant (resp. increases one-for-one with h), so we eventually get more (resp. fewer) inside roots than non-predetermined variables, and hence explosiveness (resp. multiplicity), for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$. Determinacy may still obtain for arbitrarily large $|h|$'s, but *only if* $|\phi|$ is arbitrarily close to $\underline{\phi}$, i.e. only if the portion of $\mathcal{C}$ on which the rule dominates the structural equations is arbitrarily small.

Finally, Points (c)-(iii) and (d)-(iii) of Proposition 5 generalize Point (c) of Propositions 1-2. Large positive (resp. negative) horizons h do not much *perturb* the imploding (resp. exploding) equilibrium paths obtained under a peg, as the term $\mathbb{E}_t\{v_{t+h}\}$ is small on these paths; so, these horizons *preserve* the determinacy status obtained under a peg if this status is multiplicity (resp. explosiveness).

3.3 Numerical examples

In order to illustrate Proposition 5 and the next propositions numerically, I consider, in addition to Model 1, three other simple calibrated monetary-policy models. Table 1 presents the overall four models: one is with $S_{peg} = M$, two with $S_{peg} = D$, and one with $S_{peg} = E$. The table also indicates, for each model, the *degree of indeterminacy under a peg*, $d_{peg} := \delta - q \in \mathbb{Z}$ (which is positive if there is multiplicity, zero if there is determinacy, and negative if there is explosiveness).

Table 1: Four simple calibrated monetary-policy models

No.	**Model**	**Calibration**	S_{peg}	d_{peg}
1	Basic New Keynesian Model	Woodford (2003)	M	1
2	Gabaix (2020)	Gabaix (2020)	D	0
3	Bilbiie (2008)	Bilbiie (2008)	D	0
4	Svensson (1997) and Ball (1999)	Ball (1999)	E	-1

All these models share the following "canonical" features: they have two simple structural equations; these equations are an IS equation and a Phillips curve; and the two endogenous variables set by the private sector are output and inflation. The IS equation and the Phillips curve of Models 2-3 are

$$\begin{aligned}
y_t &= \alpha \mathbb{E}_t\{y_{t+1}\} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t\{\pi_{t+1}\}\right), \\
\pi_t &= \beta \mathbb{E}_t\{\pi_{t+1}\} + \kappa y_t,
\end{aligned}$$

with $(\alpha, \beta, \sigma, \kappa) = (0.85, 0.792, 5, 0.11)$ in Model 2 and $(\alpha, \beta, \sigma, \kappa) = (1, 0.99, -0.11, 0.228)$ in Model 3. These models introduce, into the basic New Keynesian model, bounded rationality (Model 2) or limited asset-markets participation (Model 3). Compared to the basic New Keynesian model (in which α is implicitly equal to 1), Model 2 "discounts" the IS equation and the Phillips curve (i.e. reduces both α and β), and Model 3 inverts the slope of the IS equation (i.e. makes σ negative). Unlike Models 1-3, Model 4 is non-micro-founded and purely backward-looking; its IS equation and Phillips curve are

$$\begin{aligned}
y_t &= \lambda y_{t-1} - \mu\left(i_{t-1} - \pi_{t-1}\right), \\
\pi_t &= \pi_{t-1} + \chi y_{t-1},
\end{aligned}$$

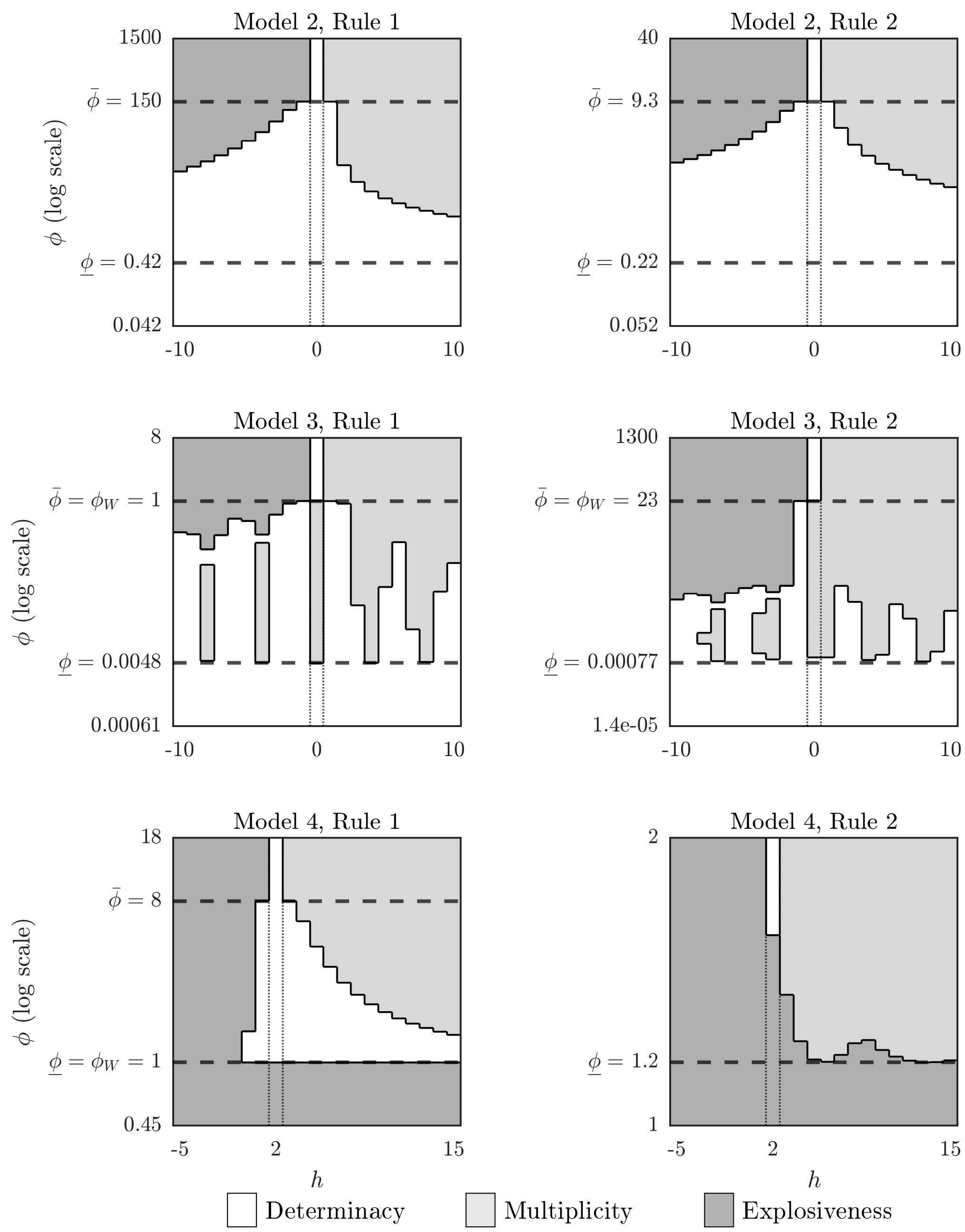
with $(\lambda, \mu, \chi) = (0.8, 1, 0.4)$.

A first numerical illustration of Proposition 5 is provided by Figure 2 in Section 2. This figure, which I have already commented upon, shows the determinacy status $S(\phi, h)$ for Model 1 and Rules 1-2, in the pseudo half-plane $(h, \phi) \in \mathbb{Z} \times \mathbb{R}_+$ with a log scale for ϕ. Since Model 1 is such that $S_{peg} = M$, Figure 2 is more specifically a numerical example of the *left* panel of Figure 3.

Figure 4 is the counterpart of Figure 2 for Models 2-4. The top four panels of this figure show the determinacy status for the systems (Model j, Rule k) with $j \in \{2, 3\}$ and $k \in \{1, 2\}$, all of which satisfy the regularity condition (8). Models 2-3, moreover, are such that $S_{peg} = D$; so, the top four panels of Figure 4 are numerical examples of the *middle* panel of Figure 3. In these panels, thus, the rule can deliver determinacy for any horizon h, unlike in Figure 2. Compared to standard values of ϕ, the lower threshold $\underline{\phi}$ is of the same order of magnitude or several orders of magnitude smaller, while the upper threshold $\bar{\phi}$ is of the same order of magnitude or several orders of magnitude larger. The "topology" of the E, D, and M regions is simple in the top two panels (for Model 2): each region is connected, and the borders between regions are monotonic functions linking h to ϕ. The topology is more complex in the middle two panels (for Model 3): the M

region is disconnected, and the borders between regions are non-monotonic, with "lace patterns."

Figure 4: Determinacy status for Models 2-4 and Rules 1-2 with $\phi > 0$



The bottom left panel of Figure 4 shows the determinacy status for the system (Model 4, Rule 1), which also satisfies the regularity condition (8). Model 4, moreover, is such that $S_{peg} = E$; so, the bottom left panel of Figure 4 is a numerical example of the *right* panel of Figure 3. Qualitatively speaking, the D region in the bottom left panel of Figure 4 looks like the mirror image, with left and right reversed, of the D region in the left panel of Figure 2. The horizon threshold at and below which the rule can no longer deliver determinacy is minus one period; so, in this example, a central bank that would react to inflation with a delay of at least one period would necessarily be behind the curve and fail to ensure determinacy, no matter how strongly or weakly it reacts to inflation. Compared to standard values of ϕ, the lower threshold is of the same order of magnitude, while the upper threshold is one order of magnitude larger.

Finally, the bottom right panel of Figure 4 shows the determinacy status for the system (Model 4, Rule 2). This system satisfies $q_{\mathcal{C}} = 0$, but not $r_{\mathcal{C}} = 0$. For this system, indeed, we have $R(z) = (1-z)/\chi$ and hence $r_{\mathcal{C}} = 1$, because the Phillips curve of Model 4 is vertical in the long run. So, this system does not satisfy the regularity condition (8), and the bottom right panel of Figure 4 is not an illustration of Proposition 5. One difference with the regular case is that $\bar{\phi} := \max_{z\in\mathcal{C}} |Q(z)/R(z)|$ no longer exists (since $R(1) = 0$); so, it does not feature in the panel. Despite this difference, however, the panel suggests that most of Proposition 5's results extend to the non-regular case $q_{\mathcal{C}} = 0$ and $r_{\mathcal{C}} = 1$; in the working paper (Loisel, 2022a), I confirm that they do.

3.4 Determinacy horizons

I now turn to the question of whether the set of determinacy horizons $\mathbb{H}_D := \{h \in \mathbb{Z} | \exists \phi \in \mathbb{R}, S(\phi, h) = D\}$ is bounded or not, below or above. This question matters for the desirability of backward- or forward-looking stabilization policy. Let $A_{min} := \operatorname{argmin}_{z\in\mathcal{C}} |Q(z)/R(z)|$. I provide the following answer to this question:

Proposition 6 (Determinacy horizons): *Consider a model of type (4) and a variable v_t of type (6). Then, under the rule $i_t = \phi \mathbb{E}_t \{v_{t+h}\}$ with $(\phi, h) \in \mathbb{R} \times \mathbb{Z}$:*
(a) if $S_{peg} = D$, then $\mathbb{H}_D = \mathbb{Z}$;
(b) if $S_{peg} = M$ (resp. $S_{peg} = E$), then:
(i) $\mathbb{H}_D$ is bounded above (resp. below);
(ii) if d_{peg} is odd and $A_{min} \subset \mathcal{C} \setminus \{-1, 1\}$, then $\mathbb{H}_D$ is bounded below (resp. above);
(iii) if $|d_{peg}| = 1$ and $A_{min} \in \{\{-1\}, \{1\}\}$, then $\mathbb{H}_D$ is unbounded below (resp. above).

Proof: See Online Appendix A.7. ■

Points (a) and (b)(i) of this proposition straightforwardly follow from Proposition 5 and Figure 3. Point (a) is illustrated in the top four panels of Figure 4, while Point (b)(i) is illustrated in the two panels of Figure 2 and the bottom left panel of Figure 4. As I discuss in the Introduction, Point (b)(i) for $S_{peg} = M$ offers an explanation for the propensity of forward-looking interest-rate rules to generate multiplicity in New Keynesian models, as found in, e.g., Levin et al. (2003). Existing results on this front are mostly numerical and sparsely distributed across calibrated models and rules; Point (b)(i) generalizes them to a broad class of models and a broad class of rules (making the policy instrument react to any expected future variable).

Point (b)(ii) of Proposition 6 states a sufficient condition for the D region to be left-bounded in the left panel of Figure 3, or right-bounded in the right panel of Figure 3. This point generalizes, along several dimensions, Point (b) of Proposition 4. Under the condition stated in this point, as $|\phi|$ crosses $\underline{\phi}$, an *even* (possibly zero) number of roots of $P(z)$ cross $\mathcal{C}$; in order to get determinacy, however, we would need an *odd* number of them. For $|h|$ sufficiently large, if no $|\phi|$ in the neighborhood of $\underline{\phi}$ can ensure determinacy, then more generally no $|\phi| \in \mathbb{R}_+$ can ensure determinacy. So, if $S_{peg} = M$ (resp. $S_{peg} = E$), then, for $-h$ (resp. h) sufficiently large, the determinacy status jumps *directly* from multiplicity to explosiveness (resp. from explosiveness to multiplicity) as $|\phi|$ goes from zero to infinity. Points (b)(i)-(ii) then imply that the set of determinacy horizons is bounded *both* below and above; *both* sufficiently backward-looking stabilization policies and sufficiently forward-looking ones will necessarily fail to deliver determinacy, no matter how strongly or weakly the policy instrument reacts to the state of the economy. This point is, again, illustrated in the right panel of Figure 2.

Finally, Point (b)(iii) of Proposition 6 states a sufficient condition for the D region to be left-unbounded in the left panel of Figure 3, or right-unbounded in the right panel of Figure 3. This point generalizes Point (a) of Proposition 3 and Point (b) of Proposition 4. Under the condition stated in this point, as $|\phi|$ crosses $\underline{\phi}$ from below, we need exactly *one* root of $P(z)$ to cross $\mathcal{C}$ in order to get determinacy, and we do get exactly *one* such root (either as ϕ crosses $\underline{\phi}$, or as ϕ crosses $-\underline{\phi}$). If $d_{peg} = 1$ (resp. $d_{peg} = -1$), we also need this root to go from outside to inside $\mathcal{C}$ (resp. from inside to outside $\mathcal{C}$) in order to get determinacy, and the root does so for $-h$ (resp. h) sufficiently large, reflecting the fact that increasing the weight on outcomes sufficiently distant in the past (resp. the future) favors exploding (resp. imploding) paths. So, the set of determinacy horizons is unbounded below (resp. above): the rule can be arbitrarily backward-looking (resp. forward-looking) and still ensure determinacy; as we know from Proposition 5 and Figure 3, however, the set of ϕ values leading to determinacy gradually shrinks to the empty set as $h \to -\infty$ (resp. $h \to +\infty$). These results are illustrated in the left panel of Figure 2 and the bottom left panel of Figure 4.

3.5 Taylor principle

I now derive the implications of the previous results for the validity of the Taylor principle as a condition for determinacy. I start by providing a formal, general definition of Woodford's (2001, 2003) long-run Taylor principle. Woodford (2001, 2003) mostly describes this principle in the specific context of the basic New Keynesian model with several alternative parametric families of interest-rate rules. He discusses how to generalize this principle to a broader context as follows: "*One observes quite generally – in the case of* any *family of policy rules that involve feedback only from inflation and output, regardless of how many lags of these might be involved – that the boundary between sets of coefficients that satisfy the Taylor principle and those that do not will consist of coefficients for which there is an eigenvalue exactly equal to* 1*. (...) It follows that a real eigenvalue crosses the unit circle as the sign of the inequality corresponding to the Taylor principle changes. This boundary is therefore one at which the number of unstable eigenvalues increases by one. Often this results in moving from a situation of indeterminacy to determinacy, though I do not seek to establish general conditions for this*" (Woodford, 2003, Chapter 4, Page 256, Footnote 27).

The system (4)-(5) has an eigenvalue equal to 1 if and only if $P(1) = 0$. In turn, since (8) implies $R(1) \neq 0$, we have $P(1) = 0$ if and only if $\phi = \phi_W := -Q(1)/R(1)$ (where again the subscript W stands for "Woodford"). In all the examples considered by Woodford (2001, 2003), ϕ_W is non-negative and the Taylor principle is $\phi > \phi_W$, not $\phi < \phi_W$ (i.e., the policy instrument should react to the variable sufficiently strongly, not sufficiently weakly). Under the regularity condition (8), moreover, we have $Q(1) \neq 0$ and hence $\phi_W \neq 0$. So, I propose the following definition of the Taylor principle:

Definition 1 (Taylor principle): *If $\phi_W := -Q(1)/R(1) > 0$, then the Taylor principle is $\phi > \phi_W$.*

This definition is a generalization of the definition considered in Section 2. The latter definition was tailored to the specific context of the basic New Keynesian model; it defined the Taylor principle as a higher-than-unitary permanent reaction of the interest rate to a permanent change in inflation. Definition 1, more generally, defines the Taylor principle as higher-than-ϕ_W coefficient ϕ in the policy-instrument rule, where ϕ_W is the only value of ϕ that makes the dynamic system have an infinity of constant equilibria (for all other values of ϕ, the dynamic system has a unique constant equilibrium; in this equilibrium, all endogenous variables are constantly equal to zero). Both definitions lead to the same Taylor principle in the basic New Keynesian model: $\phi > 1$ for Rule 1, and $\phi > \kappa/(1-\beta)$ for Rule 2. The advantage of Definition 1 is that it applies not just to the basic New Keynesian model with inflation or output in the interest-rate rule, but more generally to any stabilization-policy model and any variable v_t in the policy-instrument rule – as long as ϕ_W exists and is positive. Recent examples of Definition 1's Taylor principle in the literature, outside the context of the basic New Keynesian model, include the "income-risk augmented Taylor principle" of Acharya and Dogra (2020), the "HANK Taylor principle" of Bilbiie (2021), and the "modified Taylor principle" of Gabaix (2020) (as long as $\phi_W > 0$).

The condition that ϕ_W should exist and be positive is, of course, not always met. The system (Model 4, Rule 2), for instance, is such that ϕ_W does not exist, since $R(1) = 0$ in this system (as discussed in Subsection 3.3). The systems (Model 2, Rule 1) and (Model 2, Rule 2) are such that ϕ_W exists but is negative; in this case, whether the Taylor principle should be defined as $\phi > \phi_W$ or $\phi < \phi_W$ is unclear. These three systems correspond to the top left, top right, and bottom right panels of Figure 4; so, ϕ_W does not appear in these panels. I have featured ϕ_W in the other panels of this figure.

Let $A_{max} := \text{argmax}_{z \in \mathcal{C}} |Q(z)/R(z)|$. I can now state the results about the Taylor principle as follows:

Proposition 7 (Taylor principle): *Consider a model of type (4) and a variable v_t of type (6) such that $\phi_W > 0$. Let $h^{**} := m + R'(1)/R(1) - Q'(1)/Q(1)$. Then, under the rule $i_t = \phi \mathbb{E}_t \{v_{t+h}\}$ with $(\phi, h) \in \mathbb{R}_+ \times \mathbb{Z}$:*

(a) if $S_{peg} \in \{M, E\}$, then $\forall h \in \mathbb{Z} \setminus \mathbb{H}_D$, the Taylor principle is irrelevant for D;

(b) if $S_{peg} = D$, then $\forall h \in \mathbb{Z}$, the Taylor principle is not necessary for D;

(c) $\forall h \in \mathbb{Z} \setminus \{h^*\}$, the Taylor principle is not sufficient for D;

(d) if $A_{max} = \{1\}$, then: (i) $\phi_W = \bar{\phi}$, (ii) $\forall h \leq h^* - 1$, the Taylor principle is sufficient for E, (iii) for $h = h^*$, it is sufficient for D, (iv) $\forall h \geq h^* + 1$, it is sufficient for M;

(e) if $A_{min} = \{1\}$, then: (i) $\phi_W = \underline{\phi}$, (ii) $\forall h \in \mathbb{Z}$, the Taylor principle is locally necessary and sufficient for D if and only if ($d_{peg} = 1$ and $h < h^{**}$) or ($d_{peg} = -1$ and $h > h^{**}$).

Proof: See Online Appendix A.8. ■

Points (a)-(d) of this proposition straightforwardly follow from Proposition 5 and Figure 3 (as well as from the definition of $\mathbb{H}_D$). Point (d) of Proposition 7, in particular, is illustrated in the right panel of Figure 2 and the middle two panels of Figure 4.

Point (e) of Proposition 7 states the necessary and sufficient condition for the Taylor principle to be locally necessary and sufficient for determinacy when $\phi_W = \underline{\phi}$. This point generalizes Point (c) of Proposition 3 and Point (f) of Proposition 4. If $d_{peg} = 1$ (resp. $d_{peg} = -1$), then, as ϕ crosses $\phi_W = \underline{\phi}$ from below, we need exactly one root of $P(z)$ to go from outside to inside $\mathcal{C}$ (resp. from inside to outside $\mathcal{C}$) in order to get determinacy; and we do get exactly one such root if and only if $h < h^{**}$ (resp. $h > h^{**}$), reflecting the fact that increasing the weight on outcomes sufficiently distant in the past (resp. the future) favors exploding (resp. imploding) paths. Alternatively, if $|d_{peg}| \neq 1$, then we still have exactly *one* root of $P(z)$ crossing $\mathcal{C}$ as ϕ crosses $\phi_W = \underline{\phi}$, but we would need a *different* number of such roots in order to get determinacy. Point (e) of Proposition 7 is illustrated in the left panel of Figure 2 and the bottom left panel of Figure 4.

Points (d)-(e) of Proposition 7 show that the validity of the Taylor principle as a condition for determinacy across different horizons depends crucially on whether $\phi_W = \bar{\phi}$ or $\phi_W = \underline{\phi}$. When the Taylor principle is locally necessary and sufficient for determinacy, it is for a single horizon if $\phi_W = \bar{\phi}$ (the horizon $h = h^*$, as illustrated in the right panel of Figure 2 and the middle two panels of Figure 4), and for an infinity of horizons if $\phi_W = \underline{\phi}$ (the horizons $h < h^{**}$ or $h > h^{**}$, as illustrated in the left panel of Figure 2 and the bottom left panel of Figure 4). The reason is that when $\phi_W = \bar{\phi}$, the Taylor principle $\phi > \phi_W$ *makes the rule completely dominate* the structural equations in the system's dynamics; as a result, the degree of indeterminacy $\nu - p$ increases one-for-one with the horizon h in the rule, and determinacy obtains only for a single horizon (h^*). By contrast, when $\phi_W = \underline{\phi}$, the Taylor principle $\phi > \phi_W$ *prevents the structural equations from completely dominating* the rule in the system's dynamics; locally, for ϕ just above ϕ_W, the structural equations are still close to completely dominating the rule, so the degree of indeterminacy $\nu - p$ and the determinacy status do not change with the horizon h in the rule, except when h crosses the threshold h^{**}.

This key distinction, $\phi_W = \underline{\phi}$ vs. $\phi_W = \bar{\phi}$, also sheds light on some contrasting results in the monetary-policy literature about the Taylor principle as a guide for determinacy. In the basic New Keynesian model under Rule 1, we have $\phi_W = 1$, and for $h = 1$ the Taylor principle $\phi > \phi_W$ is locally necessary and sufficient for determinacy (as illustrated in the left panel of Figure 2). In Bilbiie's (2008) model under Rule 1, we also have $\phi_W = 1$, but for $h = 1$ the locally necessary and sufficient condition for determinacy is the "*inverted Taylor principle*" $\phi < \phi_W$ (as highlighted by Bilbiie, 2008, and as illustrated in the middle left panel of Figure 4). Key to understand these contrasting results is the fact that $\phi_W = \underline{\phi}$ in the former setup, while $\phi_W = \bar{\phi}$ in the latter setup (and $h = 1 > 0 = h^*$ in both setups). Thus, in the former setup, the Taylor principle avoids multiplicity by preventing the structural equations from completely dominating the rule; in the latter setup, the inverted Taylor principle avoids multiplicity by preventing the rule from completely dominating the structural equations.

Similarly, Benhabib et al. (2001) show that in the standard flexible-price money-in-the-utility-function model, under Rule 1 with $h = 1$, the Taylor principle (resp. the inverted Taylor principle) is locally necessary and sufficient for determinacy if consumption and real money balances are complements (resp. substitutes). What distinguishes these two cases (complements and substitutes), and can explain their contrasting implications for the Taylor principle, is not the value of ϕ_W (equal to 1 in both cases), but rather the fact that $\phi_W = \bar{\phi}$ in one case, while $\phi_W = \underline{\phi}$ in the other.[11]

4 Extensions and discussion

In this section, I extend the results of the previous section to rules involving several variables and to inertial rules (which are often considered in the applied literature). I also discuss the implications of my results for the robustness of rules across alternative models, and the application of my results to targeting rules and structural equations (instead of policy-instrument rules).

4.1 Extension to rules with several variables

I start with rules involving several variables. The rules I have considered so far were of type (5). This type of rule involves a single variable v_t, associated with a single coefficient ϕ and a single horizon h (even though v_t can itself be defined as a linear combination of several variables). I now consider rules that involve, in addition to the variable v_t, some other variables $v_{1,t}$,...,$v_{J,t}$ of type (6), each with its own coefficient and its own horizon:

$$i_t = \phi \mathbb{E}_t \{v_{t+h}\} + \sum_{j=1}^{J} \phi_j \mathbb{E}_t \{v_{j,t+h_j}\}, \tag{9}$$

where $(\phi, \phi_1, ..., \phi_J) \in \mathbb{R} \times (\mathbb{R} \backslash \{0\})^J$, $(h, h_1, ..., h_J) \in \mathbb{Z}^{J+1}$, and $J \in \mathbb{N} \backslash \{0\}$. My goal is to address the same questions as previously, about the determinacy status, the determinacy

[11] Other illustrations of the inverted Taylor principle $\phi < \phi_W$ being locally necessary and sufficient for determinacy (with $\phi_W = \bar{\phi}$) can be found in the right panel of Figure 2 for $h = -1$, and in the middle right panel of Figure 4 for $h = -1$.

horizons, and the Taylor principle, still conditionally on the coefficient ϕ and the horizon h of the variable v_t, but this time under Rule (9) instead of Rule (5).

Consider, for example, in the basic New Keynesian model (used in Section 2), the Taylor-type rule

$$i_t = \phi \mathbb{E}_t \left\{ \pi_{t+h} \right\} + \phi_1 \mathbb{E}_t \left\{ y_{t+h_1} \right\}, \tag{Rule 3}$$

where $(\phi, \phi_1) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\})$ and $(h, h_1) \in \mathbb{Z}^2$. For any given value of (ϕ_1, h_1), to analyze how the determinacy status depends on (ϕ, h), I move the term $\phi_1 \mathbb{E}_t\{y_{t+h_1}\}$ out of the rule and into the IS equation (1). The original system composed of the IS equation (1), the Phillips curve (2), and Rule 3 has the same determinacy status as the system composed of the Phillips curve (2), the modified IS equation

$$y_t = \mathbb{E}_t\{y_{t+1}\} - \frac{1}{\sigma} \left(i_t + \phi_1 \mathbb{E}_t\{y_{t+h_1}\} - \mathbb{E}_t\{\pi_{t+1}\} \right), \tag{10}$$

and the rule $i_t = \phi \mathbb{E}_t\{\pi_{t+h}\}$ (i.e. Rule 1, instead of Rule 3). So, I can apply Propositions 5-7 to the modified model (2) and (10) under Rule 1. If the modified model (2) and (10) delivers multiplicity under a peg (for instance if $\phi_1 < \underline{\phi}_1$, where $\underline{\phi}_1$ denotes the value of $\underline{\phi}$ in the original model under Rule 2), then the determinacy status $S(\phi, h)$ will be diagrammatically represented by the left panel of Figure 3. Alternatively, if the modified model delivers determinacy or explosiveness under a peg (which is possible only if $\phi_1 > \underline{\phi}_1$), then the determinacy status will be diagrammatically represented by the middle or right panel of Figure 3.

Similarly and more generally, to analyze the determinacy status $S(\phi, h)$ for a model of type (4) and Rule (9), I simply move the term $\sum_{j=1}^{J} \phi_j \mathbb{E}_t\{v_{j,t+h_j}\}$ out of the rule and into the model, and then I apply the results of the previous section to the modified model and Rule (5). To state formally the outcome of this procedure, I need to introduce some new notations. For any $j \in \{1, ..., J\}$, let m_j, $R_j(z)$, and $\underline{\phi}_j$ denote the counterparts of m, $R(z)$, and $\underline{\phi}$ under the rule $i_t = \phi_j \mathbb{E}_t\{v_{j,t+h_j}\}$ instead of the rule $i_t = \phi \mathbb{E}_t\{v_{t+h}\}$. In addition, let $g := \max[0, \max_{j \in \{1,...,J\}}(h_j - m_j)]$, $\tilde{\delta} := \delta + g$, $\tilde{m} := m + g$, $\tilde{Q}(z) := z^g[Q(z) + \sum_{j=1}^{J} \phi_j R_j(z) z^{m_j - h_j}]$, $\tilde{q} := \#\{z \in \mathbb{C} | \tilde{Q}(z) = 0, |z| < 1\}$, $\tilde{d}_{peg} := \tilde{\delta} - \tilde{q}$, and $\tilde{S}_{peg} := M$ if $\tilde{d}_{peg} \geq 1$, D if $\tilde{d}_{peg} = 0$, E if $\tilde{d}_{peg} \leq -1$. As I show in Online Appendix A.9, $\tilde{\delta}$, $\tilde{m}$, $\tilde{Q}(z)$, $\tilde{q}$, $\tilde{d}_{peg}$, and $\tilde{S}_{peg}$ are the counterparts of δ, m, $Q(z)$, q, d_{peg}, and S_{peg} for the modified model instead of the original model (both with Rule (5)). As in the previous section, I focus on the "regular case" in which now $q_{\mathcal{C}} = \tilde{q}_{\mathcal{C}} = r_{\mathcal{C}} = 0$ (where $\tilde{q}_{\mathcal{C}} := \#\{z \in \mathcal{C} | \tilde{Q}(z) = 0\}$ denotes the number of roots of $\tilde{Q}(z)$ *exactly* on $\mathcal{C}$, counting multiplicity), leaving the study of non-regular cases to the (longer) working-paper version (Loisel, 2022a). I then obtain the following results:

Proposition 8 (Determinacy status, determinacy horizons, and Taylor principle under a rule with several variables): *Propositions 5-7 still hold for Rule (9) instead of Rule (5), if δ, m, $Q(z)$, d_{peg}, and S_{peg} are respectively replaced by $\tilde{\delta}$, $\tilde{m}$, $\tilde{Q}(z)$, $\tilde{d}_{peg}$, and $\tilde{S}_{peg}$ in these propositions. In addition, if $\sum_{j=1}^{J} |\phi_j| / \underline{\phi}_j < 1$, then, as we move from Rule (5) to Rule (9):*
(a) the determinacy status for $\phi = 0$ is unchanged;
(b) $\underline{\phi}$ is multiplied by a factor not lower than $1 - \sum_{j=1}^{J} |\phi_j| / \underline{\phi}_j$;
(c) $\bar{\phi}$ is multiplied by a factor not higher than $1 + \sum_{j=1}^{J} |\phi_j| / \underline{\phi}_j$.

Proof: See Online Appendix A.9. ■

The first part of this proposition follows from the fact that the determinacy status $S(\phi, h)$ for the original model and Rule (9) is the same as for the modified model and Rule (5). In the second part of the proposition, the condition $\sum_{j=1}^{J} |\phi_j| / \underline{\phi}_j < 1$ characterizes the size of a neighborhood of $(\phi_1, ..., \phi_J) = (0, ..., 0)$ within which the modified model has the same determinacy status under a peg as the original model ($\tilde{S}_{peg} = S_{peg}$). Under this condition, therefore, the determinacy-status results in the original model are qualitatively similar under Rule (5) and under Rule (9), in the sense that these results are described by the same panel of Figure 3 under the two rules (Point (a) of the proposition). Under this condition, moreover, the values of $\underline{\phi}$ and $\bar{\phi}$ under Rule (9) lie in a neighborhood of their values under Rule (5); the size of this neighborhood is specified in Points (b)-(c) of the proposition.

4.2 Extension to inertial rules

I now turn to inertial rules, i.e. rules making the current policy instrument react to its past values in addition to a variable v_t of type (6) at horizon h with coefficient ϕ:

$$\rho(L) i_t = \phi \mathbb{E}_t \{v_{t+h}\}, \tag{11}$$

where $\phi \in \mathbb{R} \setminus \{0\}$, $h \in \mathbb{Z}$, and $\rho(z) \in \mathbb{R}[z]$ with $\rho(0) \neq 0$.[12] As I show in Online Appendix A.10, moving from Rule (5) to Rule (11) does not affect the number of non-predetermined variables ν of the dynamic system, and its only effect on the reciprocal polynomial of the characteristic polynomial, $P(z)$, is to replace $Q(z)$ by $Q(z)\rho(z)$ in the expression (7) of $P(z)$. In essence, introducing some inertia into the reaction of the

[12] I rule out the (uninteresting) case $\phi = 0$, as Blanchard and Kahn's (1980) no-decoupling condition (discussed in Footnote 10) may not be met in this case.

policy instrument to the state of the economy (i.e. replacing i_t by $\rho(L)i_t$ in the rule), leaving unchanged the reaction of the economy to the policy instrument, is equivalent, as far as determinacy analysis is concerned, to introducing the same inertia into the reaction of the economy to the policy instrument (i.e. replacing $\mathbf{A}(L)\mathbf{X}_t$ by $\mathbf{A}(L)\rho(L)\mathbf{X}_t$ in the structural equations, and hence replacing $Q(z)$ by $Q(z)\rho(z)$), leaving unchanged the reaction of the policy instrument to the state of the economy.

So, as in the previous subsection, I can reformulate the problem in a way that enables me to apply the results of Section 3: to analyze the determinacy status $S(\phi, h)$ for a model of type (4) and Rule (11), I simply move the "inertia operator" $\rho(L)$ out of the rule and into the model, and then I apply the results of Section 3 to the modified model and Rule (5). To state formally the outcome of this procedure, I need to introduce some new notations. Let $\rho := \#\{z \in \mathbb{C} | \rho(z) = 0, |z| < 1\}$ denote the number of roots of $\rho(z)$ inside $\mathcal{C}$ (counting multiplicity). Let $\hat{Q}(z)$, $\hat{q}$, $\hat{d}_{peg}$, and $\hat{S}_{peg}$ denote the counterparts of $Q(z)$, q, d_{peg}, and S_{peg} for the modified model instead of the original model: $\hat{Q}(z) := Q(z)\rho(z)$, $\hat{q} := \#\left\{z \in \mathbb{C} | \hat{Q}(z) = 0, |z| < 1\right\} = q + \rho$, $\hat{d}_{peg} := \delta - \hat{q} = d_{peg} - \rho$, and $\hat{S}_{peg} := M$ if $\hat{d}_{peg} \geq 1$, D if $\hat{d}_{peg} = 0$, E if $\hat{d}_{peg} \leq -1$. As previously, I focus on the "regular case" in which now $q_{\mathcal{C}} = \rho_{\mathcal{C}} = r_{\mathcal{C}} = 0$ (where $\rho_{\mathcal{C}} := \#\{z \in \mathcal{C} | \rho(z) = 0\}$ denotes the number of roots of $\rho(z)$ *exactly* on $\mathcal{C}$, counting multiplicity), leaving the study of non-regular cases to the (longer) working-paper version (Loisel, 2022a).[13] I thus obtain the following proposition:

Proposition 9 (Determinacy status, determinacy horizons, and Taylor principle under an inertial rule): *Propositions 5-7 still hold for Rule (11) instead of Rule (5), if $Q(z)$, d_{peg}, and S_{peg} are respectively replaced by $\hat{Q}(z)$, $\hat{d}_{peg}$, and $\hat{S}_{peg}$ in these propositions.*

Proof: See Online Appendix A.10. ■

If $\rho(z)$ has all its roots outside $\mathcal{C}$ (i.e. $\rho = 0$), then the modified model has the same degree of indeterminacy under a peg as the original model ($\hat{d}_{peg} = d_{peg}$), and hence the same determinacy status under a peg as well ($\hat{S}_{peg} = S_{peg}$). In this case, the determinacy-status results in the original model are qualitatively similar under Rule (5) and under Rule (11), in the sense that these results are diagrammatically described by the same panel of Figure 3 under the two rules. Alternatively, if $\rho(z)$ has at least one root inside

[13]The non-regular case $\rho_{\mathcal{C}} \geq 1$ arises, for instance, under "first-difference rules," i.e. rules with $\rho(z) = 1 - z$ (which are advocated in, e.g., Levin et al., 1999, and Levin and Williams, 2003).

$\mathcal{C}$ (i.e. $\rho \geq 1$), that is to say if Rule (11) is "*superinertial*" in the sense of Woodford (2003, Chapter 8) and Giannoni and Woodford (2002), then the modified model has a lower degree of indeterminacy under a peg than the original model ($\hat{d}_{peg} \leq d_{peg} - 1$). In this case, the determinacy status under a peg may differ across the two models: we may have $\hat{S}_{peg} \in \{D, E\}$ if $S_{peg} = M$, and we necessarily have $\hat{S}_{peg} = E$ if $S_{peg} = D$. So, as we move from Rule (5) to Rule (11) in the original model, we may move from the left panel of Figure 3 to its middle or right panel, or from its middle panel to its right panel. This change simply reflects the fact that superinertia favors exploding paths for small (but non-zero) coefficients ϕ in absolute value.

Proposition 9 can be used to design a rule that makes the set of determinacy horizons $\mathbb{H}_D$ *unbounded below and above* in models delivering multiplicity under a peg (i.e. models with $S_{peg} = M$, or equivalently $d_{peg} \geq 1$). In these models, under the non-inertial rule (5), $\mathbb{H}_D$ is bounded above (as shown in Point (b)(i) of Proposition 6) and may be bounded below (as shown in Point (b)(ii) of Proposition 6). To enlarge $\mathbb{H}_D$ (in order to respond to forecasts without loosing determinacy, or to ensure determinacy in the presence of inside lags), the policymaker can adopt a superinertial rule (11) with exactly d_{peg} roots of $\rho(z)$ inside $\mathcal{C}$ (i.e. $\rho = d_{peg}$). The resulting modified model delivers determinacy under a peg ($\hat{d}_{peg} = 0$ and $\hat{S}_{peg} = D$). So, replacing Rule (5) with this rule in the original model moves us from the left to the middle panel of Figure 3, and makes $\mathbb{H}_D$ unbounded below and above. Thus, because superinertia favors exploding paths, it can be used to *offset* multiplicity: with a degree of superinertia equal to the degree of indeterminacy under a peg ($\rho = d_{peg}$) and with a sufficiently small (but non-zero) coefficient ϕ in absolute value, Rule (11) ensures determinacy for any horizon h. This result echoes, and sheds light on, a result obtained by Woodford (2003, Chapter 8) and Giannoni and Woodford (2002, 2003, 2005) about the degree of superinertia of their "robustly optimal rules," which they find is equal to the degree of indeterminacy under a peg (in models delivering multiplicity under a peg).

Alternatively, for models delivering determinacy or explosiveness under a peg ($S_{peg} \in \{D, E\}$), Proposition 9 implies that any superinertial rule with a sufficiently small (but non-zero) coefficient ϕ in absolute value will necessarily lead to explosiveness. In effect, replacing the non-inertial rule (5) with a superinertial rule (11) moves us from the middle to the right panel of Figure 3 (for a model with $S_{peg} = D$), or keeps us in the right panel of Figure 3 (for a model with $S_{peg} = E$). This result offers an explanation for the propensity of superinertial rules to generate explosiveness in backward-looking models, i.e. models with $\delta = 0$ (Rudebusch and Svensson, 1999, and Levin and Williams, 2003).

4.3 Implications for the robustness of rules

The general results that I have established in this paper provide guidelines for finding rules with robust determinacy properties across alternative models. For example, if a variable v_t leads to the same h^* value across all the models considered, then a non-inertial rule (5) with $h = h^*$ and $|\phi|$ sufficiently large will deliver determinacy in all these models (Proposition 5). As another example, if the degree of indeterminacy under a peg d_{peg} takes the same positive value across all the models considered, then an inertial rule (11) with exactly d_{peg} roots of $\rho(z)$ inside $\mathcal{C}$ and with $|\phi|$ sufficiently small will deliver determinacy in all these models (Proposition 9).

Research on the robustness of interest-rate rules across alternative monetary-policy models has, over the past ten years, benefited from the development of a Macroeconomic Model Data Base (MMB) described in Wieland et al. (2012, 2016). The MMB Team (2023) writes that "there is no hard guideline for determinacy" and refers to Levin et al. (2003) for a suggestion of several characteristics of rules that deliver determinacy. Among these characteristics, which Levin et al. (2003) identify numerically using five calibrated models, are "a relatively short inflation forecast horizon" and "a moderate degree of responsiveness to the inflation forecast."

Since the five calibrated models considered in Levin et al. (2003) deliver multiplicity under an interest-rate peg, Propositions 5 and 9 offer an explanation of these two characteristics: for $h \geq h^* + 1$, two necessary conditions for determinacy are that h should be sufficiently small and that $|\phi|$ should be between $\underline{\phi}$ and $\bar{\phi}$, no matter whether the rule is non-inertial or inertial (provided that it is not superinertial). Moreover, Propositions 5 and 9 show that these two characteristics, qualitatively speaking, remain necessary for determinacy for a broad range of model calibrations (not just the ones considered in Levin et al., 2003), a broad class of models (not just the five models they consider), and a broad class of variables in the rule (not just inflation) $-$ in essence, for all the models of type (4) delivering multiplicity under a peg, and all the variables v_t of type (6).

Like the five calibrated models considered in Levin et al. (2003), most of the models in the current MMB version (3.1) deliver multiplicity under an interest-rate peg ($S_{peg} = M$, or equivalently $d_{peg} \geq 1$). Table 2 reports the distribution of d_{peg} across the 140 rational-expectations models in the base: 90% of these models are, more specifically, such that

$d_{peg} = 1$. Computing the thresholds $\underline{\phi}$, $\bar{\phi}$, h^*, and h^{**} for various models in the base and various variables in the rule could be helpful in the quest for a robust rule.

Table 2: Distribution of d_{peg} in the Macroeconomic Model Data Base 3.1

Value of d_{peg}	-1	0	1	2
Number of models	6	4	126	4

4.4 Application to targeting rules and structural equations

I have so far considered the variable i_t as the policy instrument, and focused on policy-instrument rules. However, nothing prevents us from interpreting i_t as an endogenous variable set by the private sector, and one element of the vector $\mathbf{X}_t$ as the policy instrument. So, the results that I have established in this paper could be applied to "targeting rules," instead of policy-instrument rules. They could also be applied to structural equations (which describe the behavior of the private sector), instead of targeting rules or policy-instrument rules (which describe the behavior of a policymaker). For instance, in a monetary-policy model, the results could be used to find conditions on the structural parameters for the model to deliver determinacy under an interest-rate peg, in order to solve New Keynesian puzzles and paradoxes at the zero lower bound. These conditions could take the form, for example, of lower bounds for the degree of income-risk procyclicality in models à la Acharya and Dogra (2020) and Bilbiie (2021), or lower bounds for the degree of bounded rationality in models à la Gabaix (2020).

5 Conclusion

This paper has established some simple, easily interpretable, necessary or sufficient conditions for determinacy in a broad class of dynamic rational-expectations models. These determinacy conditions lead to new, general principles for stabilization policy in terms of whether, and how strongly or weakly, to react to any variable, at any horizon, in any model. In so doing, the paper has provided new insights into the forces at work behind determinacy, multiplicity, and explosiveness; it has characterized circumstances under which the long-run Taylor principle is (not) necessary, (not) sufficient, or irrelevant for determinacy; and it has provided the first hard guidelines for finding rules with robust determinacy properties across alternative models. The results can be applied to monetary policy, fiscal policy, macroprudential policy, or any other stabilization policy. Overall, the paper thus opens new horizons for the study of stabilization policies, and paves the way for new qualitative and quantitative research.

References

[1] Acharya, S., and Dogra, K. (2020), "Understanding HANK: Insights from a PRANK," *Econometrica*, 88(3), 1113–1158.

[2] Ball, L. (1999), "Efficient Rules for Monetary Policy," *International Finance*, 2(1), 63-83.

[3] Beaudry, P., Galizia, D., and Portier, F. (2017), "Is the Macroeconomy Locally Unstable and Why Should We Care?," in Eichenbaum, M., and Parker, J.A. (eds) *NBER Macroeconomics Annual 2016*, 31, 479-530.

[4] Beaudry, P., Galizia, D., and Portier, F. (2020), "Putting the Cycle Back into Business-Cycle Analysis," *American Economic Review*, 110(1), 1-47.

[5] Benhabib, J. (2004), "Interest-Rate Policy in Continuous Time With Discrete Delays," *Journal of Money, Credit, and Banking*, 36(1), 1-15.

[6] Benhabib, J., Schmitt-Grohé, S., and Uribe, M. (2001), "Monetary Policy and Multiple Equilibria," *American Economic Review*, 91(1), 167-186.

[7] Benhabib, J., Schmitt-Grohé, S., and Uribe, M. (2002), "Avoiding Liquidity Traps," *Journal of Political Economy*, 110(3), 535-563.

[8] Benhabib, J., Schmitt-Grohé, S., and Uribe, M. (2003), "Backward-Looking Interest-Rate Rules, Interest-Rate Smoothing, and Macroeconomic Instability," *Journal of Money, Credit, and Banking*, 35(6), 1379-1412.

[9] Bernanke, B.S., and Woodford, M. (1997), "Inflation Forecasts and Monetary Policy," *Journal of Money, Credit and Banking*, 29, 653-684.

[10] Bhattarai, S., Lee, J.W., and Park, W.Y. (2014), "Price Indexation, Habit Formation, and the Generalized Taylor Principle," *Journal of Economic Dynamics and Control*, 48, 218-225.

[11] Bilbiie, F.O. (2008), "Limited Asset Markets Participation, Monetary Policy and (Inverted) Aggregate Demand Logic," *Journal of Economic Theory*, 140(1), 162-196.

[12] Bilbiie, F.O. (2021), "Monetary Policy and Heterogeneity: An Analytical Framework," mimeo.

[13] Blanchard, O.J., and Kahn, C.M. (1980), "The Solution of Linear Difference Models under Rational Expectations," *Econometrica*, 48(5), 1305-1311.

[14] Bullard, J., and Mitra, K. (2002), "Learning About Monetary Policy Rules," *Journal of Monetary Economics*, 49(6), 1105-1129.

[15] Carlstrom, C.T., and Fuerst, T.S. (2002), "Taylor Rules in a Model That Satisfies the Natural-Rate Hypothesis," *American Economic Review*, 92(2), 79-84.

[16] Clarida, R., Galí, J., and Gertler, M. (2000), "Monetary Policy Rules and Macroeconomic Stability: Evidence and Some Theory," *Quarterly Journal of Economics*, 115(1), 147-180.

[17] Cochrane, J.H. (2011), "Determinacy and Identification with Taylor Rules," *Journal of Political Economy*, 119(3), 565-615.

[18] Erdős, P., and P. Turán (1950), "On the Distribution of Roots of Polynomials," *Annals of Mathematics*, 51(1), 105-119.

[19] Gabaix, X. (2020), "A Behavioral New Keynesian Model," *American Economic Review*, 110(8), 2271-2327.

[20] Galí, J. (2015), *Monetary Policy, Inflation, and the Business Cycle: An Introduction to the New Keynesian Framework and its Applications*, Princeton: Princeton University Press, second edition.

[21] Giannoni, M.P., and Woodford, M. (2002), "Optimal Interest-Rate Rules: I. General Theory," NBER Working Paper No. 9419.

[22] Giannoni, M.P., and Woodford, M. (2003), "How Forward-Looking Is Optimal Monetary Policy?," *Journal of Money, Credit and Banking*, 35(6), 1425-1469.

[23] Giannoni, M.P., and Woodford, M. (2005), "Optimal Inflation-Targeting Rules," in Bernanke, B.S., and Woodford, M. (eds) *The Inflation-Targeting Debate*, Chicago: NBER and University of Chicago Press, 93-162.

[24] Glicksberg, I. (1976), "A Remark on Rouché's Theorem," *American Mathematical Monthly*, 83(3), 186-187.

[25] Hamilton, J.D. (1994), *Time Series Analysis*, Princeton: Princeton University Press.

[26] Henrici, P. (1988), *Applied and Computational Complex Analysis, Volume 1: Power Series, Integration, Conformal Mapping, Location of Zeros*, Wiley Classics Library.

[27] Levin, A.T., Wieland, V., and Williams, J.C. (1999), "Robustness of Simple Monetary Policy Rules Under Model Uncertainty," in Taylor, J.B. (ed) *Monetary policy rules*, Chicago: NBER and University of Chicago Press, 263-299.

[28] Levin, A.T., Wieland, V., and Williams, J.C. (2003), "The Performance of Forecast-Based Monetary Policy Rules Under Model Uncertainty," *American Economic Review*, 93(3), 622-645.

[29] Levin, A.T., and Williams, J.C. (2003), "Robust Monetary Policy With Competing Reference Models," *Journal of Monetary Economics*, 50, 945-975.

[30] Loisel, O. (2009), "Bubble-Free Policy Feedback Rules," *Journal of Economic Theory*, 144(4), 1521-1559.

[31] Loisel, O. (2022a), "New Principles For Stabilization Policy," CREST Working Paper No. 2022-16.

[32] Loisel, O. (2022b), "Stabilization Policy and Lags," mimeo.

[33] Lubik, T.A., and Schorfheide, F. (2004), "Testing for Indeterminacy: An Application to U.S. Monetary Policy," *American Economic Review*, 94(1), 190-217.

[34] Lucas, F. (1879), "Sur une Application de la Mécanique Rationnelle à la Théorie des Équations," *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences*, 89, 224-226.

[35] Macroeconomic Model Data Base Team (2023), "A Detailed Overview of Monetary Policy Rules in Modelbase," available at https://www.macromodelbase.com/files/documentation_source/mmb-mprule-description.pdf?40780101f6.

[36] Marden, M. (1966), *Geometry of Polynomials*, Providence: American Mathematical Society, second edition.

[37] McCallum, B.T. (1983), "On Non-Uniqueness in Rational Expectations Models: An Attempt at Perspective," *Journal of Monetary Economics*, 11(2), 139–168.

[38] Rouché, E. (1862), "Mémoire sur la Série de Lagrange," *Journal de l'École Polytechnique*, 39, 193-224.

[39] Rudebusch, G.D., and Svensson, L.E.O (1999), "Policy Rules for Inflation Targeting," in Taylor, J.B. (ed) *Monetary Policy Rules*, Chicago: NBER and University of Chicago Press, 203-246.

[40] Sargent, T.J., and Wallace, N. (1975), "Rational Expectations, the Optimal Monetary Intrument and the Optimal Money Supply Rule," *Journal of Political Economy*, 83(2), 241-254.

[41] Schmitt-Grohé, S., and Uribe, M. (1997), "Balanced-Budget Rules, Distortionary Taxes, and Aggregate Instability," *Journal of Political Economy*, 105(5), 976-1000.

[42] Sims, C.A. (2007), "On the Genericity of the Winding Number Criterion for Linear Rational Expectations Models," mimeo.

[43] Svensson, L.E.O. (1997), "Inflation Forecast Targeting: Implementing and Monitoring Inflation Targets," *European Economic Review*, 41(6), 1111-1146.

[44] Taylor, J.B. (1993), "Discretion Versus Policy Rules in Practice," *Carnegie-Rochester Conference Series on Public Policy*, 39, 195-214.

[45] Taylor, J.B., and Williams, J.C. (2011), "Simple and Robust Rules for Monetary Policy," in Friedman, B.M., and Woodford, M. (eds) *Handbook of Monetary Economics*, Amsterdam: Elsevier Science, 3B, 829-859.

[46] Wieland, V., Cwik, T., Müller, G.J., Schmidt, S., and Wolters, M. (2012), "A New Comparative Approach to Macroeconomic Modeling and Policy Analysis," *Journal of Economic Behavior and Organization*, 83(3), 523-541.

[47] Wieland, V., Afanasyeva, E., Kuete, M., and Yoo, J. (2016), "New Methods for Macro-Financial Model Comparison and Policy Analysis," in Taylor, J.B., and Uhlig, H. (eds) *Handbook of Macroeconomics*, Amsterdam: Elsevier Science, 2A, 1241-1319.

[48] Woodford, M. (1994), "Nonstandard Indicators for Monetary Policy: Can Their Usefulness be Judged from Forecasting Regressions?," in Mankiw, G.N. (ed) *Monetary policy*, Chicago: NBER and University of Chicago Press, 95-115.

[49] Woodford, M. (2001), "The Taylor Rule and Optimal Monetary Policy," *American Economic Review*, 91(2), 232-237.

[50] Woodford, M. (2003), *Interest and Prices: Foundations of a Theory of Monetary Policy*, Princeton: Princeton University Press.

Appendix: Proof of Points (c)-(d) of Proposition 1

Point (c). For $h \geq 2$, we have $\nu = h$ and

$$P(z) = Q(z)z^{h-2} + \frac{\phi\kappa}{\sigma}.$$

Let z_o denote the root of $Q(z)$ in $(1, +\infty)$, with the subscript "o" standing for "outside $\mathcal{C}$." Consider a Jordan curve $\mathcal{J}_o$ surrounding z_o and not intersecting nor surrounding $\mathcal{C}$.

I apply Rouché's theorem to $\mathcal{J} = \mathcal{J}_o$, $P_b(z) = Q(z)z^{h-2}$, and $P_s(z) = \phi\kappa/\sigma$. For any $|\phi| \in (\underline{\phi}, \bar{\phi})$, any

$$h \geq \bar{h} := 2 + \max\left\{0, \left\lceil \frac{\log\left(\frac{\bar{\phi}\kappa}{\sigma}\right) - \log\left(\min_{\tilde{z}\in\mathcal{J}_o} |Q(\tilde{z})|\right)}{\log\left(\min_{\tilde{z}\in\mathcal{J}_o} |\tilde{z}|\right)} \right\rceil\right\},$$

and any $z \in \mathcal{J}_o$, we have

$$\left|Q(z)z^{h-2}\right| \geq \min_{\tilde{z}\in\mathcal{J}_o} \left|Q(\tilde{z})\,\tilde{z}^{h-2}\right| \geq \left(\min_{\tilde{z}\in\mathcal{J}_o} |Q(\tilde{z})|\right)\left(\min_{\tilde{z}\in\mathcal{J}_o} |\tilde{z}|\right)^{h-2} \geq \frac{\bar{\phi}\kappa}{\sigma} > \left|\frac{\phi\kappa}{\sigma}\right|,$$

where the last but one inequality follows from the definition of $\bar{h}$. So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{J}_o$ as $Q(z)z^{h-2}$. The latter polynomial has exactly one root inside $\mathcal{J}_o$, which is z_o. Therefore, $P(z)$ has also exactly one root inside $\mathcal{J}_o$, and hence at least one root outside $\mathcal{C}$. Since the degree of $P(z)$ is h, we thus get $p \leq h - 1 < h = \nu$, and consequently $S(\phi, h) = M$ for any $|\phi| \in (\underline{\phi}, \bar{\phi})$ and any $h \geq \bar{h}$.

Point (d). For $h \leq 2$, we have $\nu = 2$ and

$$P(z) = Q(z) + \frac{\phi\kappa}{\sigma} z^{2-h}.$$

I proceed in four steps. In the first step, I show that for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$, all but one root of $P(z)$ converge uniformly to $\mathcal{C}$ as $h \to -\infty$. I get this result by applying Rouché's theorem twice. Consider an arbitrary $\epsilon \in (0, 1 - z_i)$, where z_i denotes the root of $Q(z)$ in $(0, 1)$, with the subscript "i" standing for "inside $\mathcal{C}$." For any $r \in \mathbb{R}_+$, let $\mathcal{C}_r$ denote the circle of radius r centered at the origin of the complex plane (so that in particular $\mathcal{C}_1 = \mathcal{C}$). I first apply Rouché's theorem to $\mathcal{J} = \mathcal{C}_{1-\epsilon}$, $P_b(z) = Q(z)$, and $P_s(z) = (\phi\kappa/\sigma)z^{2-h}$. For any $|\phi| \in (\underline{\phi}, \bar{\phi})$, any

$$h \leq \underline{h}_{1-\epsilon} := 2 + \min\left\{0, \left\lfloor \frac{\log\left(\min_{\tilde{z}\in\mathcal{C}_{1-\epsilon}} |Q(\tilde{z})|\right) - \log\left(\frac{\bar{\phi}\kappa}{\sigma}\right)}{-\log(1-\epsilon)} \right\rfloor\right\},$$

and any $z \in \mathcal{C}_{1-\epsilon}$, we have

$$|Q(z)| \geq \min_{\tilde{z}\in\mathcal{C}_{1-\epsilon}} |Q(\tilde{z})| \geq \frac{\bar{\phi}\kappa}{\sigma}(1-\epsilon)^{2-h} > \frac{|\phi|\,\kappa}{\sigma}(1-\epsilon)^{2-h} = \left|\frac{\phi\kappa}{\sigma} z^{2-h}\right|,$$

where the second inequality follows from the definition of $\underline{h}_{1-\epsilon}$. So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{C}_{1-\epsilon}$ as $Q(z)$. The latter polynomial has exactly one root inside $\mathcal{C}_{1-\epsilon}$, which is z_i. Therefore, $P(z)$ has also exactly one root inside $\mathcal{C}_{1-\epsilon}$ for any $|\phi| \in (\underline{\phi}, \bar{\phi})$ and any $h \leq \underline{h}_{1-\epsilon}$.

I then apply Rouché's theorem to $\mathcal{J} = \mathcal{C}_{1+\epsilon}$, $P_b(z) = (\phi\kappa/\sigma)z^{2-h}$, and $P_s(z) = Q(z)$. For any $|\phi| \in (\underline{\phi}, \bar{\phi})$, any

$$h \leq \underline{h}_{1+\epsilon} := 2 + \min\left\{0, \left\lfloor \frac{\log\left(\frac{\phi\kappa}{\sigma}\right) - \log\left(\max_{\tilde{z}\in\mathcal{C}_{1+\epsilon}} |Q(\tilde{z})|\right)}{\log(1+\epsilon)} \right\rfloor \right\},$$

and any $z \in \mathcal{C}_{1+\epsilon}$, we have

$$\left|\frac{\phi\kappa}{\sigma} z^{2-h}\right| = \frac{|\phi|\,\kappa}{\sigma}(1+\epsilon)^{2-h} > \frac{\phi\kappa}{\sigma}(1+\epsilon)^{2-h} \geq \max_{\tilde{z}\in\mathcal{C}_{1+\epsilon}} |Q(\tilde{z})| \geq |Q(z)|,$$

where the last but one inequality follows from the definition of $\underline{h}_{1+\epsilon}$. So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{C}_{1+\epsilon}$ as $(\phi\kappa/\sigma)z^{2-h}$. Therefore, $P(z)$ has exactly $2-h$ roots inside $\mathcal{C}_{1+\epsilon}$ for any $h \leq \underline{h}_{1+\epsilon}$. Since the degree of $P(z)$ is $2-h$ when $h \leq 0$, we eventually get that for any $|\phi| \in (\underline{\phi}, \bar{\phi})$ and any $h \leq \min\left(0, \underline{h}_{1-\epsilon}, \underline{h}_{1+\epsilon}\right)$, all but one root of $P(z)$ lie between $\mathcal{C}_{1-\epsilon}$ and $\mathcal{C}_{1+\epsilon}$. We conclude that for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$, all but one root of $P(z)$ converge uniformly to $\mathcal{C}$ as $h \to -\infty$.

In the second step, I show that for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$, the roots of $P(z)$ uniformly converging to $\mathcal{C}$ as $h \to -\infty$ converge in distribution to the uniform distribution on $\mathcal{C}$. This result is a direct consequence of the following theorem (stated but not proved in Marden, 1966, Page 193):

Theorem 2 (Erdős and Turán, 1950): *Let $\tilde{P}(z) = \sum_{k=0}^{d} \tilde{p}_k z^k \in \mathbb{C}[z]$ with $\tilde{p}_0\tilde{p}_d \neq 0$. Let $\varphi_k \in [0, 2\pi)$ for $1 \leq k \leq d$ denote the angular coordinates of the roots of $\tilde{P}(z)$. For any $0 \leq \underline{\alpha} < \bar{\alpha} \leq 2\pi$,*

$$\left|\#\left\{k \in \{1, ..., d\} | \underline{\alpha} \leq \varphi_k < \bar{\alpha}\right\} - \left(\frac{\bar{\alpha} - \underline{\alpha}}{2\pi}\right) d\right| \leq 16\sqrt{d \log\left(\frac{1}{\sqrt{|\tilde{p}_0\tilde{p}_d|}} \sum_{k=0}^{d} |\tilde{p}_k|\right)}.$$

Proof: See Erdős and Turán (1950). ■

I apply this theorem to $\tilde{P}(z) = P(z)$. For $\tilde{P}(z) = P(z)$ and $h \leq -1$, we have

$$\frac{1}{\sqrt{|\tilde{p}_0\tilde{p}_d|}} \sum_{k=0}^{d} |\tilde{p}_k| = 2(1+\beta) + \frac{\kappa}{\sigma}(1+\phi).$$

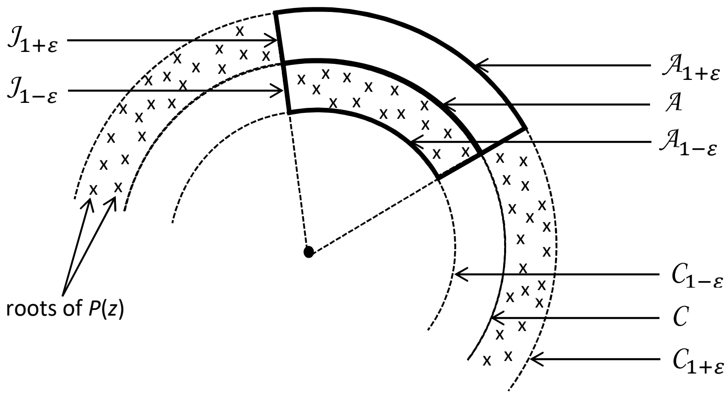
So, the Erdős-Turán theorem straightforwardly implies, together with the result of the previous step, that all but one root of $P(z)$ uniformly converge in distribution to the uniform distribution on $\mathcal{C}$ as $h \to -\infty$, for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$.

In the third step, I show that the number of roots of $P(z)$ inside $\mathcal{C}$ grows unboundedly as $h \to -\infty$, for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$. Since $|\phi| > \underline{\phi}$, there exists an arc $\mathcal{A}$ of $\mathcal{C}$ such that $\forall z \in \mathcal{A}$, $|\phi|\,\kappa/\sigma > |Q(z)|$. For any $r \in \mathbb{R}_+$, let $\mathcal{A}_r$ denote the image of $\mathcal{A}$ under the homothety whose center is the origin of the complex plane and whose ratio is r (so that in particular $\mathcal{A}_1 = \mathcal{A}$). By continuity, there exists $\varepsilon \in (0,1)$ such that $|\phi|\,\kappa/\sigma > |Q(z)|$ for all z on the Jordan curve $\mathcal{J}_{1+\varepsilon}$ made of $\mathcal{A}$, $\mathcal{A}_{1+\varepsilon}$, and the two radial line segments joining the endpoints of $\mathcal{A}$ and $\mathcal{A}_{1+\varepsilon}$ (see Figure 5). I apply Rouché's theorem to $\mathcal{J} = \mathcal{J}_{1+\varepsilon}$, $P_b(z) = (\phi\kappa/\sigma)z^{2-h}$, and $P_s(z) = Q(z)$. For any $h \leq 2$ and any $z \in \mathcal{J}_{1+\varepsilon}$, we have

$$\left|\frac{\phi\kappa}{\sigma} z^{2-h}\right| \geq \frac{|\phi|\,\kappa}{\sigma} > |Q(z)|\,.$$

So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{J}_{1+\varepsilon}$ as $(\phi\kappa/\sigma)z^{2-h}$. Therefore, $P(z)$ has no roots inside $\mathcal{J}_{1+\varepsilon}$ for any $h \leq 2$. Using the results of the first two steps, we get that the number of roots of $P(z)$ inside the Jordan curve $\mathcal{J}_{1-\varepsilon}$ made of $\mathcal{A}_{1-\varepsilon}$, $\mathcal{A}$, and the two radial line segments joining the endpoints of $\mathcal{A}_{1-\varepsilon}$ and $\mathcal{A}$, grows unboundedly as $h \to -\infty$. As a result, p grows unboundedly as $h \to -\infty$. Thus, there exists $\underline{h}\,(|\phi|)$ such that $p > 2 = \nu$ and $S(\phi, h) = E$ for all $h \leq \underline{h}\,(|\phi|)$.

Figure 5: Roots of $P(z)$ as $h \to -\infty$



In the fourth step, I just note that for any $\varepsilon \in \left(0, \bar{\phi} - \underline{\phi}\right)$ and any $|\phi| \in (\underline{\phi} + \varepsilon, \bar{\phi})$, there exists, by continuity, $\ell(\varepsilon) > 0$ such that the arc $\mathcal{A}$ can be chosen of length higher than $\ell(\varepsilon)$. As a result, $\underline{h}\,(|\phi|)$ can be chosen a bounded function of $|\phi|$ for $|\phi| \in (\underline{\phi} + \varepsilon, \bar{\phi})$.

Online Appendix

The paper derives results in two different contexts in turn: (i) the basic New Keynesian model under Rule 1 or 2; and (ii) a generic model under a generic rule. The proofs of all these results are at least briefly discussed in the main text. The complete proofs of some key results for (i), most notably Proposition 1, are in the main text and in the Appendix of the paper. This Online Appendix contains the complete proofs of the other results for (i)-(ii). As I make clear in the main text, the latter proofs are essentially variants or generalizations of the former proofs; they rest on similar arguments and use the same theorems.

A.1 Determination of $\underline{\phi}$ and $\bar{\phi}$ in the basic New Keynesian model under Rule 1

For any $z \in \mathcal{C}$, we have

$$|Q(z)| = \left|\beta - \left(1+\beta+\frac{\kappa}{\sigma}\right)z + z^2\right| \leq \beta + \left(1+\beta+\frac{\kappa}{\sigma}\right)|z| + |z|^2 = 2\,(1+\beta) + \frac{\kappa}{\sigma},$$

with equality only for $z = -1$. Therefore, $\operatorname{argmax}_{z\in\mathcal{C}} |Q(z)| = \{-1\}$, $\max_{z\in\mathcal{C}} |Q(z)| = 2\,(1+\beta) + \kappa/\sigma$, and

$$\bar{\phi} := \frac{\sigma}{\kappa}\max_{z\in\mathcal{C}} |Q(z)| = 1 + 2\,(1+\beta)\,\frac{\sigma}{\kappa}.$$

For any $z = a + ib \in \mathcal{C}$, where $(a,b) \in [-1,1]^2$ and $a^2+b^2 = 1$, some simple algebra leads to $|Q(z)|^2 = T_1(a)$, where

$$T_1(a) := 4\beta a^2 - 2\,(1+\beta)\left(1+\beta+\frac{\kappa}{\sigma}\right)a + \left[(1-\beta)^2 + \left(1+\beta+\frac{\kappa}{\sigma}\right)^2\right].$$

For any $a \in [-1,1]$, we have $T_1'(a) \leq T_1'(1) = -2(1-\beta)^2 - 2(1+\beta)\kappa/\sigma < 0$. So, $T_1(a)$ is decreasing in a over $[-1,1]$. Therefore, $\operatorname{argmin}_{a\in[-1,1]} T_1(a) = \{1\}$, $\operatorname{argmin}_{z\in\mathcal{C}} |Q(z)| = \{1\}$, $\min_{z\in\mathcal{C}} |Q(z)| = \kappa/\sigma$, and

$$\underline{\phi} := \frac{\sigma}{\kappa}\min_{z\in\mathcal{C}} |Q(z)| = 1.$$

A.2 Determination of $\underline{\phi}$ and $\bar{\phi}$ in the basic New Keynesian model under Rule 2

For any $z = a + ib \in \mathcal{C}$, where $(a,b) \in [-1,1]^2$ and $a^2+b^2 = 1$, some simple algebra leads to $|Q(z)/\,(z-\beta)|^2 = T_2(a) := T_1(a)/(1+\beta^2-2\beta a)$, where $T_1(a)$ is defined in Online

Appendix A.1. We have $T_2'(a) = T_3(a)/(1+\beta^2-2\beta a)^2$, where

$$T_3(a) := -8\beta^2 a^2 + 8\beta\left(1+\beta^2\right)a + 2\eta$$

with

$$\eta := \left[(1-\beta)^2 + \left(1+\beta+\frac{\kappa}{\sigma}\right)^2\right]\beta - (1+\beta)\left(1+\beta^2\right)\left(1+\beta+\frac{\kappa}{\sigma}\right).$$

For any $a \in [-1,1]$, we have $T_3'(a) \geq T_3'(1) = 8\beta(1-\beta)^2 > 0$. So, $T_3(a)$ is increasing in a over $[-1,1]$. There are, therefore, three possible alternative cases: (i) $T_3(-1) > 0$, (ii) $T_3(-1) < 0 < T_3(1)$, and (iii) $T_3(1) < 0$. In Case (i), $T_2(a)$ is increasing in a over $[-1,1]$; in Case (ii), $T_2(a)$ is first decreasing and then increasing in a over $[-1,1]$; and in Case (iii), $T_2(a)$ is decreasing in a over $[-1,1]$. In all three cases, $\operatorname{argmax}_{a\in[-1,1]} T_2(a) \subset \{-1,1\}$, hence $\operatorname{argmax}_{z\in\mathcal{C}} |Q(z)/(z-\beta)| \subset \{-1,1\}$, and therefore

$$\bar{\phi} := \sigma \max_{z\in\mathcal{C}}\left|\frac{Q(z)}{z-\beta}\right| = \max\left(\frac{\kappa}{1-\beta}, 2\sigma + \frac{\kappa}{1+\beta}\right).$$

The double inequality $T_3(-1) < 0 < T_3(1)$ is equivalent to

$$\left|\eta - 4\beta^2\right| < 4\beta\left(1+\beta^2\right). \tag{A.1}$$

If Condition (A.1) is not met, then we are in Case (i) or (iii), so $\operatorname{argmin}_{a\in[-1,1]} T_2(a) \subset \{-1,1\}$, and therefore $\operatorname{argmin}_{z\in\mathcal{C}} |Q(z)/(z-\beta)| \subset \{-1,1\}$. Alternatively, if Condition (A.1) is met, then we are in Case (ii), so $\operatorname{argmin}_{a\in[-1,1]} T_2(a) = \{a^*\}$, where $a^* := [(1+\beta^2) - \sqrt{(1+\beta^2)^2+\eta}]/(2\beta)$ is the root of $T_3(a)$ in $[-1,1]$, and therefore $\operatorname{argmin}_{z\in\mathcal{C}} |Q(z)/(z-\beta)| = \{a^* - i\sqrt{1-a^{*2}}, a^* + i\sqrt{1-a^{*2}}\}$. As a consequence,

$$(A.1) \implies \underline{\phi} := \sigma \min_{z\in\mathcal{C}}\left|\frac{Q(z)}{z-\beta}\right| = \frac{\sigma}{\sqrt{\beta}}\sqrt{(1+\beta)\frac{\kappa}{\sigma} - (1-\beta)^2 + 2\sqrt{\left(1+\beta^2\right)^2+\eta}},$$
$$\neg(A.1) \implies \underline{\phi} := \sigma \min_{z\in\mathcal{C}}\left|\frac{Q(z)}{z-\beta}\right| = \min\left(\frac{\kappa}{1-\beta}, 2\sigma + \frac{\kappa}{1+\beta}\right).$$

A.3 Proof of Proposition 3

The equality $\phi_W = \underline{\phi}$ stated in Proposition 3 straightforwardly follows from $\phi_W := 1$ (as explained in the main text) and $\underline{\phi} = 1$ (as shown in Online Appendix A.1).

Points (b) and (d). Point (b) of Proposition 3 straightforwardly follows from the definition of $\mathbb{H}_D$. The "if" part of Point (d) of Proposition 3 is a very well known result whose proof can be found in, e.g., Woodford (2003, Chapter 4). The "only if" part of Point (d) of Proposition 3 straightforwardly follows from Point (b) of Proposition 1.

Point (c). I start by rewriting $P(z)$ as a function of two variables: $\hat{P}(\phi, z) := Q(z)z^{\max(0,h-2)} +(\phi\kappa/\sigma)z^{\max(0,2-h)}$, where $(\phi, z) \in \mathbb{R} \times \mathbb{C}$. Simple algebra leads to $\hat{P}(1, 1) = 0$ and $\partial\hat{P}/\partial z(1, 1) = 1 - \beta + (1 - h)\kappa/\sigma$. This last expression is generically non-zero (it can be zero only if $(1 - \beta)\sigma/\kappa$ is an integer, and I ignore this zero-measure case). So, one root of the polynomial $\hat{P}(1, z)$ is 1, and this root is of multiplicity one. The implicit-function theorem implies the existence of a continuously differentiable function $\phi \mapsto Z(\phi)$ such that one real root of $P(z)$ can be written as $Z(\phi)$ in the neighborhood of $\phi = 1$, with $Z(1) = 1$ and

$$Z'(1) = \frac{-\frac{\partial \hat{P}}{\partial \phi}(1, 1)}{\frac{\partial \hat{P}}{\partial z}(1, 1)} = \frac{1}{h - \left[1 + (1 - \beta)\frac{\sigma}{\kappa}\right]}.$$

For any $h \in \{h \in \mathbb{Z}|h < 1 + (1 - \beta)\sigma/\kappa\}$, we have $Z'(1) < 0$, and therefore the root of $P(z)$ goes from outside to inside $\mathcal{C}$ as ϕ crosses 1 from below. It is the only root that crosses $\mathcal{C}$ as ϕ goes through 1. Indeed, any root $z \in \mathbb{C}$ having this property must satisfy $\hat{P}(1, z) = 0$, which implies $|Q(z)| = \kappa/\sigma$ and hence $z = 1$ (since $\min_{\tilde{z}\in\mathcal{C}} |Q(\tilde{z})| = \kappa/\sigma$ and $\operatorname{argmin}_{\tilde{z}\in\mathcal{C}} |Q(\tilde{z})| = \{1\}$, as shown in Online Appendix A.1). So, the number of roots of $P(z)$ inside $\mathcal{C}$ increases by exactly one as ϕ crosses 1 from below. We know from Subsection 2.2 that this number is $p = \max(1, h - 1) < \nu$ for ϕ just below $\underline{\phi} = 1 = \phi_W$. Therefore, we have $p = \max(2, h) = \nu$ for ϕ just above $\underline{\phi} = 1 = \phi_W$. As a result, the determinacy status moves from M to D as ϕ crosses 1 from below. Thus, the Taylor principle is locally necessary and sufficient for determinacy for any $h \in \{h \in \mathbb{Z}|h < 1 + (1 - \beta)\sigma/\kappa\}$.

Point (a). The result just above straightforwardly implies that $\{h \in \mathbb{Z}|h < 1 + (1 - \beta)\sigma/\kappa\} \subset \mathbb{H}_D$. To prove the reverse inclusion, I first show that $P'(z)$ has a real root higher than 1 for any $h > 1 + (1 - \beta)\sigma/\kappa$. If $h = 2 > 1 + (1 - \beta)\sigma/\kappa$, then $P'(z) = -(1 + \beta + \kappa/\sigma) + 2z$, the unique root of $P'(z)$ is $(1 + \beta + \kappa/\sigma)/2$, and this root is indeed higher than 1. If $h \geq 3$ and $h > 1 + (1 - \beta)\sigma/\kappa$, then $P'(z) = z^{h-3}T_4(z)$, where $T_4(z) := \beta(h - 2) - (1 + \beta + \kappa/\sigma)(h - 1)z + hz^2$; in addition, we have $T_4(1) = -(\kappa/\sigma)h+1-\beta+\kappa/\sigma < 0$ and $\lim_{z\in\mathbb{R},z\to+\infty} T_4(z) = +\infty$; therefore, $T_4(z)$ has a real root above 1, and so has $P'(z)$. I then use the so-called Gauss-Lucas theorem (first proved by Lucas, 1879, but used earlier by Gauss):

Theorem 3 (Gauss-Lucas theorem): *For any non-constant $\tilde{P}(z) \in \mathbb{C}[z]$, all the roots of $\tilde{P}'(z)$ belong to the convex hull of the set of roots of $\tilde{P}(z)$.*

Proof: See Henrici (1988, Pages 463-464) or Marden (1966, Page 22). ■

Applied to $\tilde{P}(z) = P(z)$, this theorem implies that if $P'(z)$ has a real root higher than 1, then $P(z)$ has at least one root outside $\mathcal{C}$. So, for any $h > 1 + (1-\beta)\sigma/\kappa$ and any $\phi \in \mathbb{R}$, $P(z)$ has at least one root outside $\mathcal{C}$, which implies $p < \deg(P) = \max(2, h) = \nu$, and hence $S(\phi, h) = M$.[1] As a result, $\mathbb{H}_D = \{h \in \mathbb{Z} | h < 1 + (1-\beta)\sigma/\kappa\}$.

A.4 Proof of Proposition 4

Points (a) and (c)-(d). Points (a) and (c) of Proposition 4 straightforwardly follow from, respectively, Point (c) of Proposition 2 and the definition of $\mathbb{H}_D$. The "if" part of Point (d) of Proposition 4 is a very well known result whose proof can be found in, e.g., Woodford (2003, Chapter 4). The "only if" part of Point (d) of Proposition 4 straightforwardly follows from Point (b) of Proposition 2.

Point (e). We have $\phi_W := \kappa/(1-\beta)$ and, as shown in Online Appendix A.2, $\bar{\phi} = \max[\kappa/(1-\beta), 2\sigma + \kappa/(1+\beta)]$. So, we get $\phi_W = \bar{\phi}$ if and only if $\kappa/(1-\beta) \geq 2\sigma + \kappa/(1+\beta)$, that is to say if and only if $\kappa/\sigma \geq (1+\beta)(1-\beta)^2/\beta$. This result corresponds to Point (e)(i) of Proposition 4. Points (e)(ii)-(iv) of Proposition 4 straightforwardly follow from Point (b) of Proposition 2.

Point (f). The condition stated in this point is equivalent to the condition $T_3(1) < 0$ in Online Appendix A.2. I have shown in Online Appendix A.2 that $A_{min} := \text{argmin}_{z \in \mathcal{C}} |Q(z) /(z-\beta)| = \{1\}$ and $\underline{\phi} = \kappa/(1-\beta)$ if this condition is met, and that $1 \notin A_{min}$ and $\underline{\phi} \neq \kappa/(1-\beta)$ if this condition is not met. This result, together with $\phi_W := \kappa/(1-\beta)$, corresponds to Point (f)(i) of Proposition 4.

The proof of Point (f)(ii) of Proposition 4 is similar to the proof of Point (c) of Proposition 3 in Online Appendix A.3. I rewrite again $P(z)$ as a function of two variables: $\hat{P}(\phi, z) := Q(z) z^{\max(0, h-1)} + (\phi/\sigma)(z-\beta) z^{\max(0, 1-h)}$, where $(\phi, z) \in \mathbb{R} \times \mathbb{C}$. Simple algebra leads to $\hat{P}(\phi_W, 1) = 0$ and $\partial\hat{P}/\partial z(\phi_W, 1) = 1 - \beta + [1/(1-\beta) - h]\kappa/\sigma$. This last expression is generically non-zero (it can be zero only if $1/(1-\beta) + (1-\beta)\sigma/\kappa$ is an integer, and I ignore this zero-measure case). So, one root of the polynomial $\hat{P}(\phi_W, z)$ is 1, and this root is of multiplicity one. The implicit-function theorem implies the existence of a continuously differentiable function $\phi \mapsto Z(\phi)$ such that one real root of $P(z)$ can be written as $Z(\phi)$ in the neighborhood of $\phi = \phi_W$, with $Z(\phi_W) = 1$ and

$$Z'(\phi_W) = \frac{-\frac{\partial \hat{P}}{\partial \phi}(\phi_W, 1)}{\frac{\partial \hat{P}}{\partial z}(\phi_W, 1)} = \frac{\frac{1-\beta}{\kappa}}{h - \left[\frac{1}{1-\beta} + (1-\beta)\frac{\sigma}{\kappa}\right]}.$$

[1] Throughout the Online Appendix, for any $T(z) \in \mathbb{R}[z]$, $\deg(T)$ denotes the degree of $T(z)$.

This root of $P(z)$ crosses $\mathcal{C}$ at point 1 as ϕ goes through ϕ_W. It is the only root that crosses $\mathcal{C}$ as ϕ goes through ϕ_W. Indeed, any root $z \in \mathbb{C}$ having this property must satisfy $\hat{P}(\phi_W, z) = 0$, which implies $|Q(z)/(z-\beta)| = \kappa/[(1-\beta)\sigma]$ and hence $z = 1$ (since $\min_{\tilde{z}\in\mathcal{C}} |Q(\tilde{z})/(\tilde{z}-\beta)| = \kappa/[(1-\beta)\sigma]$ and $\text{argmin}_{\tilde{z}\in\mathcal{C}} |Q(\tilde{z})/(\tilde{z}-\beta)| = \{1\}$, as follows from the analysis in Online Appendix A.2).

For any $h < 1/(1-\beta) + (1-\beta)\sigma/\kappa$, we have $Z'(\phi_W) < 0$, and therefore the root of $P(z)$ goes from outside to inside $\mathcal{C}$ as ϕ crosses ϕ_W from below. So, the number of roots of $P(z)$ inside $\mathcal{C}$ increases by exactly one, from $p = \max(1, h-1) < \nu$ to $p = \max(2, h) = \nu$, and the determinacy status moves from M to D, as ϕ crosses ϕ_W from below. Thus, the Taylor principle is locally necessary and sufficient for determinacy for any $h < 1/(1-\beta)+(1-\beta)\sigma/\kappa$. Alternatively, for $h > 1/(1-\beta)+(1-\beta)\sigma/\kappa$, we have $Z'(\phi_W) > 0$; as ϕ crosses ϕ_W from below, therefore, the root of $P(z)$ goes this time from inside to outside $\mathcal{C}$, and the determinacy status remains M. Thus, the Taylor principle is not locally necessary and sufficient for determinacy for any $h > 1/(1-\beta) + (1-\beta)\sigma/\kappa$.

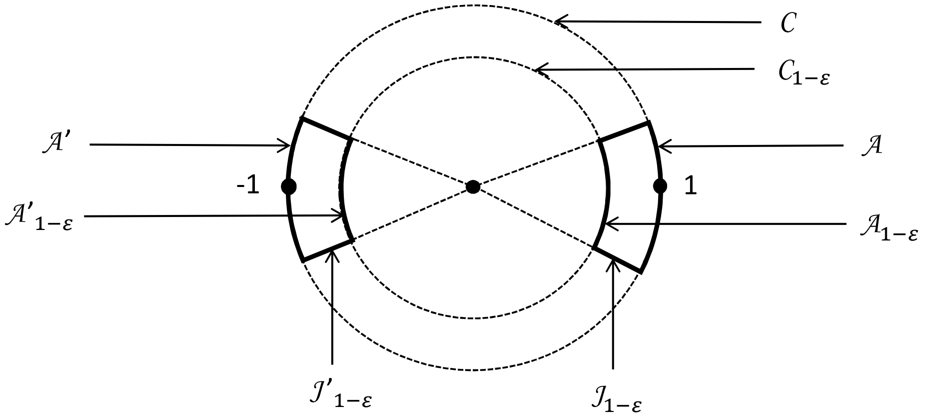
Point (b). The condition stated in this point is the same as Condition (A.1) in Online Appendix A.2. I have shown in Online Appendix A.2 that $A_{min} \subset \mathcal{C} \setminus \{-1, 1\}$ if this condition is met, and that $A_{min} \subset \{-1, 1\}$ if this condition is not met.

Suppose first that this condition is met, and hence that $A_{min} \subset \mathcal{C} \setminus \{-1, 1\}$. Then, $|Q(z)| > (\underline{\phi}/\sigma)\,|z-\beta|$ for $z \in \{-1, 1\}$. So, by continuity, there exist $\epsilon \in (0, \bar{\phi} - \underline{\phi})$ and two open arcs $\mathcal{A}$ and $\mathcal{A}'$ of $\mathcal{C}$ such that: (i) $1 \in \mathcal{A}$, (ii) $-1 \in \mathcal{A}'$, and (iii) $\forall\, |\phi| \in (\underline{\phi}, \underline{\phi}+\epsilon)$, $\forall z \in \mathcal{A}\cup\mathcal{A}'$, $|Q(z)| > (|\phi|/\sigma)\,|z-\beta|$. For any $r \in \mathbb{R}_+$, let $\mathcal{A}_r$ and $\mathcal{A}'_r$ denote respectively the images of $\mathcal{A}$ and $\mathcal{A}'$ under the homothety whose center is the origin of the complex plane and whose ratio is r (so that in particular $\mathcal{A}_1 = \mathcal{A}$ and $\mathcal{A}'_1 = \mathcal{A}'$). In addition, for any $r \in \mathbb{R}_+ \setminus \{0\}$, let $\mathcal{J}_r$ (resp. $\mathcal{J}'_r$) denote the Jordan curve made of $\mathcal{A}$ (resp. $\mathcal{A}'$), $\mathcal{A}_r$ (resp. $\mathcal{A}'_r$), and the two radial line segments joining the endpoints of $\mathcal{A}$ (resp. $\mathcal{A}'$) and $\mathcal{A}_r$ (resp. $\mathcal{A}'_r$). By continuity, there exists $\varepsilon \in (0, 1-z_i)$ such that $\forall\, |\phi| \in (\underline{\phi}, \underline{\phi}+\epsilon)$, $\forall h \leq 1$, $\forall z \in \mathcal{J}_{1-\varepsilon} \cup \mathcal{J}'_{1-\varepsilon}$ (see Figure A.1),

$$|Q(z)| > \frac{|\phi|}{\sigma}\,|z-\beta| \geq \left|\frac{\phi}{\sigma}\,(z-\beta)\,z^{1-h}\right|.$$

Applying Rouché's theorem to $P_b(z) = Q(z)$, $P_s(z) = (\phi/\sigma)(z-\beta)z^{1-h}$, and (alternatively) $\mathcal{J}_{1-\varepsilon}$ and $\mathcal{J}'_{1-\varepsilon}$, I obtain that $\forall\, |\phi| \in (\underline{\phi}, \underline{\phi}+\epsilon)$, $\forall h \leq 1$, $P(z)$ has no roots inside $\mathcal{J}_{1-\varepsilon}$ and no roots inside $\mathcal{J}'_{1-\varepsilon}$. Therefore, $P(z)$ has no real roots between $\mathcal{C}_{1-\varepsilon}$ and $\mathcal{C}$. Now, we know from the Appendix of the paper that there exists $\underline{h}_{1-\varepsilon} \in \mathbb{Z}$ such that $\forall\, |\phi| \in (\underline{\phi}, \bar{\phi})$, $\forall h \leq \underline{h}_{1-\varepsilon}$, $P(z)$ has exactly one root inside $\mathcal{C}_{1-\varepsilon}$. As a result,

Figure A.1: Jordan curves $\mathcal{J}_{1-\varepsilon}$ and $\mathcal{J}'_{1-\varepsilon}$



$\forall\,|\phi| \in (\underline{\phi}, \underline{\phi}+\epsilon)$, $\forall h \leq \min(1, \underline{h}_{1-\varepsilon})$, $P(z)$ has exactly one real root inside $\mathcal{C}$, and therefore an odd number p of roots inside $\mathcal{C}$. Since there are $\nu = 2$ non-predetermined variables for any $h \leq 1$, we have $p \neq \nu$, and hence $S(\phi, h) \neq D$, for all $|\phi| \in (\underline{\phi}, \underline{\phi}+\epsilon)$ and all $h \leq \min(1, \underline{h}_{1-\varepsilon})$. Together with Point (d) of Proposition 2, this result implies that $\mathbb{H}_D$ is bounded below.

Now suppose alternatively that the condition stated in Point (b) of Proposition 4 is not met, and hence that either $\eta - 4\beta^2 < -4\beta\,(1+\beta^2)$, or $\eta - 4\beta^2 > 4\beta\,(1+\beta^2)$. In the first case, Point (f) of Proposition 4 implies that $\mathbb{H}_D$ is unbounded below. In the second case, we have $A_{min} = \{-1\}$ and $\underline{\phi} = 2\sigma + \kappa/(1+\beta)$ (as follows from the analysis in Online Appendix A.2). For any negative and odd integer h, simple algebra leads to $\hat{P}(\underline{\phi}, -1) = 0$ and $\partial\hat{P}/\partial z(\underline{\phi}, -1) = 1 + \beta + \kappa/[(1+\beta)\sigma] - [2(1+\beta) + \kappa/\sigma]h$. This last expression is generically non-zero. So, one root of the polynomial $\hat{P}(\underline{\phi}, z)$ is -1, and this root is of multiplicity one. The implicit-function theorem implies the existence of a continuously differentiable function $\phi \mapsto Z(\phi)$ such that one real root of $P(z)$ can be written as $Z(\phi)$ in the neighborhood of $\phi = \underline{\phi}$, with $Z(\underline{\phi}) = -1$ and

$$Z'(\underline{\phi}) = \frac{-\frac{\partial\hat{P}}{\partial\phi}(\underline{\phi}, -1)}{\frac{\partial\hat{P}}{\partial z}(\underline{\phi}, -1)} = \frac{\frac{1+\beta}{\sigma}}{1+\beta+\frac{\kappa}{(1+\beta)\sigma} - \left[2(1+\beta)+\frac{\kappa}{\sigma}\right]h}.$$

For any negative and odd integer h, we have $Z'(\underline{\phi}) > 0$, and therefore the root of $P(z)$ goes from outside to inside $\mathcal{C}$ as ϕ crosses $\underline{\phi}$ from below. It is the only root that crosses $\mathcal{C}$ as ϕ goes through $\underline{\phi}$. Indeed, any root $z \in \mathbb{C}$ having this property must satisfy $\hat{P}(\underline{\phi}, z) = 0$, which implies $|Q(z)/(z-\beta)| = 2 + \kappa/[(1+\beta)\sigma]$ and hence $z = -1$ (since $\min_{\tilde{z}\in\mathcal{C}} |Q(\tilde{z})/(\tilde{z}-\beta)| = 2+\kappa/[(1+\beta)\sigma]$ and $\operatorname{argmin}_{\tilde{z}\in\mathcal{C}} |Q(\tilde{z})/(\tilde{z}-\beta)| = \{-1\}$, as follows from the analysis in Online Appendix A.2). So, the number of roots of $P(z)$ inside $\mathcal{C}$ increases by exactly one, from $p = 1 < \nu$ to $p = 2 = \nu$, and the determinacy status moves from M to D, as ϕ crosses ϕ_W from below. For any negative and odd integer h, thus, we get determinacy for ϕ just above $\underline{\phi}$. As a consequence, $\mathbb{H}_D$ is unbounded below.

A.5 Proof of Lemma 1

I start with the case of a peg ($\phi = 0$). In this case, the dynamic system boils down to $\mathbb{E}_t\{\mathbf{\Delta}(L^{-1})\mathbf{A}(L)\mathbf{X}_t\} = \mathbf{0}$. The characteristic polynomial of this system is the same as the characteristic polynomial of the corresponding perfect-foresight system. The latter system is $\mathbf{A}(L)\mathbf{X}_t = \mathbf{0}$. Since $\det[\mathbf{A}(0)] \neq 0$, using a standard result in time-series analysis (see, e.g., Hamilton, 1994, Proposition 10.1, Page 259), I get that $P(z)$, the reciprocal polynomial of this characteristic polynomial, is equal to $Q(z) := \det[\mathbf{A}(z)]$.

Since $\det[\mathbf{A}(0)] \neq 0$, the dynamic system can be rewritten as $\mathbb{E}_t\{\mathbf{\Delta}(L^{-1})\tilde{\mathbf{A}}(L)\tilde{\mathbf{X}}_t\} = \mathbf{0}$, where $\tilde{\mathbf{A}}(z) := \mathbf{A}(z)[\mathbf{A}(0)]^{-1}$ and $\tilde{\mathbf{X}}_t := \mathbf{A}(0)\mathbf{X}_t$. Let $\tilde{X}_{j,t}$ denote the j^{th} element of $\tilde{\mathbf{X}}_t$ for $j \in \{1, ..., n\}$. The non-predetermined variables of the system are the variables $\mathbb{E}_t\{\tilde{X}_{j,t+k_j}\}$ for all $j \in \{1, ..., n\}$ such that $\delta_j \geq 1$ and all $k_j \in \{0, ..., \delta_j - 1\}$. Their number, ν, is equal to $\delta := \sum_{j=1}^{n} \delta_j$.

I now turn to the case in which $\phi \neq 0$. In this case, the characteristic polynomial of the dynamic system is still the same as the characteristic polynomial of the corresponding perfect-foresight system, but the latter system is now

$$\begin{bmatrix} \mathbf{A}(L) & L^{-\gamma}\mathbf{B}(L) \\ -\phi L^{-h}\mathbf{V}(L) & 1 \end{bmatrix} \begin{bmatrix} \mathbf{X}_t \\ i_t \end{bmatrix} = \mathbf{0}.$$

Except possibly for a zero-measure set of ϕ values, I can use the same standard result in time-series analysis as above. I get that there exists $k \in \mathbb{Z}$ such that $P(z)$, the reciprocal polynomial of the characteristic polynomial, is

$$P(z) = z^k \det \begin{bmatrix} \mathbf{A}(z) & z^{-\gamma}\mathbf{B}(z) \\ -\phi z^{-h}\mathbf{V}(z) & 1 \end{bmatrix}.$$

Using the Laplace expansion and the notations introduced in the main text, I rewrite $P(z)$ as $P(z) = z^k\{\det[\mathbf{A}(z)] - \phi z^{-\gamma-h}W(z)\} = z^k[Q(z) + \phi z^{m-h}R(z)]$. As a reciprocal polynomial, $P(z)$ is such that $P(0) \neq 0$; moreover, we have $Q(0) \neq 0$ and $R(0) \neq 0$; as a consequence, we get $k = \max(0, h - m)$, and thus $P(z) = Q(z)z^{\max(0,h-m)} + \phi R(z)z^{\max(0,m-h)}$.

The number of non-predetermined variables, ν, is equal to δ when h is lower than or equal to a certain threshold, and it increases one-for-one with h when h is higher than this threshold. This threshold is equal to the highest value of h for which $P(0)$ depends on $Q(0)$, i.e. for which the most forward variable in the dynamic system is the same as under a peg (except in the zero-measure case where $\phi = -Q(0)/R(0)$). This value is m, and thus $\nu = \delta + \max(0, h - m)$.

A.6 Proof of Proposition 5

The proof of Proposition 5 is essentially a generalization of the proof of Proposition 1, using this time $P(z) = Q(z)z^{\max(0,h-m)} + \phi R(z)z^{\max(0,m-h)}$ and $\nu = \delta + \max(0, h - m)$ (as stated in Lemma 1).

Point (a). I apply Rouché's theorem to $\mathcal{J} = \mathcal{C}$, $P_b(z) = Q(z)z^{\max(0,h-m)}$, and $P_s(z) = \phi R(z)z^{\max(0,m-h)}$. For any $|\phi| < \underline{\phi}$ and any $z \in \mathcal{C}$, we have

$$\left|Q(z)z^{\max(0,h-m)}\right| = |Q(z)| \geq \min_{\tilde{z}\in\mathcal{C}} \left|\frac{Q(\tilde{z})}{R(\tilde{z})}\right| |R(z)| = \underline{\phi}\,|R(z)| > |\phi R(z)| = \left|\phi R(z)z^{\max(0,m-h)}\right|.$$

So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{C}$ as $Q(z)z^{\max(0,h-m)}$, i.e. that $p = q + \max(0, h - m)$. Since $\nu = \delta + \max(0, h - m)$, we get $\nu - p = \delta - q = d_{peg}$, and hence $S(\phi, h) = S_{peg}$, for any $|\phi| < \underline{\phi}$ and any $h \in \mathbb{Z}$.

Point (b). I apply Rouché's theorem to $\mathcal{J} = \mathcal{C}$, $P_b(z) = \phi R(z)z^{\max(0,m-h)}$, and $P_s(z) = Q(z)z^{\max(0,h-m)}$. For any $|\phi| > \bar{\phi}$ and any $z \in \mathcal{C}$, we have

$$\left|\phi R(z)z^{\max(0,m-h)}\right| = |\phi R(z)| > \bar{\phi}\,|R(z)| = \max_{\tilde{z}\in\mathcal{C}} \left|\frac{Q(\tilde{z})}{R(\tilde{z})}\right| |R(z)| \geq |Q(z)| = \left|Q(z)z^{\max(0,h-m)}\right|.$$

So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{C}$ as $\phi R(z)z^{\max(0,m-h)}$, i.e. that $p = r + \max(0, m - h)$. Since $\nu = \delta + \max(0, h - m)$, we get, for any $|\phi| > \bar{\phi}$: (i) if $h \leq h^* - 1$, then $p > \nu$ and $S(\phi, h) = E$; (ii) if $h = h^*$, then $p = \nu$ and $S(\phi, h) = D$; and (iii) if $h \geq h^* + 1$, then $p < \nu$ and $S(\phi, h) = M$.

Points (d)(i) and (d)(ii). For $h \leq m$, we have $\nu = \delta$ and $P(z) = Q(z) + \phi R(z)z^{m-h}$. I proceed in four steps.

In the first step, I show that for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$, all but $q + \deg(R) - r$ roots of $P(z)$ converge uniformly to $\mathcal{C}$ as $h \to -\infty$. I get this result by applying Rouché's theorem twice. Since $q_{\mathcal{C}} = r_{\mathcal{C}} = 0$, I can consider an arbitrary $\epsilon \in (0, 1)$ such that neither $Q(z)$ nor $R(z)$ has any root inside the annulus whose borders are $\mathcal{C}_{1-\epsilon}$ and $\mathcal{C}_{1+\epsilon}$ (where again, for any $r \in \mathbb{R}_+$, $\mathcal{C}_r$ denotes the circle of radius r centered at the origin of the complex plane).

I first apply Rouché's theorem to $\mathcal{J} = \mathcal{C}_{1-\epsilon}$, $P_b(z) = Q(z)$, and $P_s(z) = \phi R(z)z^{m-h}$. For any $|\phi| \in (\underline{\phi}, \bar{\phi})$, any

$$h \leq \underline{h}_{1-\epsilon} := m + \min \left\{ 0, \left\lfloor \frac{\log \left(\min_{\tilde{z} \in \mathcal{C}_{1-\epsilon}} |Q(\tilde{z})| \right) - \log \left(\bar{\phi} \max_{\tilde{z} \in \mathcal{C}_{1-\epsilon}} |R(\tilde{z})| \right)}{-\log(1-\epsilon)} \right\rfloor \right\},$$

and any $z \in \mathcal{C}_{1-\epsilon}$, we have

$$|Q(z)| \geq \min_{\tilde{z} \in \mathcal{C}_{1-\epsilon}} |Q(\tilde{z})| \geq \bar{\phi} \max_{\tilde{z} \in \mathcal{C}_{1-\epsilon}} |R(\tilde{z})| (1-\epsilon)^{m-h} \geq \bar{\phi} \left| R(z) z^{m-h} \right| > \left| \phi R(z) z^{m-h} \right|,$$

where the second inequality follows from the definition of $\underline{h}_{1-\epsilon}$. So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{C}_{1-\epsilon}$ as $Q(z)$. Therefore, $P(z)$ has also exactly q roots inside $\mathcal{C}_{1-\epsilon}$ for any $|\phi| \in (\underline{\phi}, \bar{\phi})$ and any $h \leq \underline{h}_{1-\epsilon}$.

I then apply Rouché's theorem to $\mathcal{J} = \mathcal{C}_{1+\epsilon}$, $P_b(z) = \phi R(z) z^{m-h}$, and $P_s(z) = Q(z)$. For any $|\phi| \in (\underline{\phi}, \bar{\phi})$, any

$$h \leq \underline{h}_{1+\epsilon} := m + \min \left\{ 0, \left\lfloor \frac{\log \left(\underline{\phi} \min_{\tilde{z} \in \mathcal{C}_{1+\epsilon}} |R(\tilde{z})| \right) - \log \left(\max_{\tilde{z} \in \mathcal{C}_{1+\epsilon}} |Q(\tilde{z})| \right)}{\log(1+\epsilon)} \right\rfloor \right\},$$

and any $z \in \mathcal{C}_{1+\epsilon}$, we have

$$\left| \phi R(z) z^{m-h} \right| = |\phi R(z)| (1+\epsilon)^{m-h} > \underline{\phi} \min_{\tilde{z} \in \mathcal{C}_{1+\epsilon}} |R(\tilde{z})| (1+\epsilon)^{m-h} \geq \max_{\tilde{z} \in \mathcal{C}_{1+\epsilon}} |Q(\tilde{z})| \geq |Q(z)|,$$

where the last but one inequality follows from the definition of $\underline{h}_{1+\epsilon}$. So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{C}_{1+\epsilon}$ as $\phi R(z) z^{m-h}$. Therefore, $P(z)$ has exactly $r + m - h$ roots inside $\mathcal{C}_{1+\epsilon}$ for any $h \leq \underline{h}_{1+\epsilon}$. As a consequence, for any $|\phi| \in (\underline{\phi}, \bar{\phi})$ and any $h \leq \min \left(\underline{h}_{1-\epsilon}, \underline{h}_{1+\epsilon} \right)$, $P(z)$ has exactly $r + m - h - q$ roots inside the annulus whose borders are $\mathcal{C}_{1-\epsilon}$ and $\mathcal{C}_{1+\epsilon}$. Now, the degree of $P(z)$ is $\deg(R) + m - h$ when $h \leq m + \deg(R) - \deg(Q)$. So, we eventually get that for any $|\phi| \in (\underline{\phi}, \bar{\phi})$ and any $h \leq \min \left[\underline{h}_{1-\epsilon}, \underline{h}_{1+\epsilon}, m + \deg(R) - \deg(Q) \right]$, all but $q + \deg(R) - r$ roots of $P(z)$ lie between $\mathcal{C}_{1-\epsilon}$ and $\mathcal{C}_{1+\epsilon}$. We conclude that for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$, all but $q + \deg(R) - r$ roots of $P(z)$ converge uniformly to $\mathcal{C}$ as $h \to -\infty$.

In the second step, I show that for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$, the roots of $P(z)$ uniformly converging to $\mathcal{C}$ as $h \to -\infty$ converge in distribution to the uniform distribution on $\mathcal{C}$. This result is a direct consequence of the Erdős-Turán theorem (stated in the Appendix of the paper). Applying this theorem to $\tilde{P}(z) = P(z)$, and using the result of the previous step, I thus get that all but $q + \deg(R) - r$ roots of $P(z)$ uniformly converge in distribution to the uniform distribution on $\mathcal{C}$ as $h \to -\infty$, for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$.

In the third step, I show that the number of roots of $P(z)$ inside $\mathcal{C}$ grows unboundedly as $h \to -\infty$, for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$. Since $|\phi| > \underline{\phi}$, there exists an arc $\mathcal{A}$ of $\mathcal{C}$ such that $\forall z \in \mathcal{A}$, $|\phi R(z)| > |Q(z)|$. For any $r \in \mathbb{R}_+$, let $\mathcal{A}_r$ denote the image of $\mathcal{A}$ under the homothety whose center is the origin of the complex plane and whose ratio is r (so that in particular $\mathcal{A}_1 = \mathcal{A}$). By continuity, there exists $\varepsilon \in (0,1)$ such that $|\phi R(z)| > |Q(z)|$ for all z on the Jordan curve $\mathcal{J}_{1+\varepsilon}$ made of $\mathcal{A}$, $\mathcal{A}_{1+\varepsilon}$, and the two radial line segments joining the endpoints of $\mathcal{A}$ and $\mathcal{A}_{1+\varepsilon}$ (see Figure 5 in the Appendix of the paper). I apply Rouché's theorem to $\mathcal{J} = \mathcal{J}_{1+\varepsilon}$, $P_b(z) = \phi R(z) z^{m-h}$, and $P_s(z) = Q(z)$. For any $h \leq m$ and any $z \in \mathcal{J}_{1+\varepsilon}$, we have

$$\left|\phi R(z) z^{m-h}\right| \geq |\phi R(z)| > |Q(z)|.$$

So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{J}_{1+\varepsilon}$ as $\phi R(z) z^{m-h}$. Therefore, $P(z)$ has at most $\deg(R)$ roots inside $\mathcal{J}_{1+\varepsilon}$ for any $h \leq m$. (Figure 5 in the Appendix of the paper represents the case in which $P(z)$ has no roots inside $\mathcal{J}_{1+\varepsilon}$; we necessarily get this case if ε is sufficiently small.) Using the results of the first two steps, we get that the number of roots of $P(z)$ inside the Jordan curve $\mathcal{J}_{1-\varepsilon}$ made of $\mathcal{A}_{1-\varepsilon}$, $\mathcal{A}$, and the two radial line segments joining the endpoints of $\mathcal{A}_{1-\varepsilon}$ and $\mathcal{A}$, grows unboundedly as $h \to -\infty$. As a result, p grows unboundedly as $h \to -\infty$. Thus, there exists $\underline{h}(|\phi|)$ such that $p > \delta = \nu$ and $S(\phi, h) = E$ for all $h \leq \underline{h}(|\phi|)$.

In the fourth step, I just note that for any $\varepsilon \in \left(0, \bar{\phi} - \underline{\phi}\right)$ and any $|\phi| \in (\underline{\phi} + \varepsilon, \bar{\phi})$, there exists, by continuity, $\ell(\varepsilon) > 0$ such that the arc $\mathcal{A}$ can be chosen of length higher than $\ell(\varepsilon)$. As a result, $\underline{h}(|\phi|)$ can be chosen a bounded function of $|\phi|$ for $|\phi| \in (\underline{\phi} + \varepsilon, \bar{\phi})$.

Points (c)(i) and (c)(ii). For $h \geq m$, we have $\nu = \delta + h - m$ and $P(z) = Q(z) z^{h-m} + \phi R(z)$. I follow the same four steps as in the proof of Points (d)(i) and (d)(ii) above, with some variants.

In the first step, I show that for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$, all but $r + \deg(Q) - q$ roots of $P(z)$ converge uniformly to $\mathcal{C}$ as $h \to +\infty$. I get this result by applying Rouché's theorem twice. Since $q_{\mathcal{C}} = r_{\mathcal{C}} = 0$, I can consider an arbitrary $\epsilon \in (0,1)$ such that neither $Q(z)$ nor $R(z)$ has any root inside the annulus whose borders are $\mathcal{C}_{1-\epsilon}$ and $\mathcal{C}_{1+\epsilon}$.

I first apply Rouché's theorem to $\mathcal{J} = \mathcal{C}_{1-\epsilon}$, $P_b(z) = \phi R(z)$, and $P_s(z) = Q(z) z^{h-m}$. For any $|\phi| \in (\underline{\phi}, \bar{\phi})$, any

$$h \geq \bar{h}_{1-\epsilon} := m + \max\left\{0, \left\lceil \frac{\log\left(\max_{\tilde{z} \in \mathcal{C}_{1-\epsilon}} |Q(\tilde{z})|\right) - \log\left(\underline{\phi} \min_{\tilde{z} \in \mathcal{C}_{1-\epsilon}} |R(\tilde{z})|\right)}{-\log(1-\epsilon)} \right\rceil \right\},$$

and any $z \in \mathcal{C}_{1-\epsilon}$, we have

$$|\phi R(z)| > \underline{\phi} \min_{\tilde{z} \in \mathcal{C}_{1-\epsilon}} |R(\tilde{z})| \geq \max_{\tilde{z} \in \mathcal{C}_{1-\epsilon}} |Q(\tilde{z})| (1-\epsilon)^{h-m} \geq \left|Q(z) z^{h-m}\right|,$$

where the second inequality follows from the definition of $\bar{h}_{1-\epsilon}$. So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{C}_{1-\epsilon}$ as $\phi R(z)$. Therefore, $P(z)$ has also exactly r roots inside $\mathcal{C}_{1-\epsilon}$ for any $|\phi| \in (\underline{\phi}, \bar{\phi})$ and any $h \geq \bar{h}_{1-\epsilon}$.

I then apply Rouché's theorem to $\mathcal{J} = \mathcal{C}_{1+\epsilon}$, $P_b(z) = Q(z)z^{h-m}$, and $P_s(z) = \phi R(z)$. For any $|\phi| \in (\underline{\phi}, \bar{\phi})$, any

$$h \geq \bar{h}_{1+\epsilon} := m + \max\left\{0, \left\lceil \frac{\log\left(\bar{\phi}\max_{\tilde{z}\in\mathcal{C}_{1+\epsilon}}|R(\tilde{z})|\right) - \log\left(\min_{\tilde{z}\in\mathcal{C}_{1+\epsilon}}|Q(\tilde{z})|\right)}{\log(1+\epsilon)}\right\rceil\right\},$$

and any $z \in \mathcal{C}_{1+\epsilon}$, we have

$$\left|Q(z)z^{h-m}\right| = |Q(z)|\,(1+\epsilon)^{h-m} \geq \min_{\tilde{z}\in\mathcal{C}_{1+\epsilon}}|Q(\tilde{z})|\,(1+\epsilon)^{h-m} \geq \bar{\phi}\max_{\tilde{z}\in\mathcal{C}_{1+\epsilon}}|R(\tilde{z})| > |\phi R(z)|,$$

where the last but one inequality follows from the definition of $\bar{h}_{1+\epsilon}$. So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{C}_{1+\epsilon}$ as $Q(z)z^{h-m}$. Therefore, $P(z)$ has exactly $q+h-m$ roots inside $\mathcal{C}_{1+\epsilon}$ for any $h \geq \bar{h}_{1+\epsilon}$. As a consequence, for any $|\phi| \in (\underline{\phi}, \bar{\phi})$ and any $h \geq \max\left(\bar{h}_{1-\epsilon}, \bar{h}_{1+\epsilon}\right)$, $P(z)$ has exactly $q+h-m-r$ roots inside the annulus whose borders are $\mathcal{C}_{1-\epsilon}$ and $\mathcal{C}_{1+\epsilon}$. Now, the degree of $P(z)$ is $\deg(Q)+h-m$ when $h \geq m + \deg(R) - \deg(Q)$. So, we eventually get that for any $|\phi| \in (\underline{\phi}, \bar{\phi})$ and any $h \geq \max\left[\bar{h}_{1-\epsilon}, \bar{h}_{1+\epsilon}, m+\deg(R)-\deg(Q)\right]$, all but $r+\deg(Q)-q$ roots of $P(z)$ lie between $\mathcal{C}_{1-\epsilon}$ and $\mathcal{C}_{1+\epsilon}$. We conclude that for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$, all but $r+\deg(Q)-q$ roots of $P(z)$ converge uniformly to $\mathcal{C}$ as $h \to +\infty$.

In the second step, I show that for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$, the roots of $P(z)$ uniformly converging to $\mathcal{C}$ as $h \to +\infty$ converge in distribution to the uniform distribution on $\mathcal{C}$. This result is, again, a direct consequence of the Erdős-Turán theorem (stated in the Appendix of the paper). Applying this theorem to $\tilde{P}(z) = P(z)$, and using the result of the previous step, I thus get that all but $r+\deg(Q)-q$ roots of $P(z)$ uniformly converge in distribution to the uniform distribution on $\mathcal{C}$ as $h \to +\infty$, for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$.

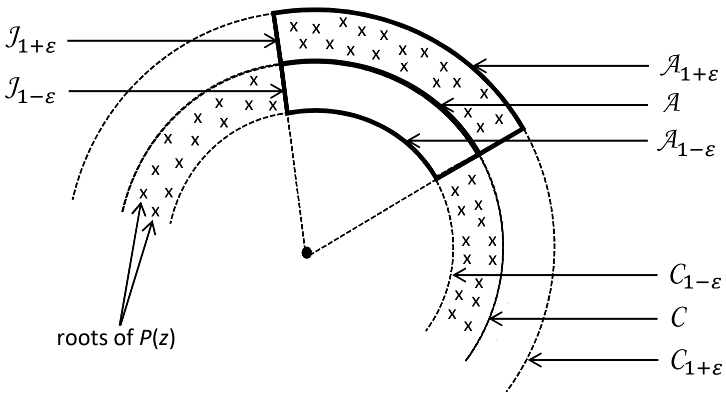
In the third step, I show that the ratio p/ν is lower than 1 as $h \to +\infty$, for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$. Since $|\phi| > \underline{\phi}$, there exists an arc $\mathcal{A}$ of $\mathcal{C}$ such that $\forall z \in \mathcal{A}$, $|\phi R(z)| > |Q(z)|$. By continuity, there exists $\varepsilon \in (0,1)$ such that $|\phi R(z)| > |Q(z)|$ for all z on the Jordan curve $\mathcal{J}_{1-\varepsilon}$ (defined above and represented in Figure A.2). I apply Rouché's theorem to $\mathcal{J} = \mathcal{J}_{1-\varepsilon}$, $P_b(z) = \phi R(z)$, and $P_s(z) = Q(z)z^{h-m}$. For any $h \geq m$ and any $z \in \mathcal{J}_{1-\varepsilon}$, we have

$$|\phi R(z)| > |Q(z)| \geq \left|Q(z)z^{h-m}\right|.$$

So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{J}_{1-\varepsilon}$ as $\phi R(z)$. Therefore, $P(z)$ has at most $\deg(R)$ roots inside $\mathcal{J}_{1-\varepsilon}$ for any $h \geq m$. (Figure

A.2 represents the case in which $P(z)$ has no roots inside $\mathcal{J}_{1-\varepsilon}$; we necessarily get this case if ε is sufficiently small.)

Figure A.2: Roots of $P(z)$ as $h \to +\infty$



Using the results of the first two steps, we get that the ratio of the number of roots of $P(z)$ inside the Jordan curve $\mathcal{J}_{1+\varepsilon}$ (defined above and represented in Figure A.2) to the total number of roots of $P(z)$ converges to $\ell(\mathcal{A})/(2\pi)$ as $h \to +\infty$, where again $\ell(.)$ denotes the standard length operator (i.e., the Lebesgue measure on $\mathcal{C}$). So, as $h \to +\infty$, the ratio of the number of roots of $P(z)$ outside $\mathcal{C}$ to the total number of roots of $P(z)$ is bounded away from 0; or, equivalently, the ratio of the number of roots of $P(z)$ inside $\mathcal{C}$ to the total number of roots of $P(z)$, i.e. the ratio $p/\deg(P)$, is bounded away from 1. Since the ratio of the number of non-predetermined variables to the total number of roots of $P(z)$, i.e. the ratio $\nu/\deg(P)$, converges to 1 as $h \to +\infty$ (given that both ν and $\deg(P)$ increase one-for-one with h), we eventually get that the ratio p/ν is lower than 1 as $h \to +\infty$. Thus, for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$, there exists $\bar{h}\left(|\phi|\right)$ such that $p < \nu$ and $S(\phi, h) = M$ for all $h \geq \bar{h}\left(|\phi|\right)$.

In the fourth step, I just note that for any $\varepsilon \in \left(0, \bar{\phi} - \underline{\phi}\right)$ and any $|\phi| \in (\underline{\phi} + \varepsilon, \bar{\phi})$, there exists, by continuity, $\ell(\varepsilon) > 0$ such that the arc $\mathcal{A}$ can be chosen of length higher than $\ell(\varepsilon)$. As a result, $\bar{h}\left(|\phi|\right)$ can be chosen a bounded function of $|\phi|$ for $|\phi| \in (\underline{\phi} + \varepsilon, \bar{\phi})$.

Point (c)(iii). For $h \geq m + \max[0, \deg(R) - \deg(Q)]$, we have $\nu = \delta + h - m$, $P(z) = Q(z)z^{h-m} + \phi R(z)$, and $\deg(P) = \deg(Q) + h - m$. Consider a Jordan curve $\mathcal{J}_o$ (where the subscript "o" stands for "outside $\mathcal{C}$") that: (i) lies entirely outside $\mathcal{C}$, (ii) surrounds the $\deg(Q) - q$ roots of $Q(z)$ outside $\mathcal{C}$ (if $\deg(Q) - q \geq 1$), and (iii) does not surround $\mathcal{C}$. I apply Rouché's theorem to $\mathcal{J} = \mathcal{J}_o$, $P_b(z) = Q(z)z^{h-m}$, and $P_s(z) = \phi R(z)$. For any

$|\phi| \in (\underline{\phi}, \bar{\phi})$, any

$$h \geq \bar{h} := m + \max\left\{0, \deg(R) - \deg(Q), \left\lceil \frac{\log\left(\bar{\phi} \max_{\tilde{z} \in \mathcal{J}_o} |R(\tilde{z})|\right) - \log\left(\min_{\tilde{z} \in \mathcal{J}_o} |Q(\tilde{z})|\right)}{\log\left(\min_{\tilde{z} \in \mathcal{J}_o} |\tilde{z}|\right)} \right\rceil \right\},$$

and any $z \in \mathcal{J}_o$, we have

$$\left|Q(z) z^{h-m}\right| \geq \min_{\tilde{z} \in \mathcal{J}_o} \left|Q(\tilde{z})\, \tilde{z}^{h-m}\right| \geq \left(\min_{\tilde{z} \in \mathcal{J}_o} |Q(\tilde{z})|\right) \left(\min_{\tilde{z} \in \mathcal{J}_o} |\tilde{z}|\right)^{h-m} \geq \bar{\phi} \max_{\tilde{z} \in \mathcal{J}_o} |R(\tilde{z})| > |\phi R(z)|,$$

where the last but one inequality follows from the definition of $\bar{h}$. So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{J}_o$ as $Q(z)z^{h-m}$. Therefore, $P(z)$ has exactly $\deg(Q) - q$ roots inside $\mathcal{J}_o$, and hence at least $\deg(Q) - q$ roots outside $\mathcal{C}$. We thus get $p \leq \deg(P) - [\deg(Q) - q] = h - m + q = \nu - (\delta - q) = \nu - d_{peg}$. Therefore, if $d_{peg} \geq 1$ (or equivalently if $S_{peg} = M$), then $p < \nu$ and consequently $S(\phi, h) = M$ for any $|\phi| \in (\underline{\phi}, \bar{\phi})$ and any $h \geq \bar{h}$.

Point (d)(iii). For $h \leq m$, we have $\nu = \delta$ and $P(z) = Q(z) + \phi R(z) z^{m-h}$. Consider a Jordan curve $\mathcal{J}_i$ (where the subscript "i" stands for "inside $\mathcal{C}$") that: (i) lies entirely inside $\mathcal{C}$, and (ii) surrounds the q roots of $Q(z)$ inside $\mathcal{C}$ (if $q \geq 1$). I apply Rouché's theorem to $\mathcal{J} = \mathcal{J}_i$, $P_b(z) = Q(z)$, and $P_s(z) = \phi R(z) z^{m-h}$. For any $|\phi| \in (\underline{\phi}, \bar{\phi})$, any

$$h \leq \underline{h} := m + \min\left\{0, \left\lfloor \frac{\log\left(\min_{\tilde{z} \in \mathcal{J}_i} |Q(\tilde{z})|\right) - \log\left(\bar{\phi} \max_{\tilde{z} \in \mathcal{J}_i} |R(\tilde{z})|\right)}{-\log\left(\max_{\tilde{z} \in \mathcal{J}_i} |\tilde{z}|\right)} \right\rfloor \right\},$$

and any $z \in \mathcal{J}_i$, we have

$$|Q(z)| \geq \min_{\tilde{z} \in \mathcal{J}_i} |Q(\tilde{z})| \geq \bar{\phi} \max_{\tilde{z} \in \mathcal{J}_i} |R(\tilde{z})| \left(\max_{\tilde{z} \in \mathcal{J}_i} |\tilde{z}|\right)^{m-h} \geq \bar{\phi} \max_{\tilde{z} \in \mathcal{J}_i} \left|R(\tilde{z})\, \tilde{z}^{m-h}\right| > \left|\phi R(z) z^{m-h}\right|,$$

where the second inequality follows from the definition of $\underline{h}$. So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{J}_i$ as $Q(z)$. Therefore, $P(z)$ has exactly q roots inside $\mathcal{J}_i$, and hence at least q roots inside $\mathcal{C}$. We thus get $p \geq q = \nu - (\delta - q) = \nu - d_{peg}$. Therefore, if $d_{peg} \leq -1$ (or equivalently if $S_{peg} = E$), then $p > \nu$ and consequently $S(\phi, h) = E$ for any $|\phi| \in (\underline{\phi}, \bar{\phi})$ and any $h \leq \underline{h}$.

A.7 Proof of Proposition 6

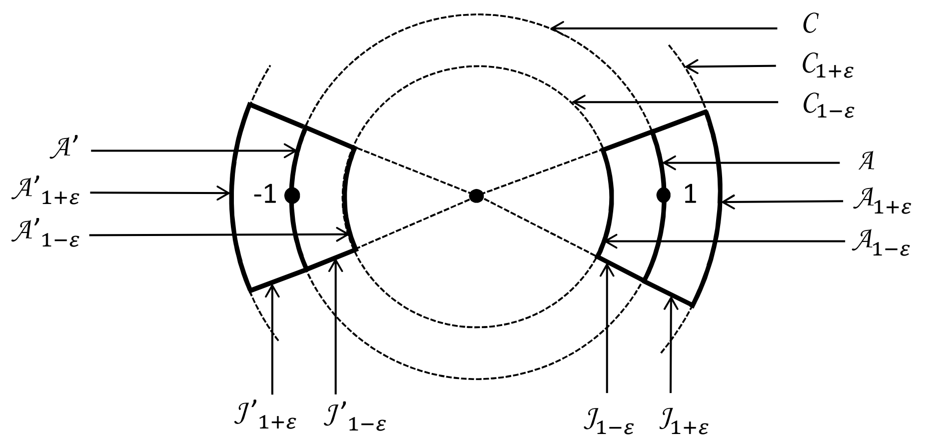
Points (a) and (b)(i). These points straightforwardly follow from Points (a), (c)(iii), and (d)(iii) of Proposition 5.

Point (b)(ii). The proof of this point is similar to (part of) the proof of Point (b) of Proposition 4. If $A_{min} \subset \mathcal{C} \setminus \{-1, 1\}$, then $|Q(z)| > \underline{\phi}\, |R(z)|$ for $z \in \{-1, 1\}$. So, by continuity, there exist $\epsilon \in (0, \bar{\phi} - \underline{\phi})$ and two open arcs $\mathcal{A}$ and $\mathcal{A}'$ of $\mathcal{C}$ such that: (i) $1 \in \mathcal{A}$, (ii) $-1 \in \mathcal{A}'$, and (iii) $\forall\, |\phi| \in (\underline{\phi}, \underline{\phi} + \epsilon)$, $\forall z \in \mathcal{A} \cup \mathcal{A}'$, $|Q(z)| > |\phi R(z)|$.

For any $r \in \mathbb{R}_+$, let $\mathcal{A}_r$ (resp. $\mathcal{A}'_r$) denote the image of $\mathcal{A}$ (resp. $\mathcal{A}'$) under the homothety whose center is the origin of the complex plane and whose ratio is r, so that in particular $\mathcal{A}_1 = \mathcal{A}$ (resp. $\mathcal{A}'_1 = \mathcal{A}'$). In addition, for any $r \in \mathbb{R}_+ \setminus \{0\}$, let $\mathcal{J}_r$ (resp. $\mathcal{J}'_r$) denote the Jordan curve made of $\mathcal{A}$ (resp. $\mathcal{A}'$), $\mathcal{A}_r$ (resp. $\mathcal{A}'_r$), and the two radial line segments joining the endpoints of $\mathcal{A}$ (resp. $\mathcal{A}'$) and $\mathcal{A}_r$ (resp. $\mathcal{A}'_r$).

By continuity, there exists $\varepsilon \in (0,1)$ such that: (i) neither $Q(z)$ nor $R(z)$ has any root inside the annulus whose borders are $\mathcal{C}_{1-\varepsilon}$ and $\mathcal{C}_{1+\varepsilon}$, (ii) $\forall\, |\phi| \in (\underline{\phi}, \underline{\phi} + \epsilon)$, $\forall h \leq m$, $\forall z \in \mathcal{J}_{1-\varepsilon} \cup \mathcal{J}'_{1-\varepsilon}$, $|Q(z)| > \left|\phi R(z) z^{m-h}\right|$, and (iii) $\forall\, |\phi| \in (\underline{\phi}, \underline{\phi} + \epsilon)$, $\forall h \geq m$, $\forall z \in \mathcal{J}_{1+\varepsilon} \cup \mathcal{J}'_{1+\varepsilon}$, $\left|Q(z) z^{h-m}\right| > |\phi R(z)|$ (see Figure A.3).

Figure A.3: Jordan curves $\mathcal{J}_{1-\varepsilon}$, $\mathcal{J}_{1+\varepsilon}$, $\mathcal{J}'_{1-\varepsilon}$, and $\mathcal{J}'_{1+\varepsilon}$



I first apply Rouché's theorem to $P_b(z) = Q(z)$, $P_s(z) = \phi R(z) z^{m-h}$, and (alternatively) $\mathcal{J}_{1-\varepsilon}$ and $\mathcal{J}'_{1-\varepsilon}$. I obtain that $\forall\, |\phi| \in (\underline{\phi}, \underline{\phi} + \epsilon)$, $\forall h \leq m$, $P(z)$ has no roots inside $\mathcal{J}_{1-\varepsilon}$ and no roots inside $\mathcal{J}'_{1-\varepsilon}$. Therefore, $P(z)$ has no real roots between $\mathcal{C}_{1-\varepsilon}$ and $\mathcal{C}$. Now, we know from Online Appendix A.6 that there exists $\underline{h}_{1-\varepsilon} \leq m$ such that $\forall\, |\phi| \in (\underline{\phi}, \bar{\phi})$, $\forall h \leq \underline{h}_{1-\varepsilon}$, $P(z)$ has exactly q roots inside $\mathcal{C}_{1-\varepsilon}$. As a result, $\forall\, |\phi| \in (\underline{\phi}, \underline{\phi} + \epsilon)$, $\forall h \leq \underline{h}_{1-\varepsilon}$, $P(z)$ has exactly $p - q$ roots between $\mathcal{C}_{1-\varepsilon}$ and $\mathcal{C}$, and none of them is real, so $p - q$ is even. Therefore, if $d_{peg} = \delta - q$ is odd, then $p - \delta$ is odd too. Since $\nu = \delta$ for $h \leq \underline{h}_{1-\varepsilon}$, $p - \nu$ is odd as well. As a consequence, $p \neq \nu$, and hence $S(\phi, h) \neq D$, for all $|\phi| \in (\underline{\phi}, \underline{\phi} + \epsilon)$ and all $h \leq \underline{h}_{1-\varepsilon}$. Together with Point (d)(ii) of Proposition 5, this result implies that $\mathbb{H}_D$ is bounded below.

I then apply Rouché's theorem to $P_b(z) = Q(z) z^{h-m}$, $P_s(z) = \phi R(z)$, and (alternatively) $\mathcal{J}_{1+\varepsilon}$ and $\mathcal{J}'_{1+\varepsilon}$. I obtain that $\forall\, |\phi| \in (\underline{\phi}, \underline{\phi} + \epsilon)$, $\forall h \geq m$, $P(z)$ has no roots inside $\mathcal{J}_{1+\varepsilon}$

and no roots inside $\mathcal{J}'_{1+\varepsilon}$. Therefore, $P(z)$ has no real roots between $\mathcal{C}$ and $\mathcal{C}_{1+\varepsilon}$. Now, we know from Online Appendix A.6 that there exists $\bar{h}_{1+\varepsilon} \geq m$ such that $\forall\, |\phi| \in (\underline{\phi}, \bar{\phi})$, $\forall h \geq \bar{h}_{1+\varepsilon}$, $P(z)$ has exactly $q + h - m$ roots inside $\mathcal{C}_{1+\varepsilon}$. As a result, $\forall\, |\phi| \in (\underline{\phi}, \underline{\phi} + \epsilon)$, $\forall h \geq \bar{h}_{1+\varepsilon}$, $P(z)$ has exactly $q + h - m - p$ roots between $\mathcal{C}$ and $\mathcal{C}_{1+\varepsilon}$, and none of them is real, so $q + h - m - p$ is even. Therefore, if $d_{peg} = \delta - q$ is odd, then $\delta + h - m - p$ is odd too. Since $\nu = \delta + h - m$ for $h \geq \bar{h}_{1+\varepsilon}$, $\nu - p$ is odd as well. As a consequence, $p \neq \nu$, and hence $S(\phi, h) \neq D$, for all $|\phi| \in (\underline{\phi}, \underline{\phi} + \epsilon)$ and all $h \geq \bar{h}_{1+\varepsilon}$. Together with Point (c)(ii) of Proposition 5, this result implies that $\mathbb{H}_D$ is bounded above.

Point (b)(iii). I consider four alternative cases in turn. First, if $A_{min} = \{1\}$ and $Q(1)R(1) < 0$, then Point (e) of Proposition 7 (proved in Online Appendix A.8 below without using Proposition 6) implies that $\mathbb{H}_D$ is unbounded below (resp. above) if $d_{peg} = 1$ (resp. $d_{peg} = -1$). Second, if $A_{min} = \{1\}$ and $Q(1)R(1) > 0$, then the same reasoning as in Online Appendix A.8 below, this time for $\phi_W = -\underline{\phi}$ instead of $\phi_W = \underline{\phi}$, straightforwardly implies that $\mathbb{H}_D$ is, again, unbounded below (resp. above) if $d_{peg} = 1$ (resp. $d_{peg} = -1$).

The third case that I consider is the case in which $A_{min} = \{-1\}$ and $Q(-1)R(-1) > 0$, and hence $\underline{\phi} = Q(-1)/R(-1)$. In this case, I rewrite $P(z)$ as a function of two variables: $\hat{P}(\phi, z) := Q(z) z^{\max(0, h-m)} + \phi R(z) z^{\max(0, m-h)}$, where $(\phi, z) \in \mathbb{R} \times \mathbb{C}$. For any $h \in \mathbb{Z}$ such that $h - m$ is odd, simple algebra leads to $\hat{P}(\underline{\phi}, -1) = 0$ and

$$\frac{\partial \hat{P}}{\partial z}(\underline{\phi}, -1) = (-1)^{\max(0, m-h)} \underline{\phi} R(-1) \left(h - \widetilde{h}\right),$$

where $\widetilde{h} := m + Q'(-1)/Q(-1) - R'(-1)/R(-1)$. This expression for $\partial \hat{P}/\partial z(\underline{\phi}, -1)$ is generically non-zero (it can be zero only if $Q'(1)/Q(1) - R'(1)/R(1)$ is an integer, and I ignore this zero-measure case). So, one root of the polynomial $\hat{P}(\underline{\phi}, z)$ is -1, and this root is of multiplicity one. The implicit-function theorem implies the existence of a continuously differentiable function $\phi \mapsto Z(\phi)$ such that one real root of $P(z)$ can be written as $Z(\phi)$ in the neighborhood of $\phi = \underline{\phi}$, with $Z(\underline{\phi}) = -1$ and

$$Z'(\underline{\phi}) = \frac{-\frac{\partial \hat{P}}{\partial \phi}(\underline{\phi}, -1)}{\frac{\partial \hat{P}}{\partial z}(\underline{\phi}, -1)} = \frac{-1}{\underline{\phi}\left(h - \widetilde{h}\right)}.$$

This root of $P(z)$ crosses $\mathcal{C}$ at point -1 as ϕ goes through $\underline{\phi}$. It is the only root that crosses $\mathcal{C}$ as ϕ goes through $\underline{\phi}$. Indeed, any root $z \in \mathbb{C}$ having this property must satisfy $\hat{P}(\underline{\phi}, z) = 0$, which implies $|Q(z)| = \underline{\phi}\, |R(z)|$ and hence $z = -1$ (since $A_{min} = \{-1\}$).

If $h < \widetilde{h}$, then $Z'(\underline{\phi}) > 0$, and therefore the root of $P(z)$ goes from outside to inside $\mathcal{C}$ as ϕ crosses $\underline{\phi}$ from below. So, the number of roots of $P(z)$ inside $\mathcal{C}$, p, increases by exactly one as ϕ crosses $\underline{\phi}$ from below. We know from Online Appendix A.6 that this number is $p = q + \max(0, h - m)$ for ϕ just below $\underline{\phi}$. We also know from Lemma 1 that $\nu = \delta + \max(0, h - m)$. So, $\nu - p = \delta - q = d_{peg}$ for ϕ just below $\underline{\phi}$, and $\nu - p = d_{peg} - 1$ for ϕ just above $\underline{\phi}$. Therefore, if $d_{peg} = 1$, then we move from $p < \nu$ to $p = \nu$, and hence from $S(\phi, h) = M$ to $S(\phi, h) = D$, as ϕ crosses $\underline{\phi}$ from below, for any $h \in \mathbb{Z}$ such that $h - m$ is odd and such that $h < \widetilde{h}$. As a consequence, $\mathbb{H}_D$ is unbounded below.

Alternatively, if $h > \widetilde{h}$, then $Z'(\underline{\phi}) < 0$, and therefore the root of $P(z)$ goes this time from inside to outside $\mathcal{C}$ as ϕ crosses $\underline{\phi}$ from below. So, p decreases by exactly one as ϕ crosses $\underline{\phi}$ from below. Again, we know from Online Appendix A.6 that $p = q + \max(0, h - m)$ for ϕ just below $\underline{\phi}$, and we know from Lemma 1 that $\nu = \delta + \max(0, h - m)$. So, we still have $\nu - p = d_{peg}$ for ϕ just below $\underline{\phi}$, but we now have $\nu - p = d_{peg} + 1$ for ϕ just above $\underline{\phi}$. Therefore, if $d_{peg} = -1$, then we move from $p > \nu$ to $p = \nu$, and hence from $S(\phi, h) = E$ to $S(\phi, h) = D$, as ϕ crosses $\underline{\phi}$ from below, for any $h \in \mathbb{Z}$ such that $h - m$ is odd and such that $h > \widetilde{h}$. Therefore, $\mathbb{H}_D$ is unbounded above.

The fourth and last case that I consider is the case in which $A_{min} = \{-1\}$ and $Q(-1)R(-1) < 0$, and hence $\underline{\phi} = -Q(-1)/R(-1)$. In this case, the analysis and the conclusion are exactly the same as in the third case, except that "$h - m$ is odd" should be replaced by "$h - m$ is even."

A.8 Proof of Proposition 7

Points (a)-(d) and (e)(i). These points straightforwardly follow from the definitions of $\mathbb{H}_D$ and ϕ_W, Points (a)-(b) of Proposition 5, and the restriction $\phi_W > 0$.

Point (e)(ii). The proof of this point is essentially a generalization of the proofs of Point (c) of Proposition 3 and Point (f)(ii) of Proposition 4. I assume that $A_{min} = \{1\}$ (as stated in this point). I rewrite $P(z)$ as a function of two variables: $\hat{P}(\phi, z) := Q(z)z^{\max(0,h-m)} + \phi R(z) z^{\max(0,m-h)}$, where $(\phi, z) \in \mathbb{R} \times \mathbb{C}$. Simple algebra leads to $\hat{P}(\phi_W, 1) = 0$ and

$$\frac{\partial \hat{P}}{\partial z}(\phi_W, 1) = Q(1)\left(h - h^{**}\right).$$

This last expression is generically non-zero (it can be zero only if $Q'(1)/Q(1) - R'(1)/R(1)$ is an integer, and I ignore this zero-measure case). So, one root of the polynomial $\hat{P}(\phi_W, z)$ is 1, and this root is of multiplicity one. The implicit-function theorem implies the existence of a continuously differentiable function $\phi \mapsto Z(\phi)$ such that one real root of $P(z)$ can be written as $Z(\phi)$ in the neighborhood of $\phi = \phi_W$, with $Z(\phi_W) = 1$ and

$$Z'(\phi_W) = \frac{-\frac{\partial \hat{P}}{\partial \phi}(\phi_W, 1)}{\frac{\partial \hat{P}}{\partial z}(\phi_W, 1)} = \frac{1}{\phi_W\,(h - h^{**})}.$$

This root of $P(z)$ crosses $\mathcal{C}$ at point 1 as ϕ goes through ϕ_W. It is the only root that crosses $\mathcal{C}$ as ϕ goes through ϕ_W. Indeed, any root $z \in \mathbb{C}$ having this property must satisfy $\hat{P}(\phi_W, z) = 0$, which implies $|Q(z)| = \underline{\phi}\,|R(z)|$ and hence $z = 1$ (since $A_{min} = \{1\}$).

For any $h < h^{**}$, we have $Z'(\phi_W) < 0$, and therefore the root of $P(z)$ goes from outside to inside $\mathcal{C}$ as ϕ crosses ϕ_W from below. So, the number of roots of $P(z)$ inside $\mathcal{C}$, p, increases by exactly one as ϕ crosses ϕ_W from below. We know from Online Appendix A.6 that this number is $p = q + \max(0, h - m)$ for ϕ just below $\underline{\phi} = \phi_W$. We also know from Lemma 1 that $\nu = \delta + \max(0, h - m)$. So, $\nu - p = \delta - q = d_{peg}$ for ϕ just below ϕ_W, and $\nu - p = d_{peg} - 1$ for ϕ just above ϕ_W. Therefore, if $d_{peg} = 1$, then we move from $p < \nu$ to $p = \nu$, and hence from $S(\phi, h) = M$ to $S(\phi, h) = D$, as ϕ crosses ϕ_W from below; in this case, the Taylor principle is locally necessary and sufficient for determinacy. Alternatively, if $d_{peg} \geq 2$ (resp. $d_{peg} \leq 0$), then we get $p < \nu$ and $S(\phi, h) = M$ (resp. $p > \nu$ and $S(\phi, h) = E$) for ϕ just above ϕ_W, and the Taylor principle is not locally necessary and sufficient for determinacy.

For any $h > h^{**}$, we have $Z'(\phi_W) > 0$, and therefore the root of $P(z)$ goes this time from inside to outside $\mathcal{C}$ as ϕ crosses ϕ_W from below. So, p decreases by exactly one as ϕ crosses ϕ_W from below. Again, we know from Online Appendix A.6 that $p = q + \max(0, h - m)$ for ϕ just below $\underline{\phi} = \phi_W$, and we know from Lemma 1 that $\nu = \delta + \max(0, h - m)$. So, we still have $\nu - p = d_{peg}$ for ϕ just below ϕ_W, but we now have $\nu - p = d_{peg} + 1$ for ϕ just above ϕ_W. Therefore, if $d_{peg} = -1$, then we move from $p > \nu$ to $p = \nu$, and hence from $S(\phi, h) = E$ to $S(\phi, h) = D$, as ϕ crosses ϕ_W from below; in this case, the Taylor principle is locally necessary and sufficient for determinacy. Alternatively, if $d_{peg} \geq 0$ (resp. $d_{peg} \leq -2$), then we get $p < \nu$ and $S(\phi, h) = M$ (resp. $p > \nu$ and $S(\phi, h) = E$) for ϕ just above ϕ_W, and the Taylor principle is not locally necessary and sufficient for determinacy.

A.9 Proof of Proposition 8

Under Rule (5) with $\phi \neq 0$, we have $P(z) = z^{\max(0,h-m)}[Q(z) + \phi R(z) z^{m-h}]$, as follows from Lemma 1. For any $j \in \{1, ..., J\}$, under the rule $i_t = \phi_j \mathbb{E}_t\{v_{j,t+h_j}\}$ with $\phi_j \neq 0$, we similarly have $P(z) = z^{\max(0,h_j-m_j)}[Q(z) + \phi_j R_j(z) z^{m_j-h_j}]$.

Under Rule (9) with $\phi = 0$, therefore, there exists $k_1 \in \mathbb{Z}$ such that $P(z) = z^{k_1}[Q(z) + \sum_{j=1}^{J} \phi_j R_j(z) z^{m_j - h_j}]$. As a reciprocal polynomial, $P(z)$ is such that $P(0) \neq 0$; moreover, we have $Q(0) \neq 0$ and $\forall j \in \{1, ..., J\}$, $R_j(0) \neq 0$; as a consequence, we get $k_1 = g := \max[0, \max_{j \in \{1,...,J\}}(h_j - m_j)]$, and thus $P(z) = \tilde{Q}(z) := z^g[Q(z) + \sum_{j=1}^{J} \phi_j R_j(z) z^{m_j - h_j}]$. In addition, the same argument as the one used at the end of Online Appendix A.5 implies that the number of non-predetermined variables, ν, is equal to $\tilde{\delta} := \delta + g$.

Under Rule (9) with $\phi \neq 0$, similarly, there exists $k_2 \in \mathbb{Z}$ such that $P(z) = z^{k_2}[Q(z) + \sum_{j=1}^{J} \phi_j R_j(z) z^{m_j - h_j} + \phi R(z) z^{m-h}]$. Again, as a reciprocal polynomial, $P(z)$ is such that $P(0) \neq 0$; moreover, we have $Q(0) \neq 0$, $\forall j \in \{1, ..., J\}$, $R_j(0) \neq 0$, and $R(0) \neq 0$; as a consequence, we get $k_2 = \max[0, \max_{j \in \{1,...,J\}}(h_j - m_j), h - m] = \max(g, h - m)$, and thus $P(z) = \tilde{Q}(z) z^{\max(0, h - \tilde{m})} + \phi R(z) z^{\max(0, \tilde{m} - h)}$, where $\tilde{m} := m + g$. In addition, the same argument as the one used at the end of Online Appendix A.5 implies that the number of non-predetermined variables, ν, is equal to $\tilde{\delta} + \max(0, h - \tilde{m})$.

Therefore, Lemma 1 still holds for Rule (9) instead of Rule (5), if δ, m, and $Q(z)$ are respectively replaced by $\tilde{\delta}$, $\tilde{m}$, and $\tilde{Q}(z)$ in this lemma. As a consequence, Propositions 5-7 still hold for Rule (9) instead of Rule (5), if δ, m, $Q(z)$, d_{peg}, and S_{peg} are respectively replaced by $\tilde{\delta}$, $\tilde{m}$, $\tilde{Q}(z)$, $\tilde{d}_{peg}$, and $\tilde{S}_{peg}$ in these propositions and if $\tilde{q}_{\mathcal{C}} = 0$, where $\tilde{q}_{\mathcal{C}} := \#\{z \in \mathcal{C} | \tilde{Q}(z) = 0\}$ denotes the number of roots of $\tilde{Q}(z)$ *exactly* on $\mathcal{C}$ (counting multiplicity).

The system composed of Model (4) and Rule (5) satisfies, by assumption, the regularity condition $q_{\mathcal{C}} = r_{\mathcal{C}} = 0$. Since $q_{\mathcal{C}} = 0$, we have $\forall j \in \{1, ..., J\}$, $\underline{\phi}_j > 0$. Suppose further that $\sum_{j=1}^{J} |\phi_j| / \underline{\phi}_j < 1$ (as stated in the proposition). For any $z \in \mathcal{C}$, we have $\tilde{Q}(z) = z^g Q(z)\{1 + \sum_{j=1}^{J} \phi_j [R_j(z)/Q(z)] z^{m_j - h_j}\}$ with $z^g \neq 0$, $Q(z) \neq 0$, and

$$\left| 1 + \sum\nolimits_{j=1}^{J} \phi_j \frac{R_j(z)}{Q(z)} z^{m_j - h_j} \right| \geq 1 - \left| \sum\nolimits_{j=1}^{J} \phi_j \frac{R_j(z)}{Q(z)} z^{m_j - h_j} \right|$$
$$\geq 1 - \sum\nolimits_{j=1}^{J} |\phi_j| \left| \frac{R_j(z)}{Q(z)} \right| \geq 1 - \sum\nolimits_{j=1}^{J} \frac{|\phi_j|}{\underline{\phi}_j} > 0.$$

So, we get $\tilde{q}_{\mathcal{C}} = 0$. Thus, the system composed of Model (4) and Rule (9) satisfies the regularity condition $\tilde{q}_{\mathcal{C}} = r_{\mathcal{C}} = 0$.

In addition, let $\tilde{q} := \#\{z \in \mathbb{C} | \tilde{Q}(z) = 0, |z| < 1\}$ denote the number of roots of $\tilde{Q}(z)$ inside $\mathcal{C}$ (counting multiplicity). To determine $\tilde{q}$, I apply Rouché's theorem to $\mathcal{J} = \mathcal{C}$,

$P_b(z) = z^g Q(z)$, and $P_s(z) = z^g \sum_{j=1}^J \phi_j R_j(z) z^{m_j-h_j}$. For any $z \in \mathcal{C}$, we have

$$|z^g Q(z)| = |Q(z)| > |Q(z)| \sum\nolimits_{j=1}^J \frac{|\phi_j|}{\underline{\phi}_j} \geq |Q(z)| \sum\nolimits_{j=1}^J |\phi_j| \left| \frac{R_j(z)}{Q(z)} \right|$$
$$= \sum\nolimits_{j=1}^J \left|\phi_j R_j(z) z^{m_j-h_j}\right| \geq \left| \sum\nolimits_{j=1}^J \phi_j R_j(z) z^{m_j-h_j} \right| = \left| z^g \sum\nolimits_{j=1}^J \phi_j R_j(z) z^{m_j-h_j} \right|.$$

So, $\tilde{Q}(z)$ has the same number of roots inside $\mathcal{C}$ as $z^g Q(z)$: $\tilde{q} = q + g$. As a consequence, $\tilde{d}_{peg} := \tilde{\delta} - \tilde{q} = \delta - q = d_{peg}$, and the determinacy status under Rule (9) with $\phi = 0$ is the same as under Rule (5) with $\phi = 0$.

Finally, let $\underline{\phi} := \min_{z\in\mathcal{C}} |Q(z)/R(z)|$ and $\bar{\phi} := \max_{z\in\mathcal{C}} |Q(z)/R(z)|$ denote the thresholds for ϕ under Rule (5), and $\tilde{\underline{\phi}} := \min_{z\in\mathcal{C}} \left|\tilde{Q}(z)/R(z)\right|$ and $\tilde{\bar{\phi}} := \max_{z\in\mathcal{C}} \left|\tilde{Q}(z)/R(z)\right|$ the thresholds for ϕ under Rule (9). We have

$$\tilde{\underline{\phi}} = \min_{z\in\mathcal{C}} \left| \frac{Q(z)}{R(z)} \left[1 + \sum\nolimits_{j=1}^J \phi_j \frac{R_j(z)}{Q(z)} z^{m_j-h_j} \right] \right| \geq \min_{z\in\mathcal{C}} \left| \frac{Q(z)}{R(z)} \right| \min_{z\in\mathcal{C}} \left| 1 + \sum\nolimits_{j=1}^J \phi_j \frac{R_j(z)}{Q(z)} z^{m_j-h_j} \right|$$
$$\geq \underline{\phi} \min_{z\in\mathcal{C}} \left[1 - \sum\nolimits_{j=1}^J |\phi_j| \left| \frac{R_j(z)}{Q(z)} \right| \right] \geq \underline{\phi} \left[1 - \sum\nolimits_{j=1}^J |\phi_j| \max_{z\in\mathcal{C}} \left| \frac{R_j(z)}{Q(z)} \right| \right] = \underline{\phi} \left(1 - \sum\nolimits_{j=1}^J \frac{|\phi_j|}{\underline{\phi}_j} \right)$$

and

$$\tilde{\bar{\phi}} = \max_{z\in\mathcal{C}} \left| \frac{Q(z)}{R(z)} \left[1 + \sum\nolimits_{j=1}^J \phi_j \frac{R_j(z)}{Q(z)} z^{m_j-h_j} \right] \right| \leq \max_{z\in\mathcal{C}} \left| \frac{Q(z)}{R(z)} \right| \max_{z\in\mathcal{C}} \left| 1 + \sum\nolimits_{j=1}^J \phi_j \frac{R_j(z)}{Q(z)} z^{m_j-h_j} \right|$$
$$\leq \bar{\phi} \max_{z\in\mathcal{C}} \left[1 + \sum\nolimits_{j=1}^J |\phi_j| \left| \frac{R_j(z)}{Q(z)} \right| \right] \leq \bar{\phi} \left[1 + \sum\nolimits_{j=1}^J |\phi_j| \max_{z\in\mathcal{C}} \left| \frac{R_j(z)}{Q(z)} \right| \right] = \bar{\phi} \left(1 + \sum\nolimits_{j=1}^J \frac{|\phi_j|}{\underline{\phi}_j} \right).$$

A.10 Proof of Proposition 9

Under Rule (5) with $\phi \neq 0$, as stated in Lemma 1, we have $\nu = \delta + \max(0, h-m)$ and $P(z) = Q(z) z^{\max(0,h-m)} + \phi R(z) z^{\max(0,m-h)}$. Under Rule (11) with $\phi \neq 0$, the number of non-predetermined variables is still $\nu = \delta + \max(0, h-m)$, since the new terms in the rule are past (as opposed to expected future) values of the policy instrument. Moreover, the characteristic polynomial of the dynamic system is still the same as the characteristic polynomial of the corresponding perfect-foresight system, but the latter system is now

$$\begin{bmatrix} \mathbf{A}(L) & L^{-\gamma}\mathbf{B}(L) \\ -\phi L^{-h}\mathbf{V}(L) & \rho(L) \end{bmatrix} \begin{bmatrix} \mathbf{X}_t \\ i_t \end{bmatrix} = \mathbf{0}.$$

Except possibly for a zero-measure set of ϕ values, I can use the same standard result in time-series analysis as in Online Appendix A.5. I get that there exists $k \in \mathbb{Z}$ such that $P(z)$, the reciprocal polynomial of the characteristic polynomial, is

$$P(z) = z^k \det \begin{bmatrix} \mathbf{A}(z) & z^{-\gamma}\mathbf{B}(z) \\ -\phi z^{-h}\mathbf{V}(z) & \rho(z) \end{bmatrix}.$$

Using the Laplace expansion and the notations introduced in the main text, I rewrite $P(z)$ as $P(z) = z^k\{\det[\mathbf{A}(z)]\rho(z) - \phi z^{-\gamma-h}W(z)\} = z^k[Q(z)\rho(z) + \phi z^{m-h}R(z)]$. As a reciprocal polynomial, $P(z)$ is such that $P(0) \neq 0$; moreover, we have $Q(0) \neq 0$, $\rho(0) \neq 0$, and $R(0) \neq 0$; as a consequence, we get $k = \max(0, h-m)$, and thus $P(z) = Q(z)\rho(z)z^{\max(0,h-m)} + \phi R(z)z^{\max(0,m-h)}$. So, Lemma 1 still holds for Rule (11) instead of Rule (5), if $Q(z)$ is replaced by $\hat{Q}(z) := Q(z)\rho(z)$ in this lemma. As a consequence, and given that $\phi \neq 0$ in Rule (11), Propositions 5-7 still hold for Rule (11) instead of Rule (5), if $Q(z)$, d_{peg}, and S_{peg} are respectively replaced by $\hat{Q}(z)$, $\hat{d}_{peg}$, and $\hat{S}_{peg}$ in these propositions and if $\rho_{\mathcal{C}} = 0$.