Monetary Policy and Herd Behavior: Leaning Against Bubbles*

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Abstract

We study the role of monetary policy when asset-price bubbles may form due to herd behavior in investment in an asset whose return is uncertain. To that aim, we build a simple general-equilibrium model whose agents are households, entrepreneurs, and a central bank. Entrepreneurs receive private signals about the productivity of the new technology and borrow from households to publicly invest in the old or the new technology. The three main results of the paper are that bubbles (informational cascades) can occur in this general equilibrium setting; that the central bank can detect them even though it has directly access to less information than the investors; and that the central bank can eliminate bubbles by manipulating the interest rate. Indeed, monetary policy, by affecting the investors' cost of resources, can make them invest in the new technology if and only if they receive an encouraging private signal about its productivity. In doing so, it makes their investment decision reveal their private signal, and therefore prevents herd behavior and the asset-price bubble. We also show that such a "leaning against the wind" monetary policy, contingent on the central bank's information set, may be preferable to laisser-faire, in terms of ex ante welfare.

Key Words : Monetary Policy – Asset Prices – Informational Cascades – Bubbles.

JEL Classification : E52, E32

1 Introduction and literature review

Should monetary policy react to perceived asset-price bubbles[1]? This question has been hotly debated since the remarkable rise and fall in stock prices in developed economies during the two last cycles. Today's conventional answer among central bankers is "no". This answer stems from the consideration of the following trade-off. On the one hand, if there is actually a bubble, then such a monetary policy reaction may reduce its size or its duration, and hence its welfare costs due to overinvestment. On

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[1] The very definition of an asset-price "bubble" is quite model dependent. We temporarily postpone the exact definition in the context of our model, and want to think of it here as the price of an asset differing from *some* benchmark present discounted value of dividends generated by the asset (possibly using a different pricing kernel than the equilibrium one).

the other hand, if alternatively there is actually no bubble, then such a monetary policy reaction will be distortive and reduce welfare. Given this trade-off, a monetary policy reaction can be viewed as an insurance-against-bubbles policy, and the two conditions most commonly stressed by central bankers for its desirability are the following ones: (i) the central bank should be sufficiently certain that there is actually a bubble; (ii) the bubble should be sufficiently sensitive to modest interest-rate hikes. Because they commonly view these conditions as unlikely to be met in practice (Bernanke [2002]), central bankers usually conclude that, in most if not all cases, such a monetary policy reaction is not desirable.

This paper seeks to challenge this view by considering a simple general-equilibrium model in which these two conditions can be met because asset-price bubbles are the result of (rational) herd behavior. We focus on bubbles in stock prices, as our argument rests on some productivity considerations that are not likely to play a key role in the development of other kinds of asset-price bubbles, *e.g.* bubbles in house prices. More precisely, we assume that a new technology becomes available whose productivity will be known with certainty only in the medium term[2]. Entrepreneurs sequentially choose whether to invest in the old or the new technology, each of them on the basis of both the previous investment decisions that she observes and a private signal that she receives about the productivity of the new technology. Herd behavior may then arise as the result of an informational cascade (Banerjee [1992], Bikhchandani, Hirshleifer, and Welch [1992]). This corresponds to a situation in which, because the first entrepreneurs choose to invest in the new technology as they receive encouraging private signals about its productivity, the following entrepreneurs rationally choose to invest in the new technology too whatever their own private signal. This gives rise to a stock-market "bubble", defined as a non-zero difference between the equilibrium share price of an entrepreneur's firm and the share price of an entrepreneur's firm that would be obtained if entrepreneurs' private signals were public information. That bubbles (informational cascades) can occur in this general-equilibrium setting is the first main result of this paper.

In our model, monetary policy tightening, by making borrowing dearer for the entrepreneurs, can make them invest in the new technology if and only if they receive an encouraging private signal about its productivity. In doing so, it prevents herd behavior and hence the stock-market bubble. With this explanation of stock-market bubbles, the two conditions mentioned above can be met: (i) the central bank can detect herd behavior with certainty, even though it knows less about the productivity of the new technology than each entrepreneur; (ii) given the fragility of informational cascades, a modest

[2] Beaudry and Portier [2006] have given some support to the existence of news about future productivity. See Beaudry and Portier [2004], Christiano, Ilut, Motto, and Rostagno [2008] or Jaimovich and Rebelo [2009] for boom-bust cycle dynamics in quantitative general-equilibrium models with news shocks.

monetary policy intervention can be enough to interrupt herd behavior in new-tech investment, even though it may not interrupt new-tech investment itself[3]. Those are the second and third main results of the paper. As a consequence, under certain conditions, such a monetary policy intervention is *ex ante* preferable, in terms of social welfare, to the laisser-faire policy, as it makes entrepreneurs internalize the externality associated with informational cascades.

Our way of modeling stock-market bubbles has some advantages over each of the following three ways in which they are modeled in the literature on monetary policy and asset-price bubbles. First, bubbles may be modeled as an exogenous boom-and-bust term in the asset-price-dynamics equation (Bernanke and Gertler [1999], Bernanke and Gertler [2001]). This modeling makes the bubble by construction insensitive to monetary policy. By contrast, our modeling enables monetary policy to affect the bubble. Second, bubbles may be modeled as the result of favourable public news about future productivity that eventually fails to materialize (Gilchrist and Leahy [2002], Christiano, Ilut, Motto, and Rostagno [2008]). In this context, given that expectations are assumed to be rational and that the central bank is assumed to have no informational advantage over the private sector, a proper unconditional assessment of the desirability of a given monetary policy stance requires to consider not only the case where the favourable news does not materialize, but also the case where it does, and to assign an occurrence probability to each case − something this branch of the literature usually does not do[4]. Modeling bubbles as the result of herd behavior enables us to do just that in a micro-founded way. Third and finally, bubbles may be modeled as the result of a permanent increase in productivity growth that economic agents gradually recognize afterwards (Gilchrist and Saito [2006]). However, in a new-technology context, this late-recognition assumption may be viewed as less relevant than the early-news assumption that we make.

Our paper is related to the literature on the role of informational cascades in the business cycle. Within this literature, the paper closest to ours is that of Chamley and Gale [1994], which models investment collapses as the result of herd behavior. A first difference between the two papers is that, unlike them, we consider a general-equilibrium model and conduct policy analysis. A second difference is that they consider an endogenous timing of investment decisions, as they are also interested in modeling strategic investment delay, while in our setup the timing of investment decisions is exogenous. And a third difference is that, in equilibrium, in their model, an investment surge is always socially optimal, unlike an investment collapse, while in ours, both new-tech and old-tech investment crazes may be socially non-optimal.

[3] The latter outcome would be expected by many a central banker, *e.g.* Bernanke [2002].

[4] Gilchrist and Leahy [2002] do actually consider both cases, without needing to assign an occurrence probability to each of them, because the monetary policy that they consider is very close to the optimal monetary policy in both cases.

When prices are endogenous, they may incorporate all available private information and lead to full social learning: this is what Chari and Kehoe [2004] have called the "price critique" to herding models. In our model, prices are endogenous but the price critique does not apply because we assume that entrepreneurs cannot trade contingent claims. Other models of cascades with endogenous prices have been studied by Glosten and Milgrom [1985], Lee [1993], Avery and Zemsky [1998], Chamley [2004], Chari and Kehoe [2004], and Decamps and Lovo [2006].

The remainder of the paper is structured as follows. Section 2 presents the model. The competitive equilibrium with exogenous information about the productivity of the new technology is described in Section 3. We introduce endogenous information, derive the results about the desirability of policy intervention in a simple case, and conduct simulations in more complex cases in Section 4. Section 5 concludes.

2 The model

We consider an economy populated with infinitely lived households, overlapping generations of finitely lived entrepreneurs, and a central bank. For simplicity, we restrict our analysis to equilibria that are symmetric across entrepreneurs and across households, *i.e.* equilibria such that there is one representative household and, in each generation, one representative entrepreneur. Time is discrete, indexed by $t \in \mathbb{Z}$, and there is a single good that is non-storable and can be consumed or invested.

2.1 Technology

A production project requires κ_t units of good at date t, the investment date, and allows to operate a firm that produces $Y_{t+N} = A_{t+N} L_{t+N}^{\alpha}$ units of good at date $t + N$, where $N \in \mathbb{N}^*$, A_{t+N} is a productivity parameter, L_{t+N} is labor services, and $0 < \alpha < 1$. A production project needs a newborn entrepreneur to be undertaken, and a newborn entrepreneur cannot undertake more than one project.

To undertake a production project, a newborn entrepreneur needs to choose a technology. We consider altogether three different technologies, which we denote by the real numbers 0, $\overline{z}$ and z, with $0 < \overline{z} < z$. Technology 0 corresponds to the absence of any production project. It is characterized by the investment $\kappa_t = 0$ and the productivity parameter $A_{t+N} = 0$. Technology $\overline{z}$ is characterized by the investment $\kappa_t = \kappa(\overline{z}) > 0$ and the productivity parameter $A_{t+N} = A(\overline{z}) > 0$. Technology z requires more investment than technology $\overline{z}$: $\kappa_t = \kappa(z) > \kappa(\overline{z})$. It may be "good" and lead to the productivity parameter $A_{t+N} = A(z) > A(\overline{z})$, or be "bad" and lead to the same productivity parameter $A_{t+N} = A(\overline{z})$ as technology $\overline{z}$.

We consider two different economies. One is an economy of tranquil times, where at each date $t \in \mathbb{Z}$ the only available technologies are 0 and $\overline{z}$ and this situation is (rightly) expected by households and entrepreneurs to last forever:

$$\forall t \in \mathbb{Z},\, \forall k \in \mathbb{N}^*,\, \mathcal{F}_t = E_{\Omega(h,t)}\mathcal{F}_{t+k} = E_{\Omega(e,t)}\mathcal{F}_{t+k} = \{0, \overline{z}\}\,,$$

where $\mathcal{F}_t$ denotes the set of technologies available at date t, $E_{\Omega(h,t)}$ the expectation operator conditional on the representative household's date-t information set $\Omega\,(h,t)$, and $E_{\Omega(e,t)}$ the expectation operator conditional on the representative newborn entrepreneur's date-t information set $\Omega\,(e,t)$. Endogenous differences in information sets will be the at the core of the model.

The other economy is one with technological change. In the latter, until date 0 included, the only available technologies are 0 and $\overline{z}$ and this situation is (wrongly) expected by households and entrepreneurs to last forever:

$$\forall t \in \mathbb{Z}^{-},\, \forall k \in \mathbb{N}^*,\, \mathcal{F}_t = E_{\Omega(h,t)}\mathcal{F}_{t+k} = E_{\Omega(e,t)}\mathcal{F}_{t+k} = \{0, \overline{z}\}\,.$$

From date 1 onwards, technology z becomes available as well and this situation is (rightly) expected by households and entrepreneurs to last forever:

$$\forall t \in \mathbb{Z}^{+*},\, \forall k \in \mathbb{N}^*,\, \mathcal{F}_t = E_{\Omega(h,t)}\mathcal{F}_{t+k} = E_{\Omega(e,t)}\mathcal{F}_{t+k} = \{0, \overline{z}, z\}\,.$$

We call $\overline{z}$ the "old technology" and z the "new technology". In period 1, Nature draws whether the new technology is good or bad: it is good with probability p, bad with $(1-p)$. We assume that whether the new technology is good or bad becomes common knowledge at some date $N+1$, where $N \in \mathbb{Z}^{+*}$, whatever the investment decisions taken at dates 1 to N. For each $t \in \{1, ..., N\}$, we note μ_t the probability that the new technology is good conditionally on $\Omega\,(h,t)$ and $\widetilde{\mu}_t$ the probability that the new technology is good conditionally on $\Omega\,(e,t)$. The endogeneity of those believes μ and $\widetilde{\mu}$ will enable us to generate herds and therefore asset-price bubbles.

2.2 Preferences

The representative household supplies inelastically one unit of labor at each date. Her preferences are represented by the following utility function:

$$U_t = E_{\Omega(h,t)} \sum_{j=0}^{\infty} \beta^j \ln(c_{t+j}),$$

where c_t denotes her consumption at date t, and $0 < \beta < 1$. We choose a logarithmic utility function to simplify the algebra. Note that if utility is linear, then the interest rate is constant and the model essentially boils down to a simple Banerjee [1992] model.

At each date, one representative entrepreneur is born. She lives for $N+1$ periods and consumes only in her last period of life. The preferences of an entrepreneur born at date t are represented by the following linear utility function:

$$V_t = \beta^N E_{\Omega(e,t)} c^e_{t+N},$$

where c^e_{t+N} denotes her consumption at date $t+N$. We assume that each generation contains a large number of entrepreneurs, so that the representative entrepreneur is price-taker.

2.3 Market organization

There are a good market, a labor market, a bond market and a stock market. All are competitive. The final good is the numéraire. A newborn entrepreneur may want to borrow κ to undertake a production project. The return from this investment will be the profit she will obtain from production N periods onwards. We assume that the only financial market to which the entrepreneurs have access is a market for N-period bonds. Households have also access to this market, and there is secondary market for those bonds. We denote B_{t+N} the number of bonds that pay in period $t+N$, and that have been subscribed by the household in period t. Each of these bonds will pay one unit of good in period $t+N$, and their price is denoted q_t. B^e_t is the number of bonds emitted by the entrepreneurs. On the stock market will be traded claims on the future profits of firms. The price of a new firm stock is denoted q^S_t. By assumption, entrepreneurs (the firms owners) do not have access to the stock market, as this would reveal their private information. Therefore, transactions will always be zero and the stock market will serve here only as a device to price firms. For that reason, and to facilitate the reading, we will omit firms shares in the households' budget constraint.

Definition 1 ***(stock market index*** $\mathcal{M}$***)*** *Firms shares (which are claims for future dividends) are traded among households. The stock market index* $\mathcal{M}_t$ *is equal to the expected discounted value of the dividends that firms created in* t *will distribute in* $t+N$*, based on the information available to households at date* t*, i.e.* $\mathcal{M}_t = E_{\Omega(h,t)}\left[q^S_t c^e_{t+N}\right]$.

2.4 Resource constraints

The resource constraint on the good market states that, at each date t, the total number of goods consumed and invested cannot be larger than the total amount of goods available:

$$c_t + c^e_t + \kappa_t \leq Y_t.$$

The resource constraint on the labor market states that, at each date t, labor services cannot exceed the total amount of labor that is supplied:

$$L_t \leq 1.$$

2.5 Monetary policy

We consider a policy that has an effect on the economy only through its effect on the real interest rate, and we interpret it as monetary policy. This amounts in effect to focusing on the real-interest-rate transmission channel of monetary policy. More specifically, we model monetary policy as a tax (or subsidy) on lending together with a positive (or negative) lump-sum transfer to the representative household. We present in an online appendix a monetary model in which an inflation-targeting monetary policy can replicate the real allocations of our model.

At each date t, the representative household lends $q_t B_{t+N}$ to the representative newborn entrepreneur and gives $(\tau_t - 1)\, q_t B_{t+N}$ to the central bank (when $\tau_t > 1$) or receives $-(\tau_t - 1)\, q_t B_{t+N}$ from the central bank (when $0 < \tau_t < 1$), while the central bank gives a lump-sum transfer $T_t \equiv (\tau_t - 1)\, q_t B_{t+N}$ to the representative household (when $\tau_t > 1$) or receives a lump-sum transfer $T_t \equiv -(\tau_t - 1)\, q_t B_{t+N}$ from the representative household (when $0 < \tau_t < 1$). The budget constraint of the representative household at date t is therefore

$$c_t + \tau_t q_t B_{t+N} \leq B_t + w_t + T_t,$$

where w_t is the wage rate at date t. We assume that there is no monetary policy intervention before date 1 and after date N: $\forall t \in \mathbb{Z} \setminus \{1, ..., N\}$, $\tau_t = 1$. This is because we will consider only monetary policy interventions that eliminate informational cascades and, in our set-up, informational cascades will potentially occur only between dates 1 and N.

2.6 Agents programs

The representative household enters period t with a portfolio $\mathcal{S}_{t-1} = (B_t, ..., B_{t+N-1})$ of bonds that pay interest if at maturity. She then decides how much to consume and how much to save, supplying inelastically one unit of labor. Her program can be written in the following recursive way:

$$\mathcal{W}\left(\mathcal{S}_{t-1}\right) = \max_{c_t, B_{t+N}} \left\{\ln(c_t) + \beta E_{\Omega(h,t)} \mathcal{W}\left(\mathcal{S}_t\right)\right\}$$
$$\text{subject to } c_t + \tau_t q_t B_{t+N} \leq B_t + w_t + T_t.$$

The corresponding optimality conditions are

$$\tau_t q_t = \beta^N E_{\Omega(h,t)} \left[\frac{c_t}{c_{t+N}}\right]$$

and a transversality condition.

The representative newborn entrepreneur borrows κ_t at date t, and hires L_{t+N} to produce Y_{t+N} at date $t + N$. Production proceeds are used to pay wages, reimburse the debt and consume. Her budget constraints are therefore

$$\begin{array}{lcll} \kappa_t & \leq & q_t B^e_{t+N} & \text{in period } t, \\ c^e_{t+N} + B^e_{t+N} & \leq & \Pi_{t+N} \equiv A_{t+N} L^{\alpha}_{t+N} - w_{t+N} L_{t+N} & \text{in period } t+N. \end{array}$$

Labor demand L_{t+N} will be set such that marginal productivity of labor equalizes the real wage w_{t+N}:

$$\alpha A_{t+N} L^{\alpha-1}_{t+N} = w_{t+N},$$

while the technology chosen at date t will be

$$z_t = \arg\max_{z_t \in \mathcal{F}_t} \beta^N E_{\Omega(e,t)} \left[\Pi_{t+N} - \frac{\kappa_t}{q_t} \right].$$

We assume entrepreneurs always play pure strategies, and do not consider non-symmetric equilibria in which entrepreneurs randomize over investment decisions.

2.7 Discussion

We have made a set of strong assumptions, which are not equally restrictive. First, we have restricted preferences. Assuming log utility for the households is crucial for our analytical results, but could be relaxed if we were to do only numerical analysis. Considering risk-neutral entrepreneurs that consume only in the last period of their life is also crucial in order to solve analytically the model when we introduce endogenous information and potential informational cascades, but would not be if we were to do only numerical simulations.

Second, we have assumed that only N-period non-contingent debt contracts are possible. The assumption that bonds are only N-period is innocuous, but the assumption that they are bonds is crucial. As entrepreneurs are risk-neutral and have some private information, they would reveal that private information by their net supply of some contingent claims (for example stocks), and informational cascades would then not be possible. What we need here is not the absence of *any* contingent claims, but only of claims contingent on the quality of the new technology[5]. As we want to think of those episodes as quite infrequent ones, and as the quality of a technology is partially soft information in the real life, we think the assumption is a reasonable description of the actual environment. Similarly, the assumption that investment is of fixed size is crucial. If entrepreneurs could choose the investment size, their private information would be revealed by their actions.

Third, we consider that entrepreneurs are exogenously ranked (by date of birth), that they cannot wait to invest, and that investment projects pay only N periods ahead. Those assumptions are made to have a simple structure of the model: after exactly N periods, uncertainty is resolved. We can therefore solve the model by backward induction, which happens to be particularly convenient.

[5] Note that, in our model, investors borrow on a bond market and the bond price is endogenous. Nevertheless, we'll show that informational cascades occur in equilibrium.

Fourth, monetary policy is modeled as a tax on real interest payments. Although such a policy could (should) be labeled tax policy in our model, we want to think of it as monetary policy for two reasons. The first reason is that it is possible to write down a particular monetary model whose real allocations are the ones of our current model. In such a model[6], the control variable of the monetary authorities is the inflation rate between period t and period $t+N$. The important assumption we have to make to recover the same real allocations is that the central bank can commit on the inflation rate between period t and period $t+N$. The second reason is that the implementation of a fiscal policy that would subsidize or tax individual firms is quite complex, requires a lot of information on who the agents are, where they are, whose turn it is to invest, etc. Monetary policy, by manipulating the cost of funds, requires very little information in the implementation phase. Obviously, it has a cost of distorting not only investors' decisions, but also the decisions of some other agents. This tradeoff is present in the paper as households' savings are distorted by real-interest-rate manipulations.

3 Competitive equilibrium with exogenous information

In this section, we consider economies with exogenous information. More specifically, we assume that the sequence of newborn entrepreneurs' believes $(\widetilde{\mu}_1, ..., \widetilde{\mu}_N)$ and social believes $(\mu_1, ..., \mu_N)$ are exogenous. Our aim is to derive necessary conditions on the parameters for the existence and uniqueness of a competitive equilibrium for all $(\mu_1, ..., \mu_N) \in [0;1]^N$ and $(\widetilde{\mu}_1, ..., \widetilde{\mu}_N) \in [0;1]^N$ and for this equilibrium to have some desirable properties. We first define a competitive equilibrium. Then we study the existence, uniqueness and local dynamic stability of the steady state in tranquil times. We then turn to the equilibrium path when there is a technological change. The results obtained will be useful for the analysis of the endogenous information case considered in the next section.

3.1 Competitive equilibrium

In this economy, a symmetric competitive equilibrium is a sequence of prices $(q_t, w_t)_{t\in\mathbb{Z}}$, quantities $(B_t, B_t^e, c_t, c_t^e, L_t)_{t\in\mathbb{Z}}$ and technology choices $(z_t)_{t\in\mathbb{Z}}$ such that, for exogenous sequences of actual and expected technological possibilities $(\mathcal{F}_t)_{t\in\mathbb{Z}}$ and $\left(E_{\Omega(h,t)}\mathcal{F}_{t+k} = E_{\Omega(e,t)}\mathcal{F}_{t+k}\right)_{t\in\mathbb{Z},k\in\mathbb{N}^*}$, for exogenous newborn entrepreneurs' believes $(\widetilde{\mu}_1, \ldots, \widetilde{\mu}_N)$ and social believes $(\mu_1, \ldots, \mu_N)$ and for an exogenous sequence of monetary policy interventions $(\tau_1, ..., \tau_N)$, *(i)* prices and quantities are positive, *(ii)* the representative household's consumption and bonds holding solve her maximization problem given prices, *(iii)* the representative newborn entrepreneur's investment decision maximizes her utility given prices, *(iv)* labor demand maximizes the representative aged $N+1$ entrepreneur's profit given prices, and *(v)* labor, bonds and good markets clear.

[6] The monetary model is presented in an online appendix.

3.2 Tranquil times

In tranquil times, the only available technologies are 0 and $\overline{z}$. This case corresponds to $\mu_t = \widetilde{\mu}_t = 0$ for all t. The following proposition gives necessary and sufficient conditions for the existence, uniqueness and dynamic local stability of a steady state $(\overline{c}, \overline{q})$. The proof of this proposition as well as the proofs of the following ones are gathered in the appendix.

Proposition 1 ***(Existence, uniqueness and dynamic local stability of the steady state)***

(i) In tranquil times, there exists an equilibrium at which households' consumption level is strictly positive and constant if and only if

$$\beta^N (1-\alpha) A(\overline{z}) - \kappa(\overline{z}) > 0 \tag{1}$$

$$\text{and } \alpha A(\overline{z}) - \kappa(\overline{z}) + \beta^{-N} \kappa(\overline{z}) > 0; \tag{2}$$

(ii) if (1) and (2) hold, then this equilibrium is the unique equilibrium at which households' consumption level is strictly positive and constant, and we call it the steady state;

(iii) if (1) and (2) hold, then: in tranquil times, the steady state is locally, dynamically stable if and only if

$$\beta^N > \frac{\kappa(\overline{z})}{|\alpha A(\overline{z}) - \kappa(\overline{z})|}, \tag{3}$$

where dynamic local stability is defined as the existence of some neighborhoods $\mathcal{N}_{\overline{c}}$ of $\overline{c}$ and $\mathcal{N}_{\overline{q}}$ of $\overline{q}$ such that if $\forall t \in \mathbb{Z}$, $z_t = \overline{z}$, $\forall t \in \mathbb{Z}^-$, $c_t \in \mathcal{N}_{\overline{c}}$ and $q_t \in \mathcal{N}_{\overline{q}}$, then $\forall t \in \mathbb{Z}^{+*}$, $c_t \in \mathcal{N}_{\overline{c}}$, $q_t \in \mathcal{N}_{\overline{q}}$ and $(c_t, q_t) \longrightarrow (\overline{c}, \overline{q})$ as $t \longrightarrow +\infty$.

3.3 Technological change

We now consider the response of the economy to the unexpected availability of the new technology z from date 1 onwards. We restrict our analysis to equilibria such that the economy is at its steady state until date 0 included, *i.e.* in particular such that $\forall t \in \mathbb{Z}^-$, $(z_t, c_t, q_t) = (\overline{z}, \overline{c}, \overline{q})$. Moreover, we assume that all these equilibria are such that $\forall t > N$, $z_t = z$ if the new technology turns out to be good and $z_t = \overline{z}$ otherwise, and will check later that this is indeed the case given the restrictions on parameters that we consider. In words, this means that the new technology will always be adopted once it is known to be good, and never once it is known to be bad. As technologies z or $\overline{z}$ can be chosen in period $t \leq N$, this implies that, $\forall t > N$,

$$c_t = \alpha A(z) - \kappa(z) + q_{t-N}^{-1} \kappa(z) \text{ if } z_{t-N} = z \text{ and the new technology is good,}$$

$$c_t = \alpha A(\overline{z}) - \kappa(\overline{z}) + q_{t-N}^{-1} \kappa(z) \text{ if } z_{t-N} = z \text{ and the new technology is bad,}$$

$$c_t = \alpha A(\overline{z}) - \kappa(z) + q_{t-N}^{-1} \kappa(\overline{z}) \text{ if } z_{t-N} = \overline{z} \text{ and the new technology is good,}$$

$$c_t = \alpha A(\overline{z}) - \kappa(\overline{z}) + q_{t-N}^{-1} \kappa(\overline{z}) \text{ if } z_{t-N} = \overline{z} \text{ and the new technology is bad.}$$

Moreover, since the representative entrepreneurs born at dates $-(N-1)$ to 0 have invested in $\overline{z}$ and pay back their debts at dates 1 to N at the interest factor $\overline{R}$, the representative household's consumption at each date $t \in \{1, ..., N\}$ is $c_t = \alpha A(\overline{z}) - \kappa(z_t) + \beta^{-N}\kappa(\overline{z})$. As a consequence, for $t \in \{1, ..., N\}$ and $z_t = \overline{z}$, the Euler equation is written

$$\tau_t q_t = \beta^N \left[\alpha A(\overline{z}) - \kappa(\overline{z}) + \frac{\kappa(\overline{z})}{\beta^N}\right] \left[\frac{\mu_t}{\alpha A(\overline{z}) - \kappa(z) + \frac{\kappa(\overline{z})}{q_t}} + \frac{1-\mu_t}{\alpha A(\overline{z}) - \kappa(\overline{z}) + \frac{\kappa(\overline{z})}{q_t}}\right]. \tag{4}$$

Alternatively, for $t \in \{1, ..., N\}$ and $z_t = z$, the Euler equation is written

$$\tau_t q_t = \beta^N \left[\alpha A(\overline{z}) - \kappa(z) + \frac{\kappa(\overline{z})}{\beta^N}\right] \left[\frac{\mu_t}{\alpha A(z) - \kappa(z) + \frac{\kappa(z)}{q_t}} + \frac{1-\mu_t}{\alpha A(\overline{z}) - \kappa(\overline{z}) + \frac{\kappa(z)}{q_t}}\right]. \tag{5}$$

In the following proposition, we find necessary and sufficient conditions for equations (4) and (5) to have a unique positive solution in q_t for any believes. We also derive some properties of the interest rate (which is inversely related to q_t), namely that it is increasing in the probability that the new technology is successful, and that a rise in the lending tax rate τ_t increases the interest rate, which corresponds to a monetary policy tightening.

Proposition 2 ***(Existence, uniqueness and some properties of the bond price q)***

(I) If (1), (2) and (3) hold, then:

(i) there exists a strictly positive real number q_t solution of (4) for all $t \in \{1, ..., N\}$ and all $(\mu_1, ..., \mu_N) \in [0;1]^N$ if and only if

$$\alpha A(\overline{z}) - \kappa(z) > 0 \tag{6}$$

$$\text{and } \forall t \in \{1, ..., N\},\ \tau_t < \tau(\overline{z}), \tag{7}$$

$$\text{where } \forall x \geq \overline{z},\ \tau(x) \equiv \frac{\beta^N [\alpha A(\overline{z}) - \kappa(x)] + \kappa(\overline{z})}{\kappa(x)};$$

(ii) if (6) and (7) hold, then $\forall t \in \{1, ..., N\}$ and $\forall (\mu_1, ..., \mu_N) \in [0;1]^N$, q_t, which we note $q(z, \tau_t, \mu_t, 0)$, is unique, and $\frac{\partial q(z,\tau_t,\mu_t,0)}{\partial \tau_t} < 0$ and $\frac{\partial q(z,\tau_t,\mu_t,0)}{\partial \mu_t} > 0$.

(II) If (1), (2), (3), (6) and (7) hold, then:

(i) there exists a strictly positive real number q_t solution of (5) for all $t \in \{1, ..., N\}$ and all $(\mu_1, ..., \mu_N) \in [0;1]^N$ if and only if

$$\forall t \in \{1, ..., N\},\ \tau_t < \tau(z); \tag{8}$$

(ii) if (8) holds, then $\forall t \in \{1, ..., N\}$ and $\forall (\mu_1, ..., \mu_N) \in [0;1]^N$, q_t, which we note $q(z, \tau_t, \mu_t, 1)$, is unique, and $\frac{\partial q(z,\tau_t,\mu_t,1)}{\partial \tau_t} < 0$;

(iii) if (8) holds, then $\frac{\partial q(z,\tau_t,\mu_t,1)}{\partial \mu_t} < 0$ *for all* $t \in \{1,...,N\}$ *and all* $(\mu_1,...,\mu_N) \in [0;1]^N$ *if and only if*

$$\alpha A(\overline{z}) - \kappa(\overline{z}) < \alpha A(z) - \kappa(z). \tag{9}$$

The results $\frac{\partial q(z,\tau,\mu_t,0)}{\partial \tau} < 0$ and $\frac{\partial q(z,\tau,\mu_t,1)}{\partial \tau} < 0$ simply illustrate the fact that a positive tax on lending (*i.e.* a monetary policy tightening) raises the interest rate and therefore lowers q_t. The result $\frac{\partial q(z,\tau,\mu_t,0)}{\partial \mu_t} > 0$ is due to the fact that if entrepreneurs invest in the old technology at date t, then, as μ_t increases, c_t remains unchanged but $E_t\{\frac{1}{c_{t+N}}\}$ increases (because the representative household is expected to lend more, and hence to consume less, at date $t+N$), so that q_t increases. The result $\frac{\partial q(z,\tau_t,\mu_t,1)}{\partial \mu_t} \lesseqgtr 0$ is due to the fact that if entrepreneurs invest in the new technology at date t, then, as μ_t increases, c_t remains unchanged but $E_t\{\frac{1}{c_{t+N}}\}$ either increases or decreases depending on the sign of $[\alpha A(z) - \alpha A(\overline{z})] - [\kappa(z) - \kappa(\overline{z})]$ (because the representative household is expected both to lend more, as $\kappa(z) > \kappa(\overline{z})$, and to receive a higher wage, as $\alpha A(z) > \alpha A(\overline{z})$, at date $t+N$), so that q_t either increases or decreases depending on the sign of $[\alpha A(z) - \alpha A(\overline{z})] - [\kappa(z) - \kappa(\overline{z})]$. In the following, we will restrict our analysis to the case where $\frac{\partial q(z,\tau_t,\mu_t,1)}{\partial \mu_t} < 0$, which seems to be the more relevant: the interest rate increases when the economy invests in a new technology whose probability of success increases.

We now derive a necessary and sufficient condition for the competitive equilibrium to have the following property (that will be used later to show that the equilibrium is symmetric): a competitive entrepreneur has no incentive not to invest in any project between dates 1 to N, in any circumstance, if all the entrepreneurs born in that period do invest.

Proposition 3 ***(Symmetry of competitive equilibrium between 1 and*** N***)*** *If (1), (2), (3), (6), (7), (8) and (9) hold, then a competitive entrepreneur has no incentive at date* t *to deviate from the other entrepreneurs' common investment decision and invest nothing for all* $t \in \{1,...,N\}$*, all* $(\mu_1,...,\mu_N) \in [0;1]^N$ *and all* $(\widetilde{\mu}_1,...,\widetilde{\mu}_N) \in [0;1]^N$*, if and only if*

$$\forall t \in \{1,...,N\}, \begin{cases} \text{either } \dfrac{\tau(z)}{1+\frac{B(z)[\alpha A(\overline{z})-\kappa(\overline{z})]}{\kappa(z)}} < \tau_t < \dfrac{\tau(\overline{z})}{1+\frac{\alpha A(\overline{z})-\kappa(\overline{z})}{(1-\alpha)A(\overline{z})}}, \\ \text{or } B(z) > \dfrac{\kappa(\overline{z})}{(1-\alpha)A(\overline{z})} \text{ and } \tau_t < \dfrac{\tau(\overline{z})}{1+\frac{\alpha A(\overline{z})-\kappa(\overline{z})}{(1-\alpha)A(\overline{z})}}, \\ \text{or } B(z) < \dfrac{\kappa(\overline{z})}{(1-\alpha)A(\overline{z})} \text{ and } \tau_t < \dfrac{\tau(z)}{1+\frac{\kappa(\overline{z})[\alpha A(z)-\kappa(z)]}{\kappa(z)(1-\alpha)A(\overline{z})}} \end{cases}\Bigg\}, \tag{10}$$

$$\text{where } B(z) \equiv \frac{\kappa(z)-\kappa(\overline{z})}{(1-\alpha)[A(z)-A(\overline{z})]}.$$

We want to restrict the analysis to a set of parameters in which the the equilibrium is well-behaved (exists, is unique, stable...) under laissez-faire, i.e. under a passive monetary policy. The next proposition derives a necessary and sufficient condition for the three constraints on the monetary policy instrument so far obtained to be satisfied in the absence of monetary policy intervention, *i.e.* when $\tau_t = 1$ for all $t \in \{1, ..., N\}$:

Proposition 4 ***(The equilibrium is well-behaved absent monetary policy intervention)*** *If (1), (2), (3), (6), (7), (8) and (9) hold, then (7), (8) and (10) hold in the absence of monetary policy intervention, i.e. when $\tau_t = 1$ for all $t \in \{1, ..., N\}$, if and only if*

$$\begin{Bmatrix} \text{either } 1 < \tau(\underline{z}) < 1 + \dfrac{B(\underline{z})\left[\alpha A(\overline{z}) - \kappa(\overline{z})\right]}{\kappa(\underline{z})}, \\ \text{or } B(\underline{z}) > \dfrac{\kappa(\overline{z})}{(1-\alpha)A(\overline{z})} \text{ and } 1 < \tau(\underline{z}), \\ \text{or } B(\underline{z}) < \dfrac{\kappa(\overline{z})}{(1-\alpha)A(\overline{z})} \text{ and } 1 + \dfrac{\kappa(\overline{z})\left[\alpha A(\underline{z}) - \kappa(\underline{z})\right]}{\kappa(\underline{z})(1-\alpha)A(\overline{z})} < \tau(\underline{z}) \end{Bmatrix}. \tag{11}$$

For each $t \in \mathbb{Z}^{+*}$, let I_t denote the representative newborn entrepreneur's investment decision at date t ($I_t = 1$ when she invests in the new technology and $I_t = 0$ when she invests in the old technology). We now derive necessary and sufficient conditions for a competitive entrepreneur to have no incentive, this time at dates $t > N$, in any circumstance, to deviate from the other entrepreneurs' common investment decision $I_t = 1$ (when the new technology is good) or $I_t = 0$ (when it is bad):

Proposition 5 ***(Symmetry of competitive equilibrium after** N**)*** *If (1), (2), (3), (6), (9) and (11) hold, then: a competitive entrepreneur has no incentive to deviate from the other entrepreneurs' common investment decision $I_t = 1$ (when the new technology is good) or $I_t = 0$ (when it is bad) for all $t > N$, all $(\mu_1, ..., \mu_N) \in [0;1]^N$, all $(\widetilde{\mu}_1, ..., \widetilde{\mu}_N) \in [0;1]^N$ and all $(\tau_1, ..., \tau_N) \in \mathbb{R}^{+*N}$ satisfying (7), (8) and (10), if and only if*

$$\beta^N - \frac{\kappa(\overline{z})}{\alpha A(\overline{z}) - \kappa(\overline{z})} > \frac{\kappa(\overline{z})}{(1-\alpha)A(\overline{z})} \tag{12}$$

$$\text{and } \beta^N \frac{\alpha A(\overline{z}) - \kappa(\underline{z})}{\alpha A(\underline{z}) - \kappa(\underline{z})} - \frac{\kappa(\underline{z})}{\alpha A(\underline{z}) - \kappa(\underline{z})} > \max\left[\frac{\kappa(\underline{z})}{(1-\alpha)A(\underline{z})}, B(\underline{z})\right]. \tag{13}$$

Up to now, we have imposed restrictions on parameters α, β, $\kappa(\overline{z})$, $\kappa(\underline{z})$, $A(\overline{z})$, $A(\underline{z})$, N and τ_t for $t \in \{1, ..., N\}$. Given that (6) and (12) imply (2) and (3), and that (8) implies (7), the conditions imposed on these parameters are $0 < \alpha < 1$, $0 < \beta < 1$, $\kappa(\underline{z}) > \kappa(\overline{z}) > 0$, $A(\underline{z}) > A(\overline{z}) > 0$, $N \in \mathbb{N}^*$, $\tau_t > 0$ for $t \in \{1, ..., N\}$, (1), (6), (8), (9), (10), (11), (12) and (13). We will show in the next section that the set of parameter values satisfying all these conditions is not empty.

We now check that under those conditions, it is indeed the case that, in equilibrium, once uncertainty is resolved ($t \geq N+1$), entrepreneurs invest in the new technology if it is a success and in the old technology if the new one is a failure (as we have assumed up to now). We also show that, under the conditions so far obtained, the dynamics of q_t, c_t and c_t^e is "well-behaved" from date 1.

Proposition 6 ***(Equilibrium investment for*** $t \geq N+1$ ***and equilibrium dynamics)*** *If (1), (2), (3), (6), (7), (8), (9), (10), (11), (12) and (13) hold, then,* $\forall \left(\mu_1, ..., \mu_N\right) \in [0;1]^N$, $\forall \left(\widetilde{\mu}_1, ..., \widetilde{\mu}_N\right) \in [0;1]^N$*:*

(i) $\forall t > N$*,* $z_t = z$ *if the new technology is good and* $z_t = \overline{z}$ *if it is bad;*

(ii) $\forall t \geq 1$*,* q_t*,* c_t *and* c_t^e *are strictly positive;*

(iii) $\lim_{t \longrightarrow +\infty} q_t = \beta^N$*,*

$$\lim_{t \longrightarrow +\infty} \left(c_t, c_t^e\right) = \left(\alpha A\left(z\right) - \kappa\left(z\right) + \beta^{-N}\kappa\left(z\right), \left(1-\alpha\right) A\left(z\right) - \beta^{-N}\kappa\left(z\right)\right)$$

if the new technology is good, and

$$\lim_{t \longrightarrow +\infty} \left(c_t, c_t^e\right) = \left(\alpha A\left(\overline{z}\right) - \kappa\left(\overline{z}\right) + \beta^{-N}\kappa\left(\overline{z}\right), \left(1-\alpha\right) A\left(\overline{z}\right) - \beta^{-N}\kappa\left(\overline{z}\right)\right)$$

if it is bad.

The conditions that we impose on the parameters are necessary for the existence and uniqueness of an equilibrium with some desirable properties, for all $\left(\mu_1, ..., \mu_N\right) \in [0;1]^N$ and $\left(\widetilde{\mu}_1, ..., \widetilde{\mu}_N\right) \in [0;1]^N$. The reason why they are not sufficient for that matter is that they do not ensure that, at each date between 1 and N, either a competitive entrepreneur has no incentive to deviate from the other entrepreneurs' common investment decision $I_t = 0$ and invest in the new technology, or a competitive entrepreneur has no incentive to deviate from the other entrepreneurs' common investment decision $I_t = 1$ and invest in the old technology, with these two possibilities being mutually exclusive. This will be ensured by an additional condition that we will derive in the next section in the context of endogenous information.

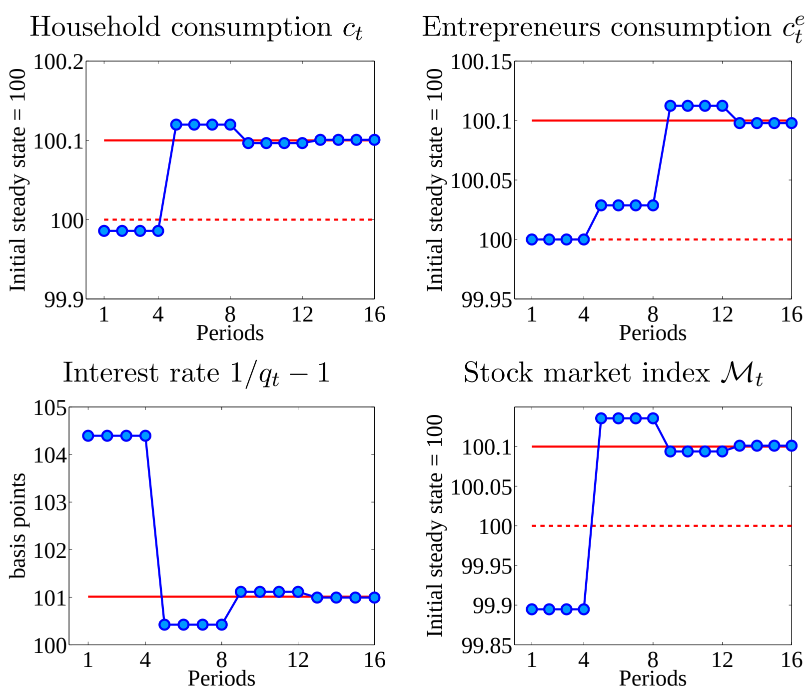
We finally show that, under the conditions so far obtained, both households and entrepreneurs gain in the long term from a good new technology:

Proposition 7 ***(Successful new technology gives higher steady-state utility)*** *If (1), (2), (3), (6), (7), (8), (9), (10), (11), (12) and (13) hold, then:* $\forall \left(\mu_1, ..., \mu_N\right) \in [0;1]^N$, $\forall \left(\widetilde{\mu}_1, ..., \widetilde{\mu}_N\right) \in [0;1]^N$*, both households' welfare* U_t *and entrepreneurs' welfare* V_t *increase in the long term if the new technology is good.*

3.4 Numerical simulations

Here we provide a numerical illustration of the model working. Parameters values are as follows. The period is one year and we assume $N = 5$, so that uncertainty is resolved after 5 years. The discount factor is $\beta = .99$, so that the real interest rate is 1.01% a year (101 basis points). The share of value added that goes to labor is $\alpha = .7$. The old technology has a TFP $A(\overline{z}) = 1$ and requires an investment of $\kappa(\overline{z}) = .1$ units of goods. The new technology requires a 10% larger investment ($\kappa(z) = .11$) and, if successful, delivers a 10% larger TFP ($A(z) = 1.1$). These parameter values fulfill the conditions imposed above. In the simulations, we check that for the exogenous believes that we have chosen, for any period t, a competitive entrepreneur that takes the interest rate as given has no incentive to deviate from the aggregate investment behavior, so that the allocations we are computing are equilibrium allocations.

Figure 1: Response of the economy to a deterministic technological change

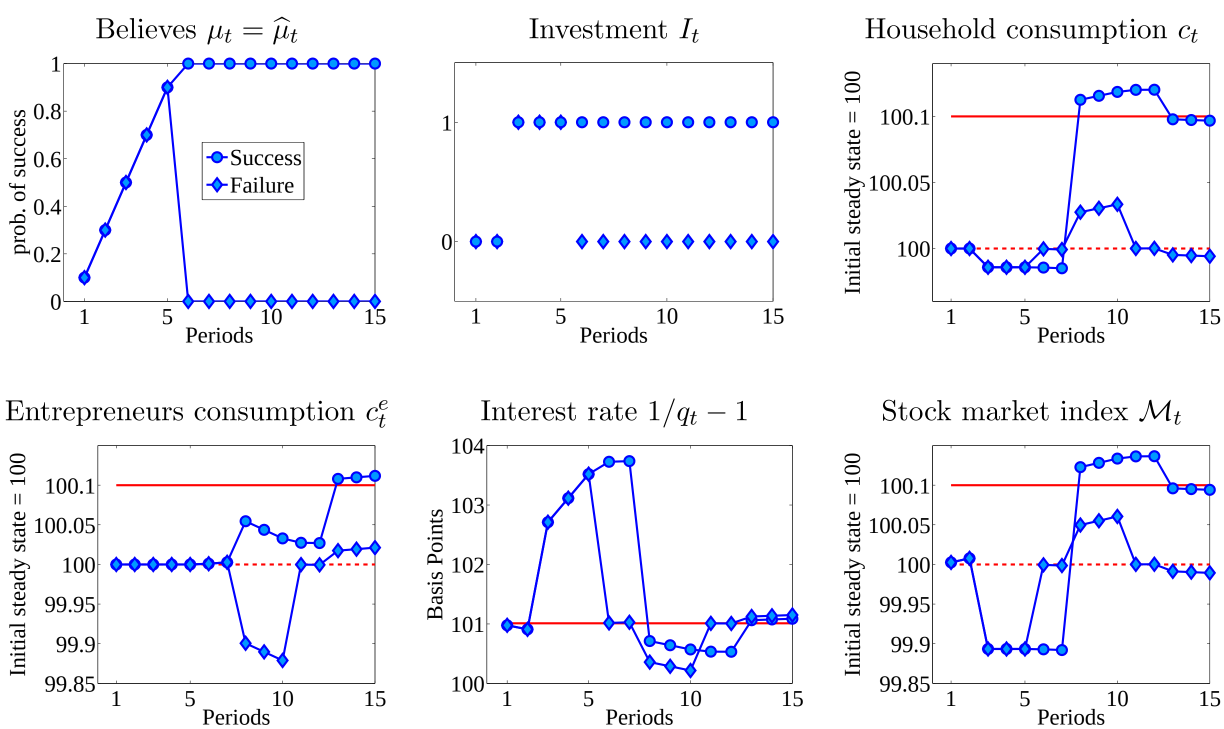


This Figure shows the response of the economy to the arrival of a new technology in period 1, which (if used) leads to a higher TFP from period 5 onwards. The dashed line represents the initial steady state level of the variable, the solid line its final steady state level.

Figure 1 corresponds to a simulation in which there is no uncertainty about the success of the new technology ($\mu_1 = ... = \mu_N = \widetilde{\mu}_1 = ... = \widetilde{\mu}_N = 1$). At the equilibrium, entrepreneurs invest in the new technology from date 1 onwards. Note that between 1 and N households consumption c_t is lower than its initial steady-state level, as investing in the new technology is relatively costly. The interest rate is initially 1.01% (101 basis points), and increases by about 3 basis points on impact. This is for the following reason: households learn that that they will have a lot of goods in $N + 1$. Therefore, the marginal utility of one extra unit of good in $N+1$ is low, and households would like to bring back some on these goods from $N+1$ to 1. To give them an incentive to lend to the entrepreneurs, rather than consuming while their marginal utility is high, the interest rate must be large. From $N+1$ to $2N$, households consumption is large (both because they receive the return from their loans of period 1 to N, which were signed at a high interest rate, and because production is higher). Consumption will not be relatively larger in period $2N+1$. Therefore, households want to save, and the interest rate is low. Note that the dynamics display an oscillating effect, that eventually vanishes. The stock market index is always above its pre-new-tech level except in the first N periods, where the interest effect dominates the dividend effect in the asset valuation.

Figure 2 corresponds to a simulation in which believes are common to entrepreneurs and households ($\forall t \in \{1, ..., N\}$, $\mu_t = \widetilde{\mu}_t$), and evolve exogenously. In period 1, the technology is unlikely to be a success (probability 10%), and this probability increases by 20 percentage points every period until period 5. In period 5, uncertainty is resolved and the technology is either a success or a failure.

Figure 2: Response of the economy to an uncertain technological change with no private information



This Figure shows the response of the economy to the arrival of a new technology in period 1, which (if used) may lead to a higher TFP from period 5 onwards. The lines with circles correspond to the case where the new technology happens to be a success, the lines with diamonds to the case where it happens to be a failure. The dashed line represents the initial steady-state level of the variable, the solid line its final steady-state level.

Recall that the first N (here 5) periods of Figure 2 do not depend on whether the technology happens to be a success or a failure. Given the evolution of believes, there is investment in the old technology in periods 1 and 2, and in the new technology in periods 3, 4 and 5. The interest rate decreases slightly in periods 1 and 2: if the technology happens to be good, the economy will invest more in $N+1$, and will not have more production because it has invested in the old technology in period 1. Therefore, c_{N+1} will be low compared to c_1. Because of this (unlikely) event, households would like to save a bit more. Since higher savings are not possible in equilibrium, the interest rate must decrease to discourage households from saving more. When investment becomes profitable in expectations (from period 3), the interest rate shoots up. Again, this discounting effect is dominant in the short-run behavior of the stock-market index, which decreases when the new technology is chosen.

4 Competitive equilibrium with endogenous information

In this section, we assume that the conditions on the parameters listed in Proposition 6 are met. We first introduce private signals, study the endogenous dynamics of the information sets and examine the role of monetary policy. We then consider a particular parametrization that enables us to solve the model analytically. We finally run numerical simulations for other parametrizations.

In this section, the competitive equilibrium is defined as in section 3.1, except that entrepreneurs' believes $\widetilde{\mu}_t$ and the social believes μ_t are endogenously and optimally set by entrepreneurs, households and the central bank in a Bayesian way. This information dynamics is now made explicit.

4.1 Information dynamics

As stated previously, Nature chooses in period 1 whether the new technology is good or bad: it is good with probability p, bad with $(1-p)$. This choice becomes common knowledge in period $N+1$. We now assume that at each date $t \in \{1, ..., N\}$, the representative new-born entrepreneur, the representative household and the central bank observe the same variables with the only exception that the representative new-born entrepreneur receives a private signal about whether the new technology is good or bad, while the representative household and the central bank receive no such private signal. As a consequence, at each date $t \in \{1, ..., N\}$, the representative household and the central bank's information sets coincide with each other and are included in the representative new-born entrepreneur's information set. We call "public information at date t" the information of the representative household and the central bank at that date. The probability that the new technology is good based on public information available at date t is therefore μ_t.

We assume that, at each date $t \in \{1, ..., N\}$, the timing of events is the following:

- The representative new-born entrepreneur starts with the public information available at date $t-1$. Therefore, she has the prior μ_{t-1} about the probability that the new technology is good. We assume that the initial prior μ_0 is exogenous.

- The central bank sets τ_t. Its intervention is public information.

- The representative new-born entrepreneur receives a private signal $S_t \in \{0,1\}$ about whether the new technology is good or bad. This signal is "good" when $S_t = 1$ and "bad" when $S_t = 0$. We note $\lambda \in \left]\frac{1}{2};1\right[$ the probability that a signal, whether good or bad, is right. Bayes' theorem implies that the representative new-born entrepreneur's posterior $\widetilde{\mu}_t$ about the probability that the new technology is good is

$$\widetilde{\mu}_t = S_t \frac{\mu_{t-1}\lambda}{\mu_{t-1}\lambda + (1-\mu_{t-1})(1-\lambda)} + (1-S_t)\frac{\mu_{t-1}(1-\lambda)}{\mu_{t-1}(1-\lambda) + (1-\mu_{t-1})\lambda}.$$

- The competitive equilibrium is determined. More precisely, the representative new-born entrepreneur takes her investment decision $I_t \in \{0,1\}$. This decision is public information, so that the probability μ_t that the new technology is good based on public information available at date t does take I_t into account. The equilibrium price is then $q_t = q(z, \tau_t, \mu_t, I_t)$.

For each $t \in \{1,...,N\}$, let $\widetilde{\mu}_t^0$ denote the value taken by $\widetilde{\mu}_t$ when $S_t = 0$ and $\widetilde{\mu}_t^1$ the value taken by $\widetilde{\mu}_t$ when $S_t = 1$. The following proposition shows that there exists at most one equilibrium, derives necessary and sufficient conditions for the existence of this equilibrium, and describes the equilibrium dynamics of I_t and μ_t.

Proposition 8 ***(Existence, uniqueness and dynamics of equilibrium)***

(i) There exists an equilibrium if and only if $\forall t \in \{1,...,N\}$, $\forall (S_1,...,S_t) \in \{0,1\}^t$, *either (a)* $\widetilde{\mu}_t^1 q(z,\tau_t,\mu_{t-1},0) < B(z)$, *or (b)* $\widetilde{\mu}_t^0 q(z,\tau_t,\mu_{t-1},1) > B(z)$, *or (c)* $\widetilde{\mu}_t^0 q\left(z,\tau_t,\widetilde{\mu}_t^0,0\right) < B(z)$ *and* $\widetilde{\mu}_t^1 q\left(z,\tau_t,\widetilde{\mu}_t^1,1\right) > B(z)$;

(ii) when there exists an equilibrium, this equilibrium is unique;

(iii) $\forall t \in \{1,...,N\}$, $\forall (S_1,...,S_t) \in \{0,1\}^t$, *at most one of the three conditions (a), (b) and (c) is met, and if it is (a) then* $\forall S_t \in \{0,1\}$, $I_t = 0$ *and* $\mu_t = \mu_{t-1}$, *if it is (b) then* $\forall S_t \in \{0,1\}$, $I_t = 1$ *and* $\mu_t = \mu_{t-1}$, *if it is (c) then* $\forall S_t \in \{0,1\}$, $I_t = S_t$ *and* $\mu_t = \widetilde{\mu}_t$.

Proposition 8 implies in particular that $\forall t \in \{1,...,N\}$, $\exists i \in \mathbb{Z}$, $\widetilde{\mu}_t = p_i$ and $\mu_t \in \{p_{i-1}, p_i, p_{i+1}\}$, where $p_0 \equiv \mu_0 \in]0;1[$ and, for $i \in \mathbb{N}^*$,

$$p_i \equiv \frac{p_{i-1}\lambda}{p_{i-1}\lambda + (1-p_{i-1})(1-\lambda)} \quad \text{and} \quad p_{-i} \equiv \frac{p_{-i+1}(1-\lambda)}{p_{-i+1}(1-\lambda) + (1-p_{-i+1})\lambda}.$$

In cases (a) and (b) of Proposition 8, herd behavior arises as the result of an informational cascade (Banerjee, 1992; Bikhchandani, Hirshleifer and Welch, 1992):

Definition 2 ***(High and low informational cascades)*** *There is an informational cascade at date $t \in \{1, ..., N\}$ when $\forall S_t \in \{0, 1\}$, $\mu_t = \mu_{t-1}$; (ii) an informational cascade is high when $I_t = 1$ and low when $I_t = 0$.*

In particular, a high cascade corresponds to a situation in which, because a sufficiently large number of past representative entrepreneurs chose to invest in the new technology as they received encouraging private signals about its productivity, the current representative entrepreneur rationally chooses to invest in the new technology too whatever her own private signal.

The existence of informational cascades is linked to the existence of what we call a stock-market bubble:

Definition 3 ***(Stock-market bubble)*** *There is a stock-market bubble at date $t \in \{1, ..., N\}$ when the stock-market index at date t differs from the value that it would have taken if all present and past private signals S_i, $1 \leq i \leq t$, had been public instead of private.*

Indeed, there is a stock-market bubble at date $t \in \{1, ..., N\}$ only if there exists $i \in \{1, ..., t\}$ such that there is an informational cascade at date i. Importantly, whether or not there is a cascade at a given date can be deduced from only the public prior and the monetary policy stance at that date. That is, one does not need to know the entrepreneurs' private signal to infer whether their investment decisions will depend on this signal or not. We therefore have our second main result, namely that the central bank can detect informational cascades, and therefore stock-market bubbles, with certainty in our model, even though it knows less about the productivity of the new technology than each entrepreneur.

4.2 Policy interventions

From (4) and (5), it is easy to check that, whatever $z \geq \overline{z}$, $\mu \in [0; 1]$ and $Q > 0$, there exists a unique $\tau > 0$ such that $q(z, \tau, \mu, 0) = Q$ and there exists a unique $\tau > 0$ such that $q(z, \tau, \mu, 1) = Q$. Let us note

$$\tau^l(z, \mu, \widetilde{\mu}) \equiv \beta^N \left[\alpha A(\overline{z}) - \kappa(\overline{z}) + \frac{\kappa(\overline{z})}{\beta^N}\right] \left[\frac{\mu}{[\alpha A(\overline{z}) - \kappa(z)] \frac{B(z)}{\widetilde{\mu}} + \kappa(\overline{z})} + \frac{1-\mu}{[\alpha A(\overline{z}) - \kappa(\overline{z})] \frac{B(z)}{\widetilde{\mu}} + \kappa(\overline{z})}\right]$$

the unique value of τ such that $q(z, \tau, \mu, 0) = \frac{B(z)}{\widetilde{\mu}}$ and

$$\tau^u(z, \mu, \widetilde{\mu}) \equiv \beta^N \left[\alpha A(\overline{z}) - \kappa(z) + \frac{\kappa(\overline{z})}{\beta^N}\right] \left[\frac{\mu}{[\alpha A(z) - \kappa(z)] \frac{B(z)}{\widetilde{\mu}} + \kappa(z)} + \frac{1-\mu}{[\alpha A(\overline{z}) - \kappa(\overline{z})] \frac{B(z)}{\widetilde{\mu}} + \kappa(z)}\right]$$

the unique value of τ such that $q(z, \tau, \mu, 1) = \frac{B(z)}{\widetilde{\mu}}$. Since $\frac{\partial q(z,\tau,\mu,0)}{\partial \tau} < 0$ and $\frac{\partial q(z,\tau,\mu,1)}{\partial \tau} < 0$ (as implied by Propositions 1 and 2), conditions (a), (b) and (c) of Proposition 8 can be rewritten in the following more policy-oriented form that singles out τ_t: (a) there exists a low cascade at date t if and only if $\tau_t > \tau^l\left(z, \mu_{t-1}, \widetilde{\mu}_t^1\right)$; (b) there exists a high cascade at date t if and only if $\tau_t < \tau^u\left(z, \mu_{t-1}, \widetilde{\mu}_t^0\right)$; (c) there exists no cascade at date t if and only if $\tau^l\left(z, \widetilde{\mu}_t^0, \widetilde{\mu}_t^0\right) < \tau_t < \tau^u\left(z, \widetilde{\mu}_t^1, \widetilde{\mu}_t^1\right)$.

In order to illustrate the mechanism of monetary policy intervention, suppose for a moment that there exists $t \in \{1, ..., N\}$ at which there is a high cascade under laissez-faire, *i.e.* that there exists $t \in \{1, ..., N\}$ such that $\widetilde{\mu}_t^0 q(z, 1, \mu_{t-1}, 1) > B(z)$. Then, as implied by Proposition 8, a necessary condition for the monetary policy intervention τ_t to get rid of the cascade at date t is $\widetilde{\mu}_t^0 q\left(z, \tau_t, \widetilde{\mu}_t^0, 0\right) < B(z)$. Now, it can be shown that $\widetilde{\mu}_t^0 q(z, 1, \mu_{t-1}, 1) > B(z)$ implies $\widetilde{\mu}_t^0 q\left(z, 1, \widetilde{\mu}_t^0, 0\right) > B(z)$.[7] Since $\frac{\partial q(z,\tau,\mu,0)}{\partial \tau} < 0$ (as implied by Proposition 3), $\tau_t > 1$ is therefore a necessary condition for the monetary policy intervention to interrupt the cascade at date t. In other words, monetary policy must be tightened to interrupt a high cascade. This is because monetary policy tightening, by making borrowing dearer for the entrepreneurs, can make them invest in the new technology if and only if they receive an encouraging private signal about its productivity. In doing so, it eliminates the high cascade.

When the conditions on the parameters listed in Proposition 6 are met, Proposition 8 proves the existence of an equilibrium with cascades under conditions (a) or (b). We need to make sure that all these conditions on parameters are not defining an empty set. This proves to be a hard task in general. We therefore prove non-emptiness first analytically in a local case, and second numerically in a numerical case.

4.3 An analytically tractable case

Here we restrict our analysis to a local case in which we can prove analytically the existence of a welfare-improving "leaning against the wind" type of monetary policy. We assume that the functions

$$\begin{array}{c} \mathbb{R}^+ \longrightarrow \mathbb{R}^+ \\ z \longmapsto \kappa(z) \end{array} \quad \text{and} \quad \begin{array}{c} \mathbb{R}^+ \longrightarrow \mathbb{R}^+ \\ z \longmapsto A(z) \end{array}$$

are twice differentiable at point $z = \overline{z}$, with $\left.\frac{d\kappa}{dz}\right|_{z=\overline{z}} > 0$ and $\left.\frac{dA}{dz}\right|_{z=\overline{z}} > 0$. We also assume that z is arbitrarily close to $\overline{z}$ and that τ_t remains arbitrarily close to 1 at dates 1 to N. The latter conditions are necessary and sufficient for q_t, c_t and c_t^e to remain arbitrarily close to their steady-state values for all $t \in \mathbb{N}^*$, all $p_0 \in]0;1[$ and all $(S_1, ..., S_N) \in \{0,1\}^N$. This, in turn, enables us to linearize the model in the neighborhood of its steady state. We also assume, for simplicity, that $N = 3$. We focus on the case examined in the following proposition:

[7] This is a straightforward consequence of Lemma 1 in the appendix.

Proposition 9 ***(Existence of cascades and of arbitrarily small monetary policy interventions eliminating these cascades)*** *There is no cascade at date* 1 *under laissez-faire* ($\tau_1 = 1$)*, there is a high cascade at date* 2 *when* $S_1 = 1$ *under laissez-faire* ($\tau_2 = 1$)*, and there exists a monetary policy intervention* τ_2 *arbitrarily close to* 1 *that ensures the absence of cascade at date* 2 *when* $S_1 = 1$*, if and only if*

$$\frac{\beta^3\left[(1-\alpha)\beta^3 p_0 \left.\frac{d^2A}{dz^2}\right|_{z=\overline{z}} - \left.\frac{d^2\kappa}{dz^2}\right|_{z=\overline{z}}\right]}{2\left(\left.\frac{d\kappa}{dz}\right|_{z=\overline{z}}\right)^2} > \frac{1+\beta^3(1-p_1)+\frac{\alpha}{1-\alpha}\frac{p_1}{p_0}}{\alpha A(\overline{z})-\kappa(\overline{z})} \tag{14}$$

$$\textit{and } B(\overline{z}) = p_0\beta^3, \tag{15}$$

$$\textit{where } B(\overline{z}) \equiv \frac{\left.\frac{d\kappa}{dz}\right|_{z=\overline{z}}}{(1-\alpha)\left.\frac{dA}{dz}\right|_{z=\overline{z}}}.$$

The relevant parameters are now α, β, $\kappa(\overline{z})$, $\left.\frac{d\kappa}{dz}\right|_{z=\overline{z}}$, $\left.\frac{d^2\kappa}{dz^2}\right|_{z=\overline{z}}$, $A(\overline{z})$, $\left.\frac{dA}{dz}\right|_{z=\overline{z}}$, $\left.\frac{d^2A}{dz^2}\right|_{z=\overline{z}}$, p_0, λ, N and $\left.\frac{d\tau_t}{dz}\right|_{z=\overline{z}}$ for $t \in \{1, ..., N\}$. The conditions imposed on these parameters are those corresponding to the conditions listed in Proposition 6, to which should be added the following conditions: $0 < p_0 < 1$, $\frac{1}{2} < \lambda < 1$, $N = 3$, (14) and (15). Because, when z is arbitrarily close to $\overline{z}$ and τ_t remains arbitrarily close to 1 at dates 1 to N, (6) and (13) imply (1), (10), (11) and (12), these conditions are altogether equivalent to the following ones: $0 < \alpha < 1$, $0 < \beta < 1$, $\kappa(\overline{z}) > 0$, $\left.\frac{d\kappa}{dz}\right|_{z=\overline{z}} > 0$, $A(\overline{z}) > 0$, $\left.\frac{dA}{dz}\right|_{z=\overline{z}} > 0$, $0 < p_0 < 1$, $\frac{1}{2} < \lambda < 1$, $N = 3$, (14), (15), $\alpha A(\overline{z}) - \kappa(\overline{z}) > 0$, $p_0\beta^3 < \frac{\alpha}{1-\alpha}$ and

$$\beta^3 - \frac{\kappa(\overline{z})}{\alpha A(\overline{z}) - \kappa(\overline{z})} > \max\left[\frac{\kappa(\overline{z})}{(1-\alpha)A(\overline{z})}, p_0\beta^3\right]. \tag{16}$$

It is easy to see that the set of parameter values satisfying all these conditions is not empty. As a consequence, neither is the set of parameter values satisfying all the conditions imposed in the general case considered in Proposition 6. We therefore have our first and third main results, namely that bubbles (i.e. informational cascades or herd behavior) can occur in this general-equilibrium setting, even though prices are endogenous, and that a modest monetary policy intervention can be enough to interrupt herd behavior in new-tech investment, while not necessarily interrupting new-tech investment itself.

Note that we have obtained our three main results assuming that all competitive entrepreneurs born at the same date receive the same private signal. This assumption is not restrictive, however, in the sense that all three results would still hold if we assumed instead that the signal realizations may differ across entrepreneurs of the same generation. The difficulty, in the latter case, would then be to show the existence of an equilibrium without cascade and characterize it. This is the primary reason why we have chosen, for simplicity, to consider the case of signals that are perfectly correlated across entrepreneurs of the same generation.

Note also that, under laissez-faire, if asked by households at date 2 whether her private signal is good or bad, a newborn entrepreneur would always have the incentive to answer that it is bad, in order to induce households to lend her (and the other entrepreneurs) at a lower interest rate[8]. This cheap-talk effect and the informational cascade prevent households from inferring the entrepreneurs' private signal whether from their words or from their deeds.

We now show that there exists a non-empty subset of parameter values satisfying all these conditions and such that the corresponding sequence of monetary policy interventions, characterized by the policy parameters $\left.\frac{d\tau_1}{dz}\right|_{z=\overline{z}}$, $\left.\frac{d\tau_2}{dz}\right|_{z=\overline{z}}$ and $\left.\frac{d\tau_3}{dz}\right|_{z=\overline{z}}$, is welfare-improving compared to laissez-faire, where the latter is defined as $\left.\frac{d\tau_1}{dz}\right|_{z=\overline{z}} = \left.\frac{d\tau_2}{dz}\right|_{z=\overline{z}} = \left.\frac{d\tau_3}{dz}\right|_{z=\overline{z}} = 0$. To that aim, we consider the following investment-decisions-contingent path of monetary policy interventions: (i) $\left.\frac{d\tau_1}{dz}\right|_{z=\overline{z}} = 0$; (ii) if $I_1 = 0$, then $\left.\frac{d\tau_2}{dz}\right|_{z=\overline{z}} = 0$; (iii) if $I_1 = 1$, then $\left.\frac{d\tau_2}{dz}\right|_{z=\overline{z}} = \min\left\{\left.\frac{d\tau_2}{dz}\right|_{z=\overline{z}}, \text{there is no cascade at date 2}\right\}$; and (iv) $\forall\,(I_1, I_2) \in \{0,1\}^2$, $\left.\frac{d\tau_3}{dz}\right|_{z=\overline{z}} = 0$.

The social welfare criterion that we consider is a weighted sum of the utility of the representative household, the utility of the current representative entrepreneur and the expected utilities of the future representative entrepreneurs:

$$W_t = E_{\Omega(h,t)\bigcup\{S_1=1\}}\left[\frac{\kappa\left(\overline{z}\right) + \beta^3\left[\alpha A\left(\overline{z}\right) - \kappa\left(\overline{z}\right)\right]}{\beta^3} U_t + \sum\nolimits_{k=0}^{+\infty} \beta^k V_{t+k}\right].$$

The weights are chosen such that the locally linearized social welfare criterion is equal to the discounted sum of current and expected future consumptions. With such a welfare function, we can prove the following proposition:

Proposition 10 ***(Welfare-improving monetary policy eliminating cascades)*** *There is a non-empty set of parameters for which monetary policy eliminates cascades and is welfare-improving, but not Pareto-improving, compared to laissez-faire.*

This proposition implies straightforwardly that there exist at least one agent that benefits from the monetary policy intervention and one that does not. As shown in the appendix, the representative entrepreneur born in the period of the monetary policy intervention necessarily loses from this intervention, as she borrows at a higher interest rate. Households and subsequent entrepreneurs may gain from the intervention, however, as they get more information about the true productivity of the new technology.

Note that, even when the particular sequence of monetary policy interventions considered is welfare-improving compared to laissez-faire, there is at least one reason why it might not be the optimal sequence of monetary policy interventions for the welfare criterion that we consider. Indeed, the

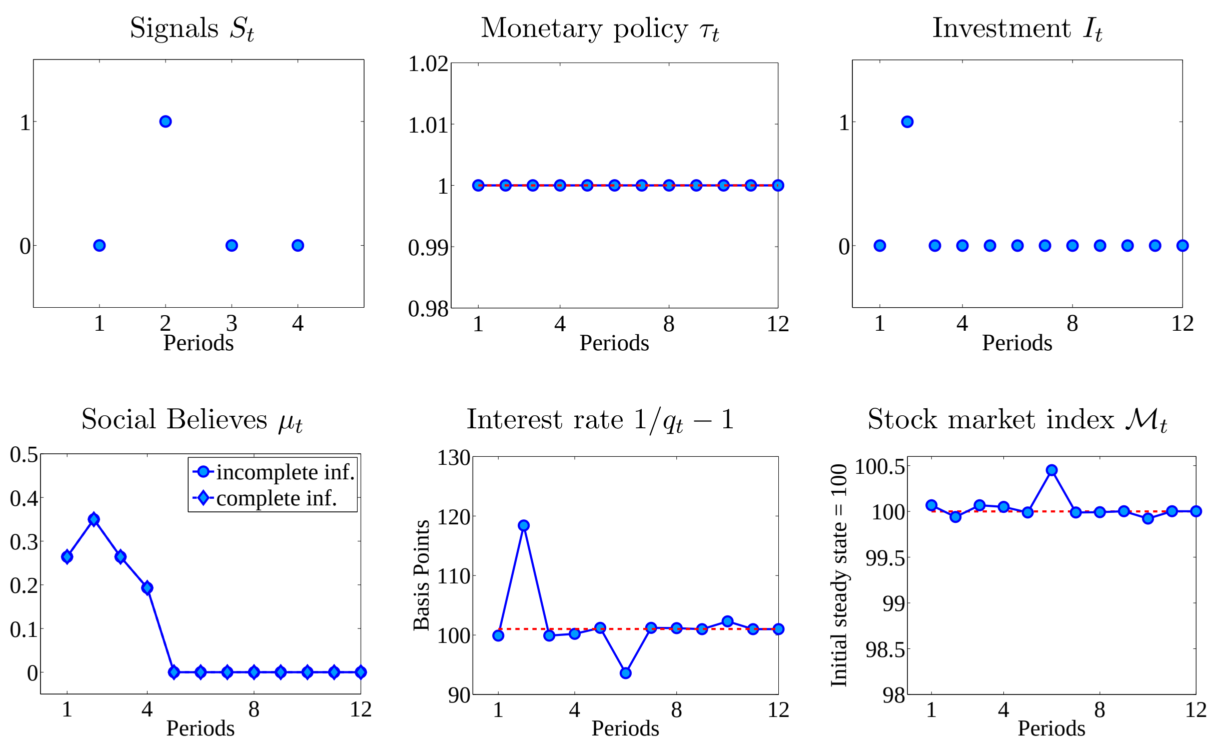
[8]That the interest rate would be lower if households believed her is a consequence of Proposition 2 and condition (9).

optimal monetary policy might set $\left.\frac{d\tau_3}{dz}\right|_{z=\overline{z}}$ higher than zero when $I_1 = I_2 = 1$ in order to eliminate the high cascade at date 3. By making S_3 public, this would not benefit future entrepreneurs, since the true productivity of the new technology is common knowledge from date 4 anyway, but it could benefit the representative household at date 3.

4.4 Numerical simulations

Here we provide a numerical illustration of the model working. Parameters values are as in the previous section, except for $N = 4$. Some new parameters are introduced: the objective probability of success of the new technology is $p = .4$, the informativeness of the signal is $\lambda = .6$, and the initial prior μ_0 is equal to the objective probability p. These parameter values fulfill the conditions imposed above. Without loss of generality for what happens between 1 and 4, it is assumed that the new technology turns out to be a failure at $t = 5$. Figures 3, 4 and 5 show three possible configurations of signals and policies: Figure 3 shows a case with no cascade and no monetary policy, Figure 4 a case with cascades and no monetary policy, and Figure 5 a case with a monetary policy that eliminates cascades.

Figure 3: A simulated path with no policy and no cascade

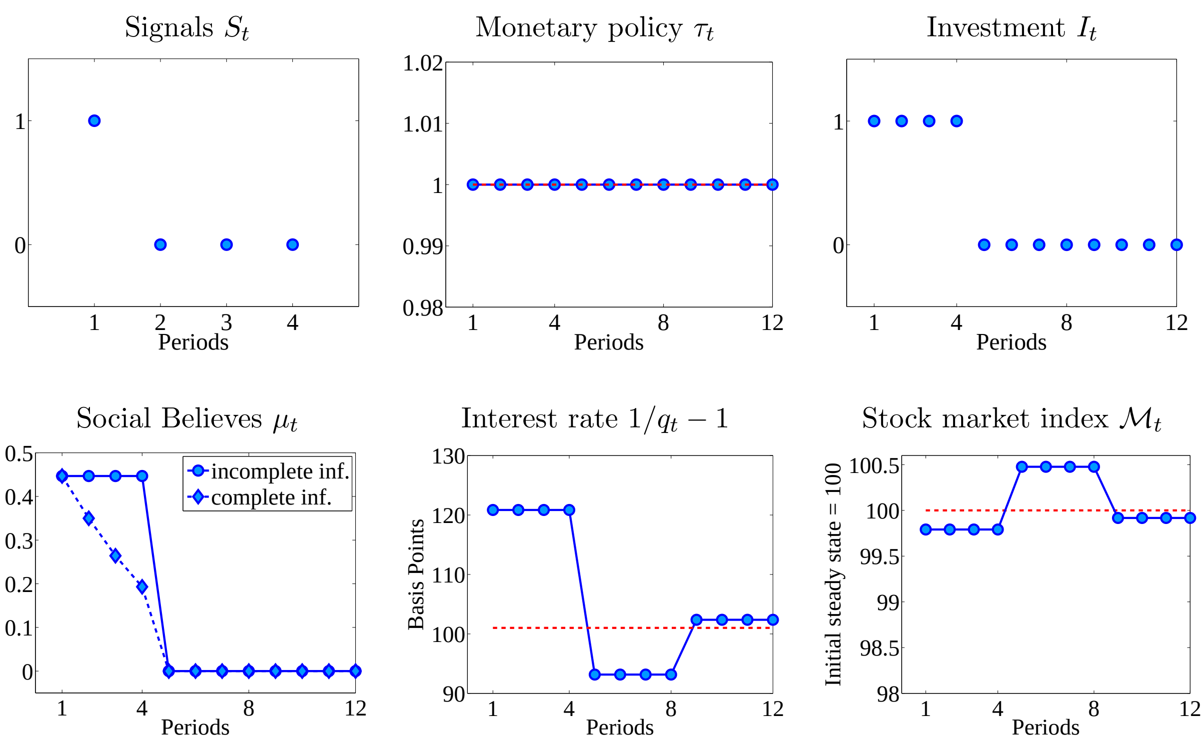


This Figure shows the response of the economy to a sequence of private signals (0 if the signal is bad, 1 if it is good) under a passive monetary policy. The dashed line represents the initial and final steady-state level of the variable.

In Figure 3, the sequence of private signals is $\{bad,\ good,\ bad,\ bad\}$. The fourth panel of this Figure compares social believes μ_t with what we call "complete information" believes, i.e. believes computed with observing the current and past signals. Any difference between the two indicates a cascade at the current date or earlier. Observe that the two series of believes are always superimposed.

In Figures 4 and 5, the sequence is $\{good,\ bad,\ bad,\ bad\}$. Absent monetary policy (Figure 4), the first good signal generates a cascade, which causes a brutal revision of believes in period 5. In Figure 5, monetary policy is tightened in period 2: the interest rate is increased by a rise in τ_2. At this higher interest rate, the entrepreneur of period 2 invests if and only if the signal is good. Social learning is then active.

Figure 4: A simulated path with no policy and a cascade

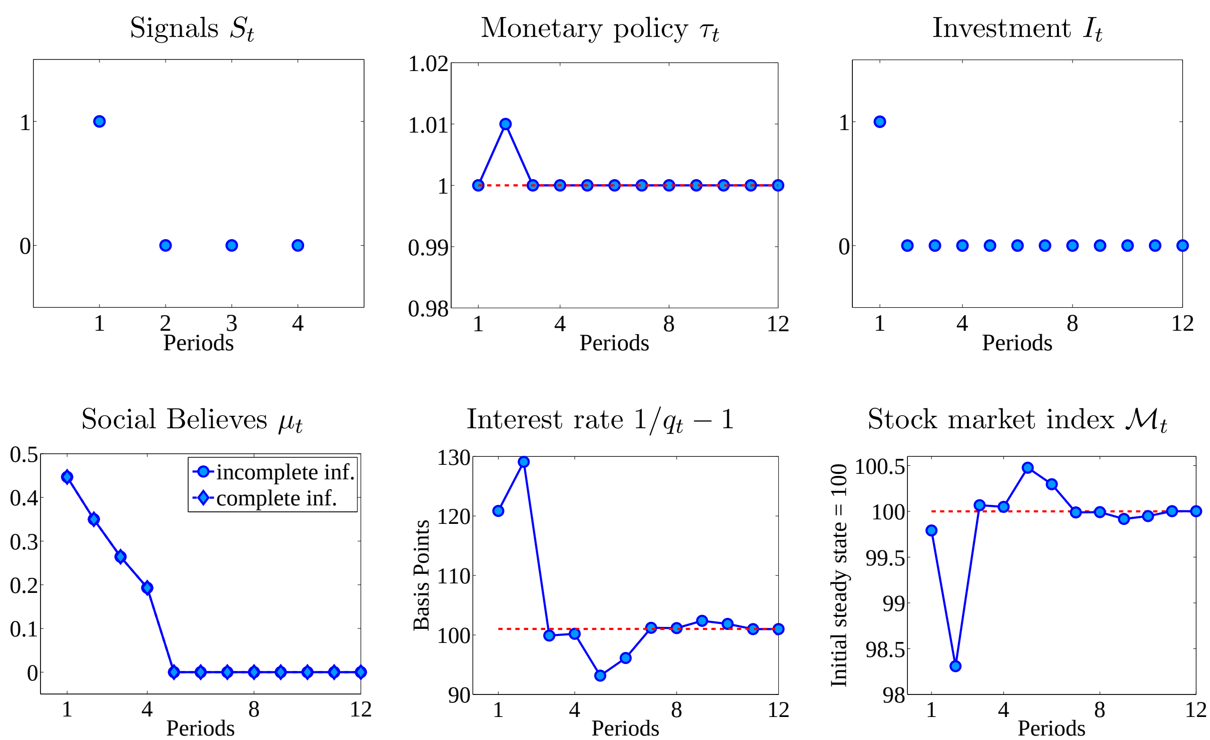


This Figure shows the response of the economy to a sequence of private signals (0 if the signal is bad, 1 if it is good) under a passive monetary policy. The dashed line represents the initial and final steady-state level of the variable.

4.5 Welfare analysis in numerical simulations

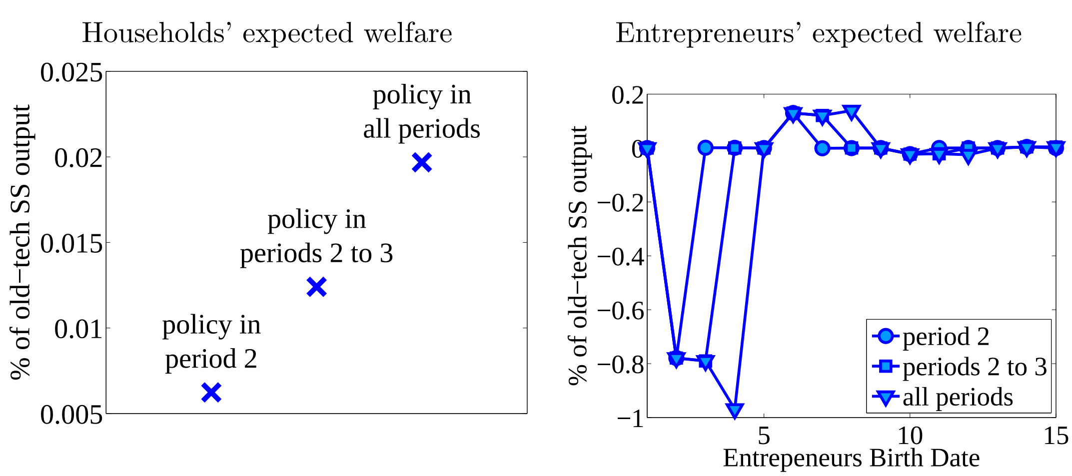
In this subsection, we compute the welfare consequences of a policy that eliminates bubbles. We consider a parametrization similar to the previous one except that the new technology is now only marginally better. The old technology has a TFP $A(\bar{z}) = 1$ and requires an investment of $\kappa(\bar{z}) = .1$

Figure 5: A simulated path with some policy and no cascade



This Figure shows the response of the economy to a sequence of private signals (0 if the signal is bad, 1 if it is good) under an active monetary policy. The dashed line represents the initial and final steady state level of the variable.

Figure 6: Expected welfare for different monetary policies



This Figure shows the expected welfare of households and entrepreneurs for different monetary policies. The expectation is taken over all possible histories of signals and realizations of the productivity of the new technology.

units of goods, while the new technology requires an .1% larger investment ($\kappa(z) = .1001$) and, if successful, delivers a .1% larger TFP ($A(z) = 1.001$). With such a calibration, a small manipulation of the interest rate is enough to prevent the occurrence of a cascade. These parameter values also fulfill the conditions imposed above. We simulate the economy for all possible histories of signals and final realization of the technology (success or failure), and compute the expected utility of households and each generation of entrepreneurs. We then repeat those simulations for different sequences of monetary policy interventions. Parameters are such that there is no cascade in period 1. The first policy follows the minimal monetary policy intervention that prevents cascades in period 2. The second one prevents cascades in periods 2 and 3, and the third one in all periods (periods 2, 3 and 4). For each of these policies, we compute the expected utility of households and each generation of entrepreneurs. The results are presented in Figure 6, where expected utility is expressed in percentage of the initial steady-state output. Households do benefit from a policy that eliminates cascades, and the more so when that policy eliminates cascades in all periods. A policy that eliminates cascades in period t is always detrimental for the entrepreneur of that period, and beneficial for the subsequent ones. In those simulations, households' gains are one order of magnitude smaller than entrepreneurs' losses, so that policies do not increase social welfare.

5 Conclusion

The first contribution of this paper has been to develop a dynamic general-equilibrium model in which informational cascades can occur in equilibrium. In this model, entrepreneurs receive private information about the productivity of a new technology, and invest or not in that new technology, borrowing from households. While entrepreneurs' information is private, entrepreneurs' actions are publicly observable. Because investment is lumpy (invest or not in the new technology), it is not always possible for households and other entrepreneurs to infer private signals from actions. When it is not possible, an informational cascade starts, social learning stops, and investment decisions are characterized by herd behavior. We call such a situation a stock-market bubble.

The second contribution has been to show that monetary policy (defined as manipulation of the real interest rate) can be used to eliminate these stock-market bubbles, even though the central bank has less information than the entrepreneurs about the productivity of the new technology (since, unlike them, it receives no private signal). In some circumstances, even a modest monetary policy intervention can be enough for that matter, and may improve social welfare from an ex ante point of view. These results suggest that, insofar as bubbles in new-tech stock prices can be modeled as the result of herd behavior, the two conditions most commonly stressed by central bankers for the desirability of a monetary policy reaction to a perceived bubble may prove less demanding than they seem at first sight.

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Appendix

A Proof of Proposition 1

Suppose that such an equilibrium exists and note $\overline{c} > 0$ households' constant consumption level at this equilibrium. Then, at this equilibrium, $\forall t \in \mathbb{Z}$, $z_t = \overline{z}$. Indeed, otherwise, if there existed $t \in \mathbb{Z}$ such that $z_t = 0$, then we would get $c_{t+N} = 0 \neq \overline{c}$. Moreover, at this equilibrium, $\forall t \in \mathbb{Z}$, $q_t = \beta^N \equiv \overline{q}$, *i.e.* the N-period interest factor is $R_t = \overline{q}^{-1} = \beta^{-N} \equiv \overline{R}$. The labor market equilibrium condition then implies that, at this equilibrium, $\forall t \in \mathbb{Z}$, $w_t L_t = \alpha A(\overline{z})$ and $\Pi_t = (1-\alpha) A(\overline{z})$, from which we deduce $\overline{c} = \alpha A(\overline{z}) - \kappa(\overline{z}) + \beta^{-N}\kappa(\overline{z})$. Therefore, this equilibrium is the unique equilibrium at which households' consumption level is strictly positive and constant. Moreover, since $\overline{c} > 0$, (2) holds. Finally, the condition that no entrepreneur is willing to deviate from this outcome[9] implies $(1-\alpha) A(\overline{z}) - \beta^{-N}\kappa(\overline{z}) > 0$, so that (1) holds. Conversely, suppose that (2) and (1) hold. Then it is easy to see that the outcome $\forall t \in \mathbb{Z}$, $c_t = \alpha A(\overline{z}) - \kappa(\overline{z}) + \beta^{-N}\kappa(\overline{z})$, $q_t = \beta^N$ and $z_t = \overline{z}$ is an equilibrium. Points (i) and (ii) follow.

Moreover, if $\forall t \in \mathbb{Z}$, $z_t = \overline{z}$, then $\forall t \in \mathbb{Z}$,

$$q_t = \beta^N \frac{\alpha A(\overline{z}) - \kappa(\overline{z}) + \frac{\kappa(\overline{z})}{q_{t-N}}}{\alpha A(\overline{z}) - \kappa(\overline{z}) + \frac{\kappa(\overline{z})}{q_t}}$$

and hence

$$q_t - \beta^N = \frac{-\kappa(\overline{z})}{\alpha A(\overline{z}) - \kappa(\overline{z})} \frac{q_{t-N} - \beta^N}{q_{t-N}},$$

so that there exists a neighborhood of $\overline{q}$ such that any sequence (q_t) originating in this neighborhood will remain in this neighborhood and converge towards $\overline{q}$ if and only if (3) holds. Point (iii) follows.

B Proof of Proposition 2

We first prove part (I) of the proposition. Let us note, for all $z > \overline{z}$ and $\tau_t > 0$,

$$D_0(\tau_t) \equiv \frac{\beta^N}{\tau_t}\left[\alpha A(\overline{z}) - \kappa(\overline{z}) + \frac{\kappa(\overline{z})}{\beta^N}\right], \; F_0(z) \equiv \alpha A(\overline{z}) - \kappa(z), \; G_0 \equiv \alpha A(\overline{z}) - \kappa(\overline{z}) \text{ and } H_0 \equiv \kappa(\overline{z}),$$

so that (4) corresponds to

$$q_t = D_0(\tau_t)\left[\frac{\mu_t}{F_0(z) + \frac{H_0}{q_t}} + \frac{1-\mu_t}{G_0 + \frac{H_0}{q_t}}\right].$$

Note that conditions (2) and (3) together imply $G_0 > 0$.

Suppose first that (4) admits a strictly positive solution q_t for all $\mu_t \in [0;1]$. Then, (4) admits a strictly positive solution q_t in particular for $\mu_t = 0$, which implies that $D_0(\tau_t) > H_0$, and for

[9] Recall that we restrict our analysis to symmetric equilibria across entrepreneurs. This condition ensures that such a symmetric equilibrium exists.

$\mu_t = 1$, which implies that $F_0(z) > 0$ (given that $D_0(\tau_t) > H_0$). These two inequalities correspond to conditions (6) and (7) in Proposition 3. Now suppose conversely that $D_0(\tau_t) > H_0$ and $F_0(z) > 0$. Then, when $\mu_t \in \{0;1\}$, (4) admits a unique solution q_t and this solution is strictly positive. When $\mu_t \notin \{0;1\}$, (4) is equivalent to

$$\Phi_0(z)\, q_t^2 + \Psi_0(z, \tau_t, \mu_t)\, q_t + \Omega_0(\tau_t) = 0$$

where, for all $z > \overline{z}$, all $\tau_t > 0$ and all $\mu_t \in]0;1[$, $\Phi_0(z) \equiv F_0(z)\, G_0$, $\Psi_0(z, \tau_t, \mu_t) \equiv [F_0(z) + G_0]\, H_0 - D_0(\tau_t)\,[G_0\mu_t + F_0(z)(1-\mu_t)]$ and $\Omega_0(\tau_t) \equiv H_0\,[H_0 - D_0(\tau_t)]$. We have: $\forall \mu_t \in]0;1[$, $[\Psi_0(z, \tau_t, \mu_t)]^2 - 4\Phi_0(z)\,\Omega_0(\tau_t) \geq -4\Phi_0(z)\,\Omega_0(\tau_t) > 0$, so that (4) admits two distinct real-number solutions and, since $\frac{\Omega_0(\tau_t)}{\Phi_0(z)} < 0$, one solution is strictly negative and the other strictly positive. Point (i) follows.

From the previous paragraph, we also get that if $D_0(\tau_t) > H_0$ and $F_0(z) > 0$, then (4) admits a unique strictly positive solution q_t for all $\mu_t \in [0;1]$, which we note $q(z, \tau_t, \mu_t, 0)$. When $\mu_t \in]0;1[$, the derivation of $\Phi_0(z)\, q(z, \tau_t, \mu_t, 0)^2 + \Psi_0(z, \tau_t, \mu_t)\, q(z, \tau_t, \mu_t, 0) + \Omega_0(\tau_t) = 0$ with respect to $x \in \{\tau_t, \mu_t\}$ leads to

$$[2\Phi_0(z)\, q(z, \tau_t, \mu_t, 0) + \Psi_0(z, \tau_t, \mu_t)]\, \frac{\partial q(z, \tau_t, \mu_t, 0)}{\partial x} + q(z, \tau_t, \mu_t, 0)\, \frac{\partial \Psi_0(z, \tau_t, \mu_t)}{\partial x} = 0,$$

where $2\Phi_0(z)\, q(z, \tau_t, \mu_t, 0) + \Psi_0(z, \tau_t, \mu_t) > 0$ by definition of $q(z, \tau_t, \mu_t, 0)$. Given that $\frac{\partial \Psi_0(z,\tau_t,\mu_t)}{\partial \tau_t} = \frac{D_0(\tau_t)}{\tau_t}[G_0\mu_t + F_0(z)(1-\mu_t)] > 0$ and $\frac{\partial \Psi_0(z,\tau_t,\mu_t)}{\partial \mu_t} = D_0(\tau_t)\,[F_0(z) - G_0] < 0$, we therefore obtain that $\frac{\partial q(z,\tau_t,\mu_t,0)}{\partial \tau_t} < 0$ and $\frac{\partial q(z,\tau_t,\mu_t,0)}{\partial \mu_t} > 0$ for $\mu_t \in]0;1[$ and by continuity for $\mu_t \in \{0;1\}$ as well. Point (ii) follows.

We now prove part (II). Let us note, for all $z > \overline{z}$ and $\tau_t > 0$,

$$D_1(z, \tau_t) \equiv \frac{\beta^N}{\tau_t}\left[\alpha A(\overline{z}) - \kappa(z) + \frac{\kappa(\overline{z})}{\beta^N}\right],\ F_1(z) \equiv \alpha A(z) - \kappa(z),\ G_1 \equiv \alpha A(\overline{z}) - \kappa(\overline{z}) \text{ and } H_1(z) \equiv \kappa(z),$$

so that (5) can be rewritten as

$$q_t = D_1(z, \tau_t)\left[\frac{\mu_t}{F_1(z) + \frac{H_1(z)}{q_t}} + \frac{1-\mu_t}{G_1 + \frac{H_1(z)}{q_t}}\right].$$

Note that conditions (2) and (3) together imply $G_1 > 0$ and that condition (6) implies $F_1(z) > 0$.

Suppose first that (5) admits a strictly positive solution q_t for all $\mu_t \in [0;1]$. Then, (5) admits a strictly positive solution q_t in particular for $\mu_t = 0$, which implies that $D_1(z, \tau_t) > H_1(z)$. The latter inequality corresponds to condition (8) in Proposition 4. Now suppose conversely that $D_1(z, \tau_t) > H_1(z)$. Then, when $\mu_t \in \{0;1\}$ or $F_1(z) = G_1$, (5) admits a unique solution q_t and this solution is strictly positive. When $\mu_t \notin \{0;1\}$ and $F_1(z) \neq G_1$, (5) is equivalent to

$$\Phi_1(z)\, q_t^2 + \Psi_1(z, \tau_t, \mu_t)\, q_t + \Omega_1(z, \tau_t) = 0$$

where, for all $z > \overline{z}$, all $\tau_t > 0$ and all $\mu_t \in]0;1[$, $\Phi_1(z) \equiv F_1(z) G_1$, $\Psi_1(z, \tau_t, \mu_t) \equiv [F_1(z) + G_1] H_1(z) - D_1(z, \tau_t)[G_1 \mu_t + F_1(z)(1-\mu_t)]$ and $\Omega_1(z, \tau_t) \equiv H_1(z)[H_1(z) - D_1(z, \tau_t)]$. We have: $\forall \mu_t \in]0;1[$, $[\Psi_1(z, \tau_t, \mu_t)]^2 - 4\Phi_1(z)\Omega_1(z, \tau_t) \geq -4\Phi_1(z)\Omega_1(z, \tau_t) > 0$, so that (5) admits two distinct real-number solutions and, since $\frac{\Omega_1(z,\tau_t)}{\Phi_1(z)} < 0$, one solution is strictly negative and the other strictly positive. Point (i) follows.

From the previous paragraph, we also get that if $D_1(z, \tau_t) > H_1(z)$, then (5) admits a unique strictly positive solution q_t for all $\mu_t \in [0;1]$, which we note $q(z, \tau_t, \mu_t, 1)$. When $\mu_t \in]0;1[$, the derivation of $\Phi_1(z) q(z, \tau_t, \mu_t, 1)^2 + \Psi_1(z, \tau_t, \mu_t) q(z, \tau_t, \mu_t, 1) + \Omega_1(z, \tau_t) = 0$ with respect to $x \in \{\tau_t, \mu_t\}$ leads to

$$[2\Phi_1(z) q(z, \tau_t, \mu_t, 1) + \Psi_1(z, \tau_t, \mu_t)] \frac{\partial q(z, \tau_t, \mu_t, 1)}{\partial x} + q(z, \tau_t, \mu_t, 1) \frac{\partial \Psi_1(z, \tau_t, \mu_t)}{\partial x} = 0,$$

where $2\Phi_1(z) q(z, \tau_t, \mu_t, 1) + \Psi_1(z, \tau_t, \mu_t) > 0$ by definition of $q(z, \tau_t, \mu_t, 1)$. Given that $\frac{\partial \Psi_1(z,\tau_t,\mu_t)}{\partial \tau_t} = \frac{D_1(z,\tau_t)}{\tau_t}[G_1 \mu_t + F_1(z)(1-\mu_t)] > 0$ and $\frac{\partial \Psi_1(z,\tau_t,\mu_t)}{\partial \mu_t} = D_1(z, \tau_t)[F_1(z) - G_1] < 0$, we therefore obtain that, for $\mu_t \in]0;1[$ and by continuity for $\mu_t \in \{0;1\}$ as well, $\frac{\partial q(z,\tau_t,\mu_t,1)}{\partial \tau_t} < 0$, $\frac{\partial q(z,\tau_t,\mu_t,1)}{\partial \mu_t} < 0$ if $F_1(z) > G_1$, $\frac{\partial q(z,\tau_t,\mu_t,1)}{\partial \mu_t} = 0$ if $F_1(z) = G_1$, and $\frac{\partial q(z,\tau_t,\mu_t,1)}{\partial \mu_t} > 0$ if $F_1(z) < G_1$. Since inequality $F_1(z) > G_1$ corresponds to condition (9) in Proposition 2, points (ii) and (iii) follow.

C Proof of Proposition 3

The necessary and sufficient condition for a competitive entrepreneur to have no incentive to deviate from the other entrepreneurs' common investment decision and invest nothing is that (a) if this decision is to invest in the new technology and if a competitive entrepreneur has no incentive to deviate from this decision and invest in the old technology, then a competitive entrepreneur has no incentive to deviate from this decision and invest nothing, and (b) if this decision is to invest in the old technology and if a competitive entrepreneur has no incentive to deviate from this decision and invest in the new technology, then a competitive entrepreneur has no incentive to deviate from this decision and invest nothing. Therefore, a competitive entrepreneur has no incentive at date t to deviate from the other entrepreneurs' common investment decision and invest nothing for all $t \in \{1, ..., N\}$, all $(\mu_1, ..., \mu_N) \in [0;1]^N$ and all $(\widetilde{\mu}_1, ..., \widetilde{\mu}_N) \in [0;1]^N$, if and only if (a) $\forall t \in \{1, ..., N\}$, $\forall (\mu_1, ..., \mu_N) \in [0;1]^N$, $\forall (\widetilde{\mu}_1, ..., \widetilde{\mu}_N) \in [0;1]^N$,

$$(1-\alpha) A(\overline{z}) - \frac{\kappa(\overline{z})}{q(z, \tau_t, \mu_t, 0)} > \widetilde{\mu}_t (1-\alpha) A(z) + (1-\widetilde{\mu}_t)(1-\alpha) A(\overline{z}) - \frac{\kappa(z)}{q(z, \tau_t, \mu_t, 0)}$$
$$\Longrightarrow (1-\alpha) A(\overline{z}) - \frac{\kappa(\overline{z})}{q(z, \tau_t, \mu_t, 0)} > 0,$$

and (b) $\forall t \in \{1, ..., N\}$, $\forall (\mu_1, ..., \mu_N) \in [0;1]^N$, $\forall (\widetilde{\mu}_1, ..., \widetilde{\mu}_N) \in [0;1]^N$,

$$\widetilde{\mu}_t (1-\alpha) A(z) + (1-\widetilde{\mu}_t)(1-\alpha) A(\overline{z}) - \frac{\kappa(z)}{q(z, \tau_t, \mu_t, 1)} > (1-\alpha) A(\overline{z}) - \frac{\kappa(\overline{z})}{q(z, \tau_t, \mu_t, 1)}$$
$$\Longrightarrow \widetilde{\mu}_t (1-\alpha) A(z) + (1-\widetilde{\mu}_t)(1-\alpha) A(\overline{z}) - \frac{\kappa(z)}{q(z, \tau_t, \mu_t, 1)} > 0,$$

which is equivalent to (a) $\forall t \in \{1, ..., N\}$, $\forall (\mu_1, ..., \mu_N) \in [0;1]^N$, $\forall (\widetilde{\mu}_1, ..., \widetilde{\mu}_N) \in [0;1]^N$,

$$\widetilde{\mu}_t q(z, \tau_t, \mu_t, 0) < B(z) \Longrightarrow q(z, \tau_t, \mu_t, 0) > \frac{\kappa(\overline{z})}{(1-\alpha) A(\overline{z})},$$

and (b) $\forall t \in \{1, ..., N\}$, $\forall (\mu_1, ..., \mu_N) \in [0;1]^N$, $\forall (\widetilde{\mu}_1, ..., \widetilde{\mu}_N) \in [0;1]^N$,

$$\widetilde{\mu}_t q(z, \tau_t, \mu_t, 1) > B(z) \Longrightarrow q(z, \tau_t, \mu_t, 1) > \frac{\kappa(z)}{\widetilde{\mu}_t (1-\alpha) A(z) + (1-\widetilde{\mu}_t)(1-\alpha) A(\overline{z})},$$

which is in turn equivalent to (a) $\forall t \in \{1, ..., N\}$, $\forall (\mu_1, ..., \mu_N) \in [0;1]^N$,

$$q(z, \tau_t, \mu_t, 0) > \frac{\kappa(\overline{z})}{(1-\alpha) A(\overline{z})},$$

and (b) $\forall t \in \{1, ..., N\}$, $\forall (\mu_1, ..., \mu_N) \in [0;1]^N$,

$$q(z, \tau_t, \mu_t, 1) > B(z) \Longrightarrow q(z, \tau_t, \mu_t, 1) > \frac{\kappa(z)}{\frac{B(z)}{q(z,\tau_t,\mu_t,1)} (1-\alpha) A(z) + \left(1 - \frac{B(z)}{q(z,\tau_t,\mu_t,1)}\right)(1-\alpha) A(\overline{z})},$$

which, given Proposition 2, is in turn equivalent to (a) $\forall t \in \{1, ..., N\}$,

$$q(z, \tau_t, 0, 0) > \frac{\kappa(\overline{z})}{(1-\alpha) A(\overline{z})},$$

and (b) $\forall t \in \{1, ..., N\}$, $\forall (\mu_1, ..., \mu_N) \in [0;1]^N$,

$$q(z, \tau_t, \mu_t, 1) > B(z) \Longrightarrow q(z, \tau_t, \mu_t, 1) > \frac{\kappa(\overline{z})}{(1-\alpha) A(\overline{z})},$$

which, given (4) and Proposition 2, is in turn equivalent to (a) $\forall t \in \{1, ..., N\}$,

$$\tau_t < \frac{\tau(\overline{z})}{1 + \frac{\alpha A(\overline{z}) - \kappa(\overline{z})}{(1-\alpha) A(\overline{z})}},$$

and (b) $\forall t \in \{1, ..., N\}$,

$$\left[q(z, \tau_t, 0, 1) > B(z) \text{ and } B(z) < \frac{\kappa(\overline{z})}{(1-\alpha) A(\overline{z})} \right] \Longrightarrow q(z, \tau_t, 1, 1) > \frac{\kappa(\overline{z})}{(1-\alpha) A(\overline{z})}.$$

Given (5), condition (b) holds if and only if $\forall t \in \{1, ..., N\}$,

$$\text{either } \frac{\tau(z)}{1 + \frac{B(z)[\alpha A(\overline{z}) - \kappa(\overline{z})]}{\kappa(z)}} < \tau_t, \text{ or } B(z) > \frac{\kappa(\overline{z})}{(1-\alpha) A(\overline{z})},$$
$$\text{or } \left[\tau_t < \frac{\tau(z)}{1 + \frac{B(z)[\alpha A(\overline{z}) - \kappa(\overline{z})]}{\kappa(z)}}, \, B(z) < \frac{\kappa(\overline{z})}{(1-\alpha) A(\overline{z})} \text{ and } \tau_t < \frac{\tau(z)}{1 + \frac{\kappa(\overline{z})[\alpha A(z) - \kappa(z)]}{\kappa(z)(1-\alpha) A(\overline{z})}} \right].$$

Now given conditions (6), (9) and $\tau(z) < \tau(\overline{z})$, we have

$$B(z) < \frac{\kappa(\overline{z})}{(1-\alpha)A(\overline{z})} \Longrightarrow \left[\frac{\tau(z)}{1+\frac{\kappa(\overline{z})[\alpha A(z)-\kappa(z)]}{\kappa(z)(1-\alpha)A(\overline{z})}} < \frac{\tau(z)}{1+\frac{B(z)[\alpha A(\overline{z})-\kappa(\overline{z})]}{\kappa(z)}} \text{ and } \frac{\tau(z)}{1+\frac{\kappa(\overline{z})[\alpha A(z)-\kappa(z)]}{\kappa(z)(1-\alpha)A(\overline{z})}} < \frac{\tau(\overline{z})}{1+\frac{\alpha A(\overline{z})-\kappa(\overline{z})}{(1-\alpha)A(\overline{z})}}\right].$$

Proposition 3 follows.

D Proof of Proposition 4

Note that (1) and (6) imply

$$1 < \frac{\tau(\overline{z})}{1+\frac{\alpha A(\overline{z})-\kappa(\overline{z})}{(1-\alpha)A(\overline{z})}},$$

which in turn implies $1 < \tau(\overline{z})$, and that (6) and

$$1 < \frac{\tau(z)}{1+\frac{\kappa(\overline{z})[\alpha A(z)-\kappa(z)]}{\kappa(z)(1-\alpha)A(\overline{z})}}$$

imply $1 < \tau(z)$. Proposition 4 follows.

E Proof of Proposition 5

In this proof, for simplicity, for any pair $(x, x') \in \{\overline{z}, z\} \times \{0, \overline{z}, z\}$ such that $x \neq x'$, by "a competitive entrepreneur has no incentive to deviate from x to x'", we mean that a competitive entrepreneur has no incentive to deviate from the other entrepreneurs' common investment decision $I_t = 0$ (when $x = \overline{z}$) or $I_t = 1$ (when $x = z$), in order to invest in the old technology (when $x' = \overline{z}$) or to invest in the new technology (when $x' = z$) or not to invest (when $x' = 0$).

First, it is straightforward that $\forall t > N$, a competitive entrepreneur has no incentive to deviate from $\overline{z}$ to z when the new technology is bad. Then, we have that $\forall t \in \{N+1, ..., 2N\}$,

$$q_t = \beta^N \frac{\alpha A(\overline{z}) - \kappa(z)}{\alpha A(z) - \kappa(z)} + \frac{1}{\alpha A(z) - \kappa(z)} \left[\frac{\beta^N}{q(z, \tau_{t-N}, \mu_{t-N}, 0)} \kappa(\overline{z}) - \kappa(z)\right]$$

if $z_{t-N} = \overline{z}$ and the new technology is good,

$$q_t = \beta^N + \frac{\kappa(z)}{\alpha A(z) - \kappa(z)} \left[\frac{\beta^N}{q(z, \tau_{t-N}, \mu_{t-N}, 1)} - 1\right]$$

if $z_{t-N} = z$ and the new technology is good,

$$q_t = \beta^N + \frac{\kappa(\overline{z})}{\alpha A(\overline{z}) - \kappa(\overline{z})} \left[\frac{\beta^N}{q(z, \tau_{t-N}, \mu_{t-N}, 0)} - 1\right]$$

if $z_{t-N} = \overline{z}$ and the new technology is bad, and

$$q_t = \beta^N + \frac{1}{\alpha A(\overline{z}) - \kappa(\overline{z})} \left[\frac{\beta^N}{q(z, \tau_{t-N}, \mu_{t-N}, 1)} \kappa(z) - \kappa(\overline{z})\right]$$

if $z_{t-N} = z$ and the new technology is bad. Since τ_{t-N} can be arbitrarily close to zero, $q(z, \tau_{t-N}, \mu_{t-N}, 0)$ and $q(z, \tau_{t-N}, \mu_{t-N}, 1)$ can be arbitrarily large and therefore q_{t-N} can be arbitrarily large in equilibrium. Moreover, since $\widetilde{\mu}_{t-N}$ can be arbitrarily close to zero, we can have in equilibrium both z_{t-N} being equal to $\overline{z}$ and $q_{t-N} = q(z, \tau_{t-N}, \mu_{t-N}, 0)$ being arbitrarily large. As a consequence, $\forall t \in \{N+1, ..., 2N\}$,

$$\inf_{\tau_{t-N}, \mu_{t-N}} q_t = \beta^N \frac{\alpha A(\overline{z}) - \kappa(z)}{\alpha A(z) - \kappa(z)} - \frac{\kappa(z)}{\alpha A(z) - \kappa(z)}$$

if the new technology is good and

$$\inf_{\tau_{t-N}, \mu_{t-N}} q_t = \beta^N - \frac{\kappa(\overline{z})}{\alpha A(\overline{z}) - \kappa(\overline{z})}$$

if the new technology is bad. As a consequence, $\forall t \in \{N+1, ..., 2N\}$, a competitive entrepreneur has no incentive to deviate from $\overline{z}$ to 0 when the new technology is bad, nor from z to 0 or $\overline{z}$ when the new technology is good, if and only if conditions (12) and (13) are met. Moreover, $\forall t > 2N$,

$$q_t = \beta^N + \frac{\beta^N \kappa(z)}{\alpha A(z) - \kappa(z)} \left(\frac{1}{q_{t-N}} - \frac{1}{\beta^N} \right)$$

if the new technology is good, and

$$q_t = \beta^N + \frac{\beta^N \kappa(\overline{z})}{\alpha A(\overline{z}) - \kappa(\overline{z})} \left(\frac{1}{q_{t-N}} - \frac{1}{\beta^N} \right)$$

if the new technology is bad. Therefore, condition (6) and the fact that q is always strictly positive in equilibrium together imply that $\forall t > 2N$,

$$q_t > \beta^N - \frac{\kappa(z)}{\alpha A(z) - \kappa(z)}$$

if the new technology is good, and

$$q_t > \beta^N - \frac{\kappa(\overline{z})}{\alpha A(\overline{z}) - \kappa(\overline{z})}$$

if the new technology is bad. If conditions (12) and (13) are met, then, $\forall t > 2N$,

$$q_t > \max \left[\frac{\kappa(z)}{(1-\alpha) A(z)}, B(z) \right]$$

if the new technology is good, and

$$q_t > \frac{\kappa(\overline{z})}{(1-\alpha) A(\overline{z})}$$

if the new technology is bad. As a consequence, $\forall t > 2N$, a competitive entrepreneur has no incentive to deviate from $\overline{z}$ to 0 when the new technology is bad, nor from z to 0 or $\overline{z}$ when it is good. Proposition 5 follows.

F Proof of Proposition 6

Let us prove point (i). First, there exists no equilibrium such that entrepreneurs choose to invest nothing at some date $t > N$. Indeed, if entrepreneurs chose to invest nothing at some date $t > N$, then q_t would be infinite, so that a competitive entrepreneur would prefer to deviate from the other entrepreneurs' common decision and invest in the old or the new technology.

Second, there exists no equilibrium such that entrepreneurs choose to invest in the new technology at some date $t > N$ when this technology is bad. Indeed, if entrepreneurs chose to invest in the new technology at some date $t > N$ when this technology is bad, then a competitive entrepreneur would prefer to deviate from the other entrepreneurs' common decision and invest in the old technology, as the latter requires less investment and leads to the same productivity.

Third, if there existed an equilibrium such that entrepreneurs choose to invest in the old technology at some date $t > N$ when the new technology is good, then at this equilibrium we would have

$$q_t = \beta^N \frac{\alpha A(z_{t-N}) - \kappa(\overline{z}) + \frac{\kappa(z_{t-N})}{q_{t-N}}}{\alpha A(\overline{z}) - \kappa(z_{t+N}) + \frac{\kappa(\overline{z})}{q_t}},$$

which would then imply

$$\begin{aligned} q_t &= \frac{\beta^N \left[\alpha A(z_{t-N}) - \kappa(\overline{z}) + \frac{\kappa(z_{t-N})}{q_{t-N}}\right] - \kappa(\overline{z})}{\alpha A(\overline{z}) - \kappa(z_{t+N})} \\ &\geq \frac{\beta^N \left[\alpha A(\overline{z}) - \kappa(\overline{z})\right] - \kappa(\overline{z})}{\alpha A(\overline{z}) - \kappa(\overline{z})} = \beta^N - \frac{\kappa(\overline{z})}{\alpha A(\overline{z}) - \kappa(\overline{z})} \end{aligned}$$

since $\frac{\kappa(z_{t-N})}{q_{t-N}} > 0$ in equilibrium. Now, a competitive entrepreneur would prefer to deviate from the other entrepreneurs' common decision and invest in the new technology if and only if

$$(1-\alpha) A(z) - \frac{\kappa(z)}{q_t} > (1-\alpha) A(\overline{z}) - \frac{\kappa(\overline{z})}{q_t},$$

that is to say if and only if $q_t > B(z)$. Therefore, a sufficient condition for the non-existence of an equilibrium such that entrepreneurs choose to invest in the old technology at some date $t > N$ when the new technology is good is

$$\beta^N - \frac{\kappa(\overline{z})}{\alpha A(\overline{z}) - \kappa(\overline{z})} > B(z).$$

Now, the latter condition is met since (13) implies

$$\beta^N \frac{\alpha A(\overline{z}) - \kappa(z)}{\alpha A(z) - \kappa(z)} - \frac{\kappa(z)}{\alpha A(z) - \kappa(z)} > B(z),$$

which, given (6), implies in turn

$$\begin{aligned} \beta^N &> \frac{\kappa(z)}{\alpha A(\overline{z}) - \kappa(z)} + B(z) \frac{\alpha A(z) - \kappa(z)}{\alpha A(\overline{z}) - \kappa(z)} \\ &> \frac{\kappa(\overline{z})}{\alpha A(\overline{z}) - \kappa(\overline{z})} + B(z). \end{aligned}$$

Point (i) follows.

Let us now prove point (ii). Proposition 2 implies that, $\forall t \in \{1, ..., N\}$, $q_t > 0$. Moreover, as shown in Appendix E, $\forall t \in \{N+1, ..., 2N\}$,

$$q_t > \beta^N \frac{\alpha A(\overline{z}) - \kappa(z)}{\alpha A(z) - \kappa(z)} - \frac{\kappa(z)}{\alpha A(z) - \kappa(z)}$$

if the new technology is good, and

$$q_t > \beta^N - \frac{\kappa(\overline{z})}{\alpha A(\overline{z}) - \kappa(\overline{z})}$$

if the new technology is bad. Therefore, conditions (12) and (13) imply that $\forall t \in \{N+1, ..., 2N\}$, $q_t > 0$. Finally, using the equations

$$q_t = \beta^N + \frac{\beta^N \kappa(z)}{\alpha A(z) - \kappa(z)} \left(\frac{1}{q_{t-N}} - \frac{1}{\beta^N} \right)$$

if the new technology is good, and

$$q_t = \beta^N + \frac{\beta^N \kappa(\overline{z})}{\alpha A(\overline{z}) - \kappa(\overline{z})} \left(\frac{1}{q_{t-N}} - \frac{1}{\beta^N} \right)$$

if the new technology is bad, which hold $\forall t > 2N$, we get by recurrence that $\forall t > 2N$, $q_t > 0$. To sum up, we get that $\forall t \geq 1$, $q_t > 0$. Together with (6), this implies in turn that $\forall t \geq 1$, $c_t > 0$. Besides, condition (1) and Propositions 3 and 5 imply that $\forall t \geq 1$, $c_t^e > 0$. Point (ii) follows.

Let us finally prove point (iii). If the new technology is good, then $\forall t > 3N$,

$$q_t - \beta^N = \frac{[\kappa(z)]^2}{[\alpha A(z) - \kappa(z)]^2 q_{t-N} q_{t-2N}} \left(q_{t-2N} - \beta^N \right).$$

Using the results: (a) $\forall t > N$, $q_t > 0$; (b) $\forall t > 2N$,

$$q_t - \beta^N = \frac{\beta^N \kappa(z)}{\alpha A(z) - \kappa(z)} \left(\frac{1}{q_{t-N}} - \frac{1}{\beta^N} \right);$$

and (c)

$$\beta^N - \frac{\kappa(z)}{\alpha A(z) - \kappa(z)} > 0,$$

which follows from (13), we get that $\forall t > 3N$,

$$\begin{aligned} & \left[\beta^N - \frac{\kappa(z)}{\alpha A(z) - \kappa(z)} \right] \left[q_{t-2N} + \frac{\kappa(z)}{\alpha A(z) - \kappa(z)} \right] > 0 \\ \Longrightarrow & \left[\beta^N - \frac{\kappa(z)}{\alpha A(z) - \kappa(z)} \right] q_{t-2N} + \frac{\beta^N \kappa(z)}{\alpha A(z) - \kappa(z)} > \frac{[\kappa(z)]^2}{[\alpha A(z) - \kappa(z)]^2} \\ \Longrightarrow & \, q_{t-N} q_{t-2N} > \frac{[\kappa(z)]^2}{[\alpha A(z) - \kappa(z)]^2} \\ \Longrightarrow & \left| \frac{[\kappa(z)]^2}{[\alpha A(z) - \kappa(z)]^2 q_{t-N} q_{t-2N}} \right| < 1, \end{aligned}$$

from which we conclude that $\lim_{t \longrightarrow +\infty} q_t = \beta^N$. Alternatively, if the new technology is bad, then $\forall t > 3N$,

$$q_t - \beta^N = \frac{\left[\kappa\left(\overline{z}\right)\right]^2}{\left[\alpha A\left(\overline{z}\right) - \kappa\left(\overline{z}\right)\right]^2 q_{t-N} q_{t-2N}} \left(q_{t-2N} - \beta^N\right).$$

Using the results: (a) $\forall t > N$, $q_t > 0$; (b) $\forall t > 2N$,

$$q_t - \beta^N = \frac{\beta^N \kappa\left(\overline{z}\right)}{\alpha A\left(\overline{z}\right) - \kappa\left(\overline{z}\right)} \left(\frac{1}{q_{t-N}} - \frac{1}{\beta^N}\right);$$

and (c)

$$\beta^N - \frac{\kappa\left(\overline{z}\right)}{\alpha A\left(\overline{z}\right) - \kappa\left(\overline{z}\right)} > 0,$$

which follows from (12), we get that $\forall t > 3N$,

$$\begin{aligned}
&\left[\beta^N - \frac{\kappa\left(\overline{z}\right)}{\alpha A\left(\overline{z}\right) - \kappa\left(\overline{z}\right)}\right] \left[q_{t-2N} + \frac{\kappa\left(\overline{z}\right)}{\alpha A\left(\overline{z}\right) - \kappa\left(\overline{z}\right)}\right] > 0 \\
\Longrightarrow &\left[\beta^N - \frac{\kappa\left(\overline{z}\right)}{\alpha A\left(\overline{z}\right) - \kappa\left(\overline{z}\right)}\right] q_{t-2N} + \frac{\beta^N \kappa\left(\overline{z}\right)}{\alpha A\left(\overline{z}\right) - \kappa\left(\overline{z}\right)} > \frac{\left[\kappa\left(\overline{z}\right)\right]^2}{\left[\alpha A\left(\overline{z}\right) - \kappa\left(\overline{z}\right)\right]^2} \\
\Longrightarrow &\ q_{t-N} q_{t-2N} > \frac{\left[\kappa\left(\overline{z}\right)\right]^2}{\left[\alpha A\left(\overline{z}\right) - \kappa\left(\overline{z}\right)\right]^2} \\
\Longrightarrow &\left|\frac{\left[\kappa\left(\overline{z}\right)\right]^2}{\left[\alpha A\left(\overline{z}\right) - \kappa\left(\overline{z}\right)\right]^2 q_{t-N} q_{t-2N}}\right| < 1,
\end{aligned}$$

from which we conclude that $\lim_{t \longrightarrow +\infty} q_t = \beta^N$. To sum up, we get that $\lim_{t \longrightarrow +\infty} q_t = \beta^N$ whether the new technology is good or bad. Then, since $\forall t > 2N$,

$$\begin{aligned}
\left(c_t, c_t^e\right) &= \left(\alpha A\left(z\right) - \kappa\left(z\right) + q_{t-N}^{-1} \kappa\left(z\right), \left(1-\alpha\right) A\left(z\right) - q_{t-N}^{-1} \kappa\left(z\right)\right) \text{ if the new technology is good,} \\
\left(c_t, c_t^e\right) &= \left(\alpha A\left(\overline{z}\right) - \kappa\left(\overline{z}\right) + q_{t-N}^{-1} \kappa\left(\overline{z}\right), \left(1-\alpha\right) A\left(\overline{z}\right) - q_{t-N}^{-1} \kappa\left(\overline{z}\right)\right) \text{ if the new technology is bad,}
\end{aligned}$$

we get that $\lim_{t \longrightarrow +\infty} \left(c_t, c_t^e\right) = \left(\alpha A\left(z\right) - \kappa\left(z\right) + \beta^{-N} \kappa\left(z\right), \left(1-\alpha\right) A\left(z\right) - \beta^{-N} \kappa\left(z\right)\right)$ if the new technology is good and $\lim_{t \longrightarrow +\infty} \left(c_t, c_t^e\right) = \left(\alpha A\left(\overline{z}\right) - \kappa\left(\overline{z}\right) + \beta^{-N} \kappa\left(\overline{z}\right), \left(1-\alpha\right) A\left(\overline{z}\right) - \beta^{-N} \kappa\left(\overline{z}\right)\right)$ if it is bad. Point (iii) follows.

G Proof of Proposition 7

(6) and (13) together imply that $\beta^N > B\left(z\right)$ and hence that $\left(1-\alpha\right) A\left(z\right) - \frac{\kappa(z)}{\beta^N} > \left(1-\alpha\right) A\left(\overline{z}\right) - \frac{\kappa(\overline{z})}{\beta^N}$, so that entrepreneurs' welfare is higher in the long term when the new technology is good than it is initially. Moreover, (9) implies that $\alpha A\left(z\right) - \kappa\left(z\right) + \frac{\kappa(z)}{\beta^N} > \alpha A\left(\overline{z}\right) - \kappa\left(\overline{z}\right) + \frac{\kappa(\overline{z})}{\beta^N}$, so that households' welfare is also higher in the long term when the new technology is good than it is initially. Proposition 7 follows.

H Proof of Proposition 8

For each $t \in \{1, ..., N\}$, let μ_t^0 denote the value taken by μ_t when $I_t = 0$ and μ_t^1 the value taken by μ_t when $I_t = 1$. Since entrepreneurs take the interest rate as given when deciding in which technology to invest, $I_t = 0$ is supported by an equilibrium only if

$$(1-\alpha) A(\overline{z}) - \frac{\kappa(\overline{z})}{q(z, \tau_t, \mu_t^0, 0)} > \widetilde{\mu}_t \left[(1-\alpha) A(z) - \frac{\kappa(z)}{q(z, \tau_t, \mu_t^0, 0)} \right] + (1-\widetilde{\mu}_t) \left[(1-\alpha) A(\overline{z}) - \frac{\kappa(z)}{q(z, \tau_t, \mu_t^0, 0)} \right],$$

that is to say only if

$$\widetilde{\mu}_t q(z, \tau_t, \mu_t^0, 0) < B(z). \tag{17}$$

Similarly, $I_t = 1$ is supported by an equilibrium only if

$$(1-\alpha) A(\overline{z}) - \frac{\kappa(\overline{z})}{q(z, \tau_t, \mu_t^1, 1)} < \widetilde{\mu}_t \left[(1-\alpha) A(z) - \frac{\kappa(z)}{q(z, \tau_t, \mu_t^1, 1)} \right] + (1-\widetilde{\mu}_t) \left[(1-\alpha) A(\overline{z}) - \frac{\kappa(z)}{q(z, \tau_t, \mu_t^1, 1)} \right],$$

that is to say only if

$$\widetilde{\mu}_t q(z, \tau_t, \mu_t^1, 1) > B(z). \tag{18}$$

Here we introduce a lemma showing that, under the conditions considered in Proposition 6, the interest rate maximized over $\mu_t \in [0;1]$ that prevails when the entrepreneurs borrow little (as they invest in the old technology) is strictly lower than the interest rate minimized over $\mu_t \in [0;1]$ that prevails when the entrepreneurs borrow much (as they invest in the new technology):

Lemma 1 *If (1), (2), (3), (6), (7), (8) and (9) hold, then:* $\forall t \in \{1, ..., N\}$, $\forall (p, p') \in [0;1]^2$, $q(z, \tau_t, p, 0) > q(z, \tau_t, p', 1)$.

The proof is presented in an online appendix. This Lemma implies that $\forall (\mu_t^0, \mu_t^1) \in [0;1]^2$, conditions (17) and (18) cannot hold for the same values of the parameters. This implies that at most one of the following four cases can occur in equilibrium at each date $t \in \{1, ..., N\}$: $S_t = 0 \Longrightarrow I_t = 0$ and $S_t = 1 \Longrightarrow I_t = 0$ (case a), $S_t = 0 \Longrightarrow I_t = 1$ and $S_t = 1 \Longrightarrow I_t = 1$ (case b), $S_t = 0 \Longrightarrow I_t = 0$ and $S_t = 1 \Longrightarrow I_t = 1$ (case c), $S_t = 0 \Longrightarrow I_t = 1$ and $S_t = 1 \Longrightarrow I_t = 0$ (case d).

Note first that case d is in fact impossible, as it would require $\widetilde{\mu}_t^0 q(z, \tau_t, \mu_t^1, 1) > B(z)$ and $\widetilde{\mu}_t^1 q(z, \tau_t, \mu_t^0, 0) < B(z)$, where $\widetilde{\mu}_t^0 < \widetilde{\mu}_t^1$, which contradicts Lemma 1. Note then that cases a and b both lead to $\mu_t = \mu_{t-1}$, while case c leads to $\mu_t = \widetilde{\mu}_t$. As a consequence, case a is supported by an equilibrium if and only if $\widetilde{\mu}_t^0 q\left(z, \tau_t, \mu_{t-1}, 0\right) < B\left(z\right)$ and $\widetilde{\mu}_t^1 q\left(z, \tau_t, \mu_{t-1}, 0\right) < B\left(z\right)$, that is to say if and only if

$$\widetilde{\mu}_t^1 q\left(z, \tau_t, \mu_{t-1}, 0\right) < B\left(z\right); \tag{19}$$

case b is supported by an equilibrium if and only if $\widetilde{\mu}_t^0 q\left(z, \tau_t, \mu_{t-1}, 1\right) > B\left(z\right)$ and $\widetilde{\mu}_t^1 q\left(z, \tau_t, \mu_{t-1}, 1\right) > B\left(z\right)$, that is to say if and only if

$$\widetilde{\mu}_t^0 q\left(z, \tau_t, \mu_{t-1}, 1\right) > B\left(z\right); \tag{20}$$

and case c is supported by an equilibrium if and only if

$$\widetilde{\mu}_t^0 q\left(z, \tau_t, \widetilde{\mu}_t^0, 0\right) < B\left(z\right) \text{ and } \widetilde{\mu}_t^1 q\left(z, \tau_t, \widetilde{\mu}_t^1, 1\right) > B\left(z\right). \tag{21}$$

Given that $\widetilde{\mu}_t^0 < \widetilde{\mu}_t^1$, Lemma 1 implies that at most one of the three conditions (19), (20) and (21) holds for some given values of the parameters. Proposition 8 follows.

I Proof of Proposition 9

Given Proposition 8, there is a high cascade at date 2 when $S_1 = 1$ under Laissez-faire ($\tau_2 = 1$) if and only if $p_0 q\left(z, 1, p_1, 1\right) > B\left(z\right)$. Moreover, since z is arbitrarily close to $\overline{z}$, $q\left(z, 1, p_1, 1\right)$ and $B\left(z\right)$ are arbitrarily close to β^3 and $B\left(\overline{z}\right)$ respectively. As a consequence, there is a high cascade at date 2 when $S_1 = 1$ under Laissez-faire ($\tau_2 = 1$) and there exists a monetary policy intervention τ_2 arbitrarily close to 1 that ensures the absence of cascade at date 2 when $S_1 = 1$ if and only if

$$p_0 \beta^3 = B\left(\overline{z}\right)$$

and

$$p_0 \left.\frac{\partial q\left(z, 1, p_1, 1\right)}{\partial z}\right|_{z=\overline{z}} > \left.\frac{dB}{dz}\right|_{z=\overline{z}}, \tag{22}$$

where the first of these two conditions correspond to (15). The partial derivative of (5) at date 2 for $\tau_2 = 1$ and $\mu_2 = p_1$ with respect to z, taken at point $z = \overline{z}$, and the use of (15) lead to

$$\left.\frac{\partial q\left(z, 1, p_1, 1\right)}{\partial z}\right|_{z=\overline{z}} = -\frac{\left[1 + \beta^3\left(1 - p_1\right)\right] + \frac{\alpha p_1}{(1-\alpha) p_0}}{\alpha A\left(\overline{z}\right) - \kappa\left(\overline{z}\right)} \left.\frac{d\kappa}{dz}\right|_{z=\overline{z}}.$$

Besides, using (15), we also get

$$\left.\frac{dB}{dz}\right|_{z=\overline{z}} = \frac{\beta^3 p_0 \left[\left.\frac{d^2\kappa}{dz^2}\right|_{z=\overline{z}} - \left(1 - \alpha\right)\beta^3 p_0 \left.\frac{d^2 A}{dz^2}\right|_{z=\overline{z}}\right]}{2 \left.\frac{d\kappa}{dz}\right|_{z=\overline{z}}}. \tag{23}$$

These last two results can then be used to rewrite (22) as (14). Therefore, there is a high cascade at date 2 when $S_1 = 1$ under Laissez-faire ($\tau_2 = 1$) and there exists a monetary policy intervention τ_2

arbitrarily close to 1 that ensures the absence of cascade at date 2 when $S_1 = 1$ if and only if (14) and (15) hold. Now, given Proposition 8, there is no cascade at date 1 if and only if

$$\begin{aligned} p_{-1} q\left(z, 1, p_{-1}, 0\right) &< B\left(z\right), \\ \text{and } p_1 q\left(z, 1, p_1, 1\right) &> B\left(z\right). \end{aligned}$$

If (15) holds, then these two conditions hold as well, since $p_{-1} < p_0 < p_1$ and $q\left(z, 1, p_{-1}, 0\right)$, $q\left(z, 1, p_1, 1\right)$ and $B\left(z\right)$ are arbitrarily close to β^3, β^3 and $B\left(\bar{z}\right)$ respectively. Proposition 9 follows.

J Proof of Proposition 10

Let us note $\widehat{V}_t \equiv \sum_{k=0}^{+\infty} \beta^k V_{t+k}$, and let $\left(U_t^{LF}\left(z\right), \widehat{V}_t^{LF}\left(z\right), W_t^{LF}\left(z\right)\right)$ and $\left(U_t^{I}\left(z\right), \widehat{V}_t^{I}\left(z\right), W_t^{I}\left(z\right)\right)$ denote the values taken by

$$\left(E\left\{U_t \middle| S_1 = 1\right\}, E\left\{\widehat{V}_t \middle| S_1 = 1\right\}, W_t\right)$$

respectively under Laissez-faire and under the intervention considered. We first obtain the following Lemma:

Lemma 2 *The welfare effects of Laissez-faire are characterized by*

$$\begin{aligned} \left.\frac{dU_1^{LF}}{dz}\right|_{z=\bar{z}} &> 0, \\ \left.\frac{d\widehat{V}_1^{LF}}{dz}\right|_{z=\bar{z}} &> 0 \text{ if } \left(p_0, \lambda\right) \text{ is sufficiently close to } \left(0, 1\right), \\ \left.\frac{d\widehat{V}_1^{LF}}{dz}\right|_{z=\bar{z}} &< 0 \text{ if } p_0 \text{ is sufficiently close to } 1, \\ \left.\frac{dW_1^{LF}}{dz}\right|_{z=\bar{z}} &= \frac{\left.\frac{d\kappa}{dz}\right|_{z=\bar{z}}}{1-\beta}\left[\frac{p_1}{\left(1-\alpha\right)p_0} - 1 + \beta^3\left(1-p_1\right)\right] > 0. \end{aligned}$$

The proof is presented in an online appendix. It is worth noting in particular that we can get $\left.\frac{d\widehat{V}_1^{LF}}{dz}\right|_{z=\bar{z}} < 0$ even though each entrepreneur individually gains from investing in the new technology. There are at least two possible reasons for this result. First, the existence of overlapping generations of entrepreneurs may create a negative externality. Indeed, at each date $t \in \mathbb{N}^*$, new-born entrepreneurs do not internalize the possible costs, in terms of interest-rate fluctuations, that their investment decision imposes on the entrepreneurs born at date $t - N$ and on those born at date $t + N$. Second, our simplifying discrete-choice assumption and our focus on symmetric equilibria may play a role. Indeed, if there were only one entrepreneur *per* generation, then she would choose between borrowing little at a low rate or borrowing much at a high rate, and might prefer to borrow little at a low rate.

But there are many of them, so that each of them, taking the interest rate as given, has either to choose between borrowing little or much at a low rate, or to choose between borrowing little or much at a high rate. If in both cases she prefers to borrow much, then the only symmetric equilibrium is that all entrepreneurs borrow much at a high rate.

Now let p_A denote the probability of receiving a signal $S_2 = 1$ conditionally on $S_1 = 1$, and p_B the probability of receiving a signal $S_3 = 1$ conditionally on $S_1 = 1$ and $S_2 = 0$, *i.e.* $p_A = p_1\lambda + (1-p_1)(1-\lambda)$ and $p_B = p_0\lambda + (1-p_0)(1-\lambda)$. We then obtain the following Lemma:

Lemma 3 *The welfare effects of the intervention considered are characterized by*

$$\begin{aligned}
\left.\frac{dU_1^I}{dz}\right|_{z=\overline{z}} &> 0, \\
\left.\frac{d\widehat{V}_1^I}{dz}\right|_{z=\overline{z}} &> 0 \text{ if } (p_0,\lambda) \text{ is sufficiently close to } (0,1), \\
\left.\frac{d\widehat{V}_1^I}{dz}\right|_{z=\overline{z}} &< 0 \text{ if } p_0 \text{ is sufficiently close to } 1, \\
\left.\frac{dW_1^I}{dz}\right|_{z=\overline{z}} &= \left\{\left[\frac{p_1}{(1-\alpha)p_0}-1\right]+\frac{\beta^3}{1-\beta}p_1\left[\frac{1}{(1-\alpha)p_0}-1\right]\right. \\
&\left. +\beta(1+\beta)p_A\left[\frac{p_2}{(1-\alpha)p_0}-1\right]+\beta^2(1-p_A)p_B\left[\frac{p_1}{(1-\alpha)p_0}-1\right]\right\}\left.\frac{d\kappa}{dz}\right|_{z=\overline{z}} \\
&> 0.
\end{aligned}$$

The proof is presented in an online appendix. The reasons why we can get $\left.\frac{d\widehat{V}_1^I}{dz}\right|_{z=\overline{z}} < 0$ are pretty much the same as under laissez-faire. We can then examine whether the intervention considered is welfare-improving compared to Laissez-faire by computing

$$\left.\frac{dW_1^I}{dz}\right|_{z=\overline{z}} - \left.\frac{dW_1^{LF}}{dz}\right|_{z=\overline{z}} = \frac{\beta(1-p_A)}{1-\alpha}\left[\beta(1-p_0)(2\lambda-1)-\alpha\left[1+\beta(1-p_B)\right]\right]\left.\frac{d\kappa}{dz}\right|_{z=\overline{z}},$$

$$\begin{aligned}
\left.\frac{dU_1^I}{dz}\right|_{z=\overline{z}} - \left.\frac{dU_1^{LF}}{dz}\right|_{z=\overline{z}} &= \frac{\beta^3 \left.\frac{d\kappa}{dz}\right|_{z=\overline{z}}}{\kappa(\overline{z})+\beta^3\left[\alpha A(\overline{z})-\kappa(\overline{z})\right]}\left\{\frac{1-p_A}{(1-\alpha)\beta}\left[\frac{p_B(p_1-p_0)}{p_0}\left[\alpha\beta^3+\frac{(1-\beta^3)\kappa(\overline{z})}{A(\overline{z})}\right]\right.\right. \\
&\left. -\frac{1+\beta(1-p_B)}{\beta}\left[\alpha\beta^3+\frac{\kappa(\overline{z})}{\alpha A(\overline{z})}\left[(1-\alpha)+\alpha\left(1-\beta^3\right)\right]\right]\right]+\frac{\beta p_0\kappa(\overline{z})}{\alpha A(\overline{z})} \\
&\left. +\frac{\beta\kappa(\overline{z})\left[\alpha A(\overline{z})-\kappa(\overline{z})\right]}{2\alpha A(\overline{z})}\left[\frac{(1-\alpha)\beta^3 p_0 \left.\frac{d^2A}{dz^2}\right|_{z=\overline{z}} - \left.\frac{d^2\kappa}{dz^2}\right|_{z=\overline{z}}}{\left(\left.\frac{d\kappa}{dz}\right|_{z=\overline{z}}\right)^2}\right]\right\},
\end{aligned}$$

$$\left.\frac{d\widehat{V}_1^I}{dz}\right|_{z=\overline{z}}-\left.\frac{d\widehat{V}_1^{LF}}{dz}\right|_{z=\overline{z}} = \left.\frac{d\kappa}{dz}\right|_{z=\overline{z}}\left\{\frac{1-p_A}{(1-\alpha)\beta}\left[\frac{p_B(p_1-p_0)}{p_0}\left[(1-\alpha)\beta^3-\frac{\left(1-\beta^3\right)\kappa(\overline{z})}{A(\overline{z})}\right]\right.\right.$$
$$\left.+\frac{1+\beta(1-p_B)}{\beta}\frac{\kappa(\overline{z})}{\alpha A(\overline{z})}\left[(1-\alpha)+\alpha\left(1-\beta^3\right)\right]\right]-\frac{\beta p_0\kappa(\overline{z})}{\alpha A(\overline{z})}$$
$$\left.-\frac{\beta\kappa(\overline{z})\left[\alpha A(\overline{z})-\kappa(\overline{z})\right]}{2\alpha A(\overline{z})}\left[\frac{(1-\alpha)\beta^3 p_0\left.\frac{d^2A}{dz^2}\right|_{z=\overline{z}}-\left.\frac{d^2\kappa}{dz^2}\right|_{z=\overline{z}}}{\left(\left.\frac{d\kappa}{dz}\right|_{z=\overline{z}}\right)^2}\right]\right\}.$$

It is clear that there exist some parameter values satisfying all the conditions listed above and such that p_0 is arbitrarily close to zero, λ is arbitrarily close to one, and $\alpha < \frac{\beta}{1+\beta}$. These results imply that, for those parameter values, the sequence of monetary policy interventions considered increases social welfare W_t relatively to laissez-faire.

Now, using

$$V_2^{LF} = \beta^4\left[p_1(1-\alpha)A(z)+(1-p_1)(1-\alpha)A(\overline{z})-\frac{\kappa(z)}{q(z,1,p_1,1)}\right],$$
$$V_2^I = \beta^4\left[p_Ap_2(1-\alpha)A(z)+(1-\alpha)A(\overline{z})\left[p_A(1-p_2)+(1-p_A)\right]\right.$$
$$\left.-p_A\frac{\kappa(z)}{q(z,\tau_2(z),p_2,1)}-(1-p_A)\frac{\kappa(\overline{z})}{q(z,\tau_2(z),p_0,0)}\right],$$

we obtain

$$\left.\frac{\partial V_2^I}{\partial z}\right|_{z=\overline{z}}-\left.\frac{\partial V_2^F}{\partial z}\right|_{z=\overline{z}} = \beta^4\left\{\underbrace{\frac{\kappa(\overline{z})}{\beta^6}\left.\frac{\partial\kappa}{\partial z}\right|_{z=\overline{z}}\frac{1}{\alpha A(\overline{z})-\kappa(\overline{z})}\left[1+\beta^3(1-p_1)+\frac{\alpha}{1-\alpha}\frac{p_1}{p_0}\right]}_{=A}\right.$$
$$\underbrace{-\frac{\kappa(\overline{z})}{\beta^6}\frac{p_A}{\alpha A(\overline{z})-\kappa(\overline{z})}\left[1+\beta^3(1+p_0-p_2)+\frac{\alpha}{1-\alpha}\frac{p_2}{p_0}\right]\left.\frac{\partial\kappa}{\partial z}\right|_{z=\overline{z}}}_{<0\text{ because of condition (6)}}$$
$$\left.\underbrace{+\frac{\kappa(\overline{z})}{\beta^6}\frac{\beta^3}{2\left.\frac{\partial\kappa}{\partial z}\right|_{z=\overline{z}}}\left[\frac{\partial^2\kappa}{\partial z^2}-\beta^3(1-\alpha)p_0\left.\frac{\partial^2A}{\partial z^2}\right|_{z=\overline{z}}\right]}_{=B}\right\},$$

and (14) implies that $A+B<0$. Therefore,

$$\left.\frac{\partial V_2^I}{\partial z}\right|_{z=\overline{z}}-\left.\frac{\partial V_2^F}{\partial z}\right|_{z=\overline{z}}<0,$$

that is to say that the representative entrepreneur born in period 2 is worse off under the monetary policy intervention. As a consequence, the latter is not Pareto-improving. Proposition 10 follows.

Monetary Policy and Herd Behavior: Leaning Against Bubbles
Online Appendix

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Abstract

This is an online appendix to the paper "Monetary Policy and Herd Behavior: Leaning Against Bubbles". Equations, propositions and lemmas correspond to those of the paper.

A A monetary model

Here we construct a monetary model that shows that a sequence of taxes $\{\tau_t\}$ can be implemented with a properly chosen path of money supply.

In the paper, equilibrium allocations are given by a resource constraint and two optimality conditions in every period

$$c_t + c_t^e + \kappa(z_t) = A(z_{t-N}), \tag{A.1}$$

$$\tau_t q_t = \beta^N E_{\Omega(h,t)}\left[\frac{c_t}{c_{t+N}}\right], \tag{A.2}$$

$$z_t = \underset{z_t \in \mathcal{F}_t}{\arg\max} \beta^N E_{\Omega(e,t)}\left[(1-\alpha)A(z_t) - \frac{\kappa(z_t)}{q_t}\right]. \tag{A.3}$$

In this model, we are interpreting changes in the tax rate τ as monetary policy action. Here we propose a monetary model in which the central bank sets the growth rate of money supply, and whose allocations replicate the ones of the paper. The main ingredients of the model are the following: households derive utility from real balances, from which we obtain a money demand equation. Households have only access to a market for nominal bonds. Entrepreneurs have only access to a market for real bonds. Banks are playing the role of intermediaries: they have access to both financial markets and transform nominal bonds into real ones. Banks are constrained to keep a share of their nominal liabilities in cash. In such a setup, a proper choice of the money supply sequence (that determines inflation) allows to replicate real allocation for a given sequence of taxes $\{\tau_t\}$.

A.1 Households

Households derive utility from real balances and consumption:

$$U_t = E_{\Omega(h,t)} \sum_{j=0}^{\infty} \beta^j \left(\ln(c_{t+j}) + \nu \ln \left(\frac{M_{t+j}}{P_{t+j}} \right) \right),$$

where M_t are nominal balances held by the household, P_t is the price in units of money of the consumption good and ν is a positive parameter. Households only have access to a financial market on which nominal bonds are traded. The budget constraint of a given period t is

$$P_t c_t + Q_t \mathcal{B}_{t+N} + M_t \leq \mathcal{B}_t + P_t w_t L_t + \overline{M}_t.$$

$\mathcal{B}_{t+N}$ is the number of nominal bonds bought by the household in period t. Their nominal price is Q_t and they pay for sure one unit of money in period $t+N$; $\overline{M}_t$ represents the amount of money created by the central bank in period t. This money supply is distributed to the household in a lump-sum way. First order conditions of this program are

$$\begin{aligned} Q_t &= \beta^N E_{\Omega(h,t)} \left[\frac{c_t}{c_{t+N}} \frac{1}{\pi_{t+N}} \right], \qquad \text{(A.4)} \\ M_t &= \nu P_t c_t, \end{aligned}$$

where $\pi_{t+N} = \frac{P_{t+N}}{P_t}$ is the inflation factor between t and $t+N$.

A.2 Entrepreneurs

The problem of the entrepreneurs is unchanged compared to the model of the paper. They only have access to a financial market for real bonds. In period t, they issue real bonds b_{t+N} to finance their investment, and their optimal behavior is characterized by:

$$z_t = \arg\max_{z_t \in \mathcal{F}_t} \beta^N E_{\Omega(e,t)} \left[(1-\alpha) A(z_t) - \frac{\kappa(z_t)}{q_t} \right]. \qquad \text{(A.5)}$$

A.3 Banks

In each period t, a bank is created, that behaves competitively, and that will be active only in periods t and $t+N$. It is owned by the household, that receives the dividends from the bank (dividends will be zero in equilibrium). The period t bank is the only economic entity that can access both nominal and real bonds of maturity N markets in period t. In period t, it will therefore issue nominal bonds (subscribed by the households) and subscribe real bonds (issued by entrepreneurs). In period $t+N$, it will collect revenues from real bonds and repay nominal ones. The source of money non-neutrality comes from the fact that the bank is required to hold a fraction $\mu > 0$ of its nominal liabilities $\mathcal{B}_{t+N}$ in cash, and we denote R_t this amount of cash reserves:

$$R_t = \mu \mathcal{B}_{t+N}. \qquad \text{(A.6)}$$

The budget constraints of the representative period t bank are:

$$P_t q_t b_{t+N} + R_t = Q_t \mathcal{B}_{t+N} \qquad \text{in period } t, \tag{A.7}$$

$$\mathcal{B}_{t+N} = P_{t+N} b_{t+N} + R_t \qquad \text{in period } t+N. \tag{A.8}$$

A.4 Central bank

The central bank sets the sequence of money supply $\{\overline{M}_t\}$. In order to get an equivalence result between *some* monetary policy and a sequence of taxes, we restrict the central bank to policies that makes deterministic the inflation rate between t and $t+N$. This is achieved by the choice of an appropriate policy rule that makes $\{\overline{M}_{t+N}\}$ contingent to the state of the economy in period $t+N$. The central bank is assumed to be able to commit to this rule. We will show in the next subsection that such a rule does exist. Finally, we assume without loss of generality that money supply $\overline{M}_0$ to $\overline{M}_{N-1}$ are given and equal to the pre-new technology steady state level.

When inflation is fully predictable, $(E_t\pi_{t+N} = \pi_{t+N})$, equation (A.4) rewrites

$$Q_t \pi_{t+N} = \beta^N E_{\Omega(h,t)} \left[\frac{c_t}{c_{t+N}} \right]. \tag{A.9}$$

A.5 Real equilibrium allocations

Equations (A.6), (A.7) and (A.8) imply

$$Q_t \pi_{t+N} = (1-\mu) q_t + \mu \pi_{t+N} \tag{A.10}$$

For a given sequence of inflation rates, real allocations (consumption c_t, investment z_t and the real price of real bonds q_t) are given by equations:

$$c_t + c_t^e + \kappa(z_t) = A(z_{t-N}), \tag{A.11}$$

$$(1-\mu) q_t + \mu \pi_{t+N} = \beta^N E_{\Omega(h,t)} \left[\frac{c_t}{c_{t+N}} \right], \tag{A.12}$$

$$z_t = \underset{z_t \in \mathcal{F}_t}{\arg\max} \beta^N E_{\Omega(e,t)} \left[(1-\alpha) A(z_t) - \frac{\kappa(z_t)}{q_t} \right]. \tag{A.13}$$

(A.11) and (A.13) are identical to equations (A.1) and (A.3). Allocations will be therefore the same in the real and the monetary economy if and only if (A.12) is identical to equation (A.2), which implies

$$\pi_{t+N} = \frac{\tau_t - 1 + \mu}{\mu} q_t. \tag{A.14}$$

To summarize, real allocations of the initial model, that are indexed by a sequence $\{\tau_t\}$ can be replicated in the monetary economy provided that inflation is determined by equation (A.14). We now determine the equilibrium level of π_{t+N} and show how it is determined by money supply.

A.6 Equilibrium prices and inflation

The money market equilibrium of period t writes

$$\underbrace{M_t + R_t}_{\text{money demand}} = \underbrace{\overline{M}_t + R_{t-N}}_{\text{money supply}} \tag{A.15}$$

Using (A.4), (A.6), we get an expression for the price of period t:

$$P_t = \frac{\overline{M}_t + \mu(\mathcal{B}_t - \mathcal{B}_{t+N})}{\nu c_t} \tag{A.16}$$

As $\mathcal{B}_{t+N} = \frac{1}{1-\mu} P_{t+N} b_{t+N}$ and $P_{t+N} b_{t+N} = \kappa(z_t)$ in equilibrium, (A.16) implies

$$P_t = \frac{1}{\nu c_t} \left(\overline{M}_t + \frac{\mu}{1-\mu} (\kappa(z_{t-N}) - \kappa(z_t)) \right) \tag{A.17}$$

from which we obtain an expression for equilibrium inflation:

$$\pi_{t+N} = \frac{c_t}{c_{t+N}} \left(\frac{\overline{M}_{t+N} + \frac{\mu}{1-\mu}(\kappa(z_t) - \kappa(z_{t+N}))}{\overline{M}_t + \frac{\mu}{1-\mu}(\kappa(z_{t-N}) - \kappa(z_t))} \right) \tag{A.18}$$

The central bank can make π_{t+N} deterministic by committing in period t to a properly chosen state contingent policy $\overline{M}_{t+N} = \overline{M}\left(c_t, c_{t+N}, z_{t-N}, z_t, z_{t+N}, \overline{M}_t\right)$.

A.7 Equivalence result

The following proposition summarizes the previous results

Proposition 11 *Consider real allocations and prices* $\mathcal{A} = \{\widehat{c}_t, \widehat{c}^e_t, \widehat{z}_t, \widehat{q}_t\}_{t=1}^{\infty}$ *that satisfies equations (A.1), (A.2) and (A.3) for a given tax rule announced in zero. Then* $\mathcal{A}$ *is also an equilibrium allocation of the monetary model when the central bank commits to a monetary supply rule* $\overline{M}_{t+N} = \overline{M}\left(c_t, c_{t+N}, z_{t-N}, z_t, z_{t+N}, \overline{M}_t\right)$ *that satisfies (i)* $\overline{M}_1$ *to* $\overline{M}_N$ *are arbitrarily chosen, (ii) inflation satisfies* $\pi_{t+N} = \frac{\tau_t - 1 + \mu}{\mu} \widehat{q}_t$ *and (iii) the level of inflation in (ii) implies the choice of a money supply rule* $\overline{M}_{t+N} = \overline{M}\left(c_t, c_{t+N}, z_{t-N}, z_t, z_{t+N}, \overline{M}_t\right)$ *according to* $\pi_{t+N} = \frac{\widehat{c}_t}{\widehat{c}_{t+N}} \left(\frac{\overline{M}_{t+N} + \frac{\mu}{1-\mu}(\kappa(\widehat{z}_t) - \kappa(\widehat{z}_{t+N}))}{\overline{M}_t + \frac{\mu}{1-\mu}(\kappa(\widehat{z}_{t-N}) - \kappa(\widehat{z}_t))} \right)$.

When ν is driven arbitrarily close to zero, the welfare properties of monetary economy with the properly chosen money supply rule are the same as the ones of real economy with taxes.

B Proofs of Lemmas 1 to 3

B.1 Proof of Lemma 1

From (4) and (5) we easily get, using the notations of appendix B,

$$q(z, \tau_t, 0, 0) = \frac{D_0(\tau_t) - H_0}{G_0} \text{ and } q(z, \tau_t, 0, 1) = \frac{D_1(z, \tau_t) - H_1(z)}{G_1}.$$

Since $D_0(\tau_t) > D_1(z,\tau_t)$, $H_0 < H_1(z)$ and $G_0 = G_1 > 0$, we obtain that

$$q(z,\tau_t,0,0) > q(z,\tau_t,0,1).$$

Lemma 1 follows.

B.2 Proof of Lemma 2

Concerning households, we have

$$\begin{aligned} U_1^{LF}(z) &= (1+\beta+\beta^2)\ln\left[\alpha A(\overline{z}) + \frac{\kappa(\overline{z})}{\beta^3} - \kappa(z)\right] + p_1\beta^3 \textstyle\sum_{i=0}^{+\infty} \beta^i \ln\left[\alpha A(z) + \dfrac{\kappa(z)}{q_{i+1}^{(1)}(z)} - \kappa(z)\right] \\ &\quad + (1-p_1)\beta^3 \textstyle\sum_{i=0}^{2} \beta^i \ln\left[\alpha A(\overline{z}) + \dfrac{\kappa(z)}{q_{i+1}^{(2)}(z)} - \kappa(\overline{z})\right] \\ &\quad + (1-p_1)\beta^3 \textstyle\sum_{i=3}^{+\infty} \beta^i \ln\left[\alpha A(\overline{z}) + \dfrac{\kappa(\overline{z})}{q_{i+1}^{(2)}(z)} - \kappa(\overline{z})\right], \end{aligned}$$

where superscripts (1), resp. (2), indicates that the new technology turns out to be good, resp. bad. Computations then lead to

$$\begin{aligned} q_i^{(1)}(z) &= q_i^{(2)}(z) = q(z,1,p_1,1) \text{ for } i \in \{1,2,3\}, \\ q_i^{(1)}(z) &= \beta^3 + \frac{\beta^3\kappa(z)}{\alpha A(z) - \kappa(z)}\left[\frac{1}{q_{i-3}^{(1)}(z)} - \frac{1}{\beta^3}\right] \text{ for } i \geq 4, \\ q_i^{(2)}(z) &= \beta^3 + \frac{\beta^3\kappa(z)}{\alpha A(\overline{z}) - \kappa(\overline{z})}\left[\frac{1}{q_{i-3}^{(2)}(z)} - \frac{1}{\beta^3}\right] + \frac{\kappa(z) - \kappa(\overline{z})}{\alpha A(\overline{z}) - \kappa(\overline{z})} \text{ for } i \in \{4,5,6\}, \\ q_i^{(2)}(z) &= \beta^3 + \frac{\beta^3\kappa(\overline{z})}{\alpha A(\overline{z}) - \kappa(\overline{z})}\left[\frac{1}{q_{i-3}^{(2)}(z)} - \frac{1}{\beta^3}\right] \text{ for } i \geq 7. \end{aligned}$$

Using (15) and the fact that $\forall i \geq 1$, $q_i^{(1)}(\overline{z}) = q_i^{(2)}(\overline{z}) = \beta^3$, we get

$$\begin{aligned} \left.\frac{dq_j^{(k)}}{dz}\right|_{z=\overline{z}} &= \frac{-1}{\alpha A(\overline{z}) - \kappa(\overline{z})}\left[1 + \beta^3(1-p_1) + \frac{\alpha p_1}{(1-\alpha)p_0}\right]\left.\frac{d\kappa}{dz}\right|_{z=\overline{z}} \text{ for } j \in \{1,2,3\} \text{ and } k \in \{1,2\}, \\ \left.\frac{dq_{3i+j}^{(1)}}{dz}\right|_{z=\overline{z}} &= \left[\frac{-\kappa(\overline{z})}{\beta^3[\alpha A(\overline{z}) - \kappa(\overline{z})]}\right]^i \left.\frac{dq_j^{(1)}}{dz}\right|_{z=\overline{z}} \text{ for } i \geq 1 \text{ and } j \in \{1,2,3\}, \\ \left.\frac{dq_{3i+j}^{(2)}}{dz}\right|_{z=\overline{z}} &= \left[\frac{-\kappa(\overline{z})}{\beta^3[\alpha A(\overline{z}) - \kappa(\overline{z})]}\right]^{i-1}\left\{\frac{1}{[\alpha A(\overline{z}) - \kappa(\overline{z})]}\left.\frac{d\kappa}{dz}\right|_{z=\overline{z}} - \frac{\kappa(\overline{z})}{\beta^3[\alpha A(\overline{z}) - \kappa(\overline{z})]}\left.\frac{dq_j^{(1)}}{dz}\right|_{z=\overline{z}}\right\} \\ &\quad \text{for } i \geq 1 \text{ and } j \in \{1,2,3\}. \end{aligned}$$

We end up with

$$\begin{aligned} \left.\frac{dU_1^{LF}}{dz}\right|_{z=\overline{z}} &= \frac{\left.\frac{d\kappa}{dz}\right|_{z=\overline{z}}}{(1-\beta)[\kappa(\overline{z}) + \beta^3[\alpha A(\overline{z}) - \kappa(\overline{z})]]}\left\{\frac{\alpha\beta^3 p_1}{(1-\alpha)p_0} + \frac{\kappa(\overline{z})\left(1-\beta^3\right)}{\alpha A(\overline{z})}\left[1 + \frac{\alpha p_1}{(1-\alpha)p_0}\right]\right\} \\ &> 0. \end{aligned}$$

Concerning entrepreneurs, we have

$$\widehat{V}_1^{LF}(z) = p_1 \sum_{i=0}^{+\infty} \beta^{3+i} \left[(1-\alpha) A(z) - \frac{\kappa(z)}{q_{i+1}^{(1)}(z)}\right]$$
$$+ (1-p_1) \sum_{i=0}^{2} \beta^{3+i} \left[(1-\alpha) A(\overline{z}) - \frac{\kappa(z)}{q_{i+1}^{(2)}(z)}\right] + (1-p_1) \sum_{i=3}^{+\infty} \beta^{3+i} \left[(1-\alpha) A(\overline{z}) - \frac{\kappa(\overline{z})}{q_{i+1}^{(2)}(z)}\right],$$

from which we get, using (15),

$$\left.\frac{d\widehat{V}_1^{LF}}{dz}\right|_{z=\overline{z}} = \frac{-\left.\frac{d\kappa}{dz}\right|_{z=\overline{z}}}{1-\beta} \left\{1 - \beta^3 + \beta^3 p_1 - \frac{p_1}{p_0} + \frac{\kappa(\overline{z})\left(1-\beta^3\right)}{\alpha A(\overline{z})\beta^3} \left[1 + \frac{\alpha p_1}{(1-\alpha)p_0}\right]\right\}.$$

The coefficient of $\frac{p_1}{p_0}$ in this expression linear in $\frac{p_1}{p_0}$ is

$$\frac{-\left.\frac{d\kappa}{dz}\right|_{z=\overline{z}}}{1-\beta} \left[-1 + \frac{\kappa(\overline{z})\left(1-\beta^3\right)}{(1-\alpha)A(\overline{z})\beta^3}\right] > \frac{\beta^3 \left.\frac{d\kappa}{dz}\right|_{z=\overline{z}}}{1-\beta} > 0,$$

given the conditions $\alpha A(\overline{z}) - \kappa(\overline{z}) > 0$ and (16), so that we get

$$\lim_{(p_0,\lambda)\longrightarrow(0,1)} \left.\frac{d\widehat{V}_1^{LF}}{dz}\right|_{z=\overline{z}} = \lim_{\frac{p_1}{p_0}\longrightarrow+\infty} \left.\frac{d\widehat{V}_1^{LF}}{dz}\right|_{z=\overline{z}} = +\infty.$$

Moreover,

$$\lim_{p_0\longrightarrow1} \left.\frac{d\widehat{V}_1^{LF}}{dz}\right|_{z=\overline{z}} = \frac{-\left.\frac{d\kappa}{dz}\right|_{z=\overline{z}}}{1-\beta} \left[\frac{\kappa(\overline{z})\left(1-\beta^3\right)}{\alpha(1-\alpha)A(\overline{z})\beta^3}\right] < 0.$$

Lemma 2 follows.

B.3 Proof of Lemma 3

Let us first derive the intervention $\left.\frac{d\tau_2}{dz}\right|_{z=\overline{z}}$ and the corresponding interest rates. Since $\tau^l(\overline{z}, p_0, p_0) = 1 < \tau^u(\overline{z}, p_2, p_2)$, we get

$$\left.\frac{d\tau_2}{dz}\right|_{z=\overline{z}} = \left.\frac{\partial\tau^l(z, p_0, p_0)}{\partial z}\right|_{z=\overline{z}}$$

which, using (23), leads to

$$\left.\frac{d\tau_2}{dz}\right|_{z=\overline{z}} = \frac{\beta^3 \left.\frac{d\kappa}{dz}\right|_{z=\overline{z}}}{\beta^3\left[\alpha A(\overline{z}) - \kappa(\overline{z})\right] + \kappa(\overline{z})} \left[p_0 - \frac{\left[\alpha A(\overline{z}) - \kappa(\overline{z})\right]\left[\left.\frac{d^2\kappa}{dz^2}\right|_{z=\overline{z}} - (1-\alpha)\beta^3 p_0 \left.\frac{d^2A}{dz^2}\right|_{z=\overline{z}}\right]}{2\left(\left.\frac{d\kappa}{dz}\right|_{z=\overline{z}}\right)^2}\right]. \tag{B.19}$$

Moreover, Proposition 8 and (23) imply

$$\left.\frac{dq(z, \tau_2(z), p_0, 0)}{dz}\right|_{z=\overline{z}} = \frac{1}{p_0} \left.\frac{dB}{dz}\right|_{z=\overline{z}} = \frac{\beta^3\left[\left.\frac{d^2\kappa}{dz^2}\right|_{z=\overline{z}} - (1-\alpha)\beta^3 p_0 \left.\frac{d^2A}{dz^2}\right|_{z=\overline{z}}\right]}{2\left.\frac{d\kappa}{dz}\right|_{z=\overline{z}}}. \tag{B.20}$$

Finally, the total derivative of (5) at date 2 for $\tau_2 = \tau_2(z)$ and $\mu_2 = p_2$ with respect to z, taken at point $z = \overline{z}$, and the use of (B.19) lead to

$$\left.\frac{dq\left(z,\tau_2\left(z\right),p_2,1\right)}{dz}\right|_{z=\overline{z}} =$$
$$-\left.\frac{d\kappa}{dz}\right|_{z=\overline{z}}\left\{\frac{1+\beta^3\left(1+p_0-p_2\right)+\frac{\alpha}{1-\alpha}\frac{p_2}{p_0}}{\alpha A\left(\overline{z}\right)-\kappa\left(\overline{z}\right)}+\frac{\beta^3\left[\left(1-\alpha\right)\beta^3 p_0\left.\frac{d^2A}{dz^2}\right|_{z=\overline{z}}-\left.\frac{d^2\kappa}{dz^2}\right|_{z=\overline{z}}\right]}{2\left(\left.\frac{d\kappa}{dz}\right|_{z=\overline{z}}\right)^2}\right\}. \tag{B.21}$$

Concerning households, we have

$$\begin{aligned}
U_1^I(z) \;=\;& \ln\left[\alpha A\left(\overline{z}\right)+\frac{\kappa\left(\overline{z}\right)}{\beta^3}-\kappa\left(z\right)\right] \\
&+p_A\left\{\textstyle\sum_{i=1}^{2}\beta^i\ln\left[\alpha A(\overline{z})+\frac{\kappa\left(\overline{z}\right)}{\beta^3}-\kappa\left(z\right)\right]+p_2\sum_{i=3}^{+\infty}\beta^i\ln\left[\alpha A\left(z\right)+\frac{\kappa\left(z\right)}{q_{i-2}^{(1)}\left(z\right)}-\kappa\left(z\right)\right]\right. \\
&+\left(1-p_2\right)\left[\textstyle\sum_{i=3}^{5}\beta^i\ln\left[\alpha A\left(\overline{z}\right)+\frac{\kappa\left(z\right)}{q_{i-2}^{(2)}\left(z\right)}-\kappa\left(\overline{z}\right)\right]\right. \\
&\left.\left.+\textstyle\sum_{i=6}^{+\infty}\beta^i\ln\left[\alpha A\left(\overline{z}\right)+\frac{\kappa\left(\overline{z}\right)}{q_{i-2}^{(2)}\left(z\right)}-\kappa\left(\overline{z}\right)\right]\right]\right\} \\
&+\left(1-p_A\right)\left\{\beta\ln\left[\alpha A(\overline{z})+\frac{\kappa\left(\overline{z}\right)}{\beta^3}-\kappa\left(\overline{z}\right)\right]+p_B\left[\beta^2\ln\left[\alpha A(\overline{z})+\frac{\kappa\left(\overline{z}\right)}{\beta^3}-\kappa\left(z\right)\right]\right.\right. \\
&+p_1\left[\textstyle\sum_{i\in\mathbb{N}\setminus\{0,1,2,4\}}\beta^i\ln\left[\alpha A(z)+\frac{\kappa\left(z\right)}{q_{i-2}^{(3)}\left(z\right)}-\kappa\left(z\right)\right]+\beta^4\ln\left[\alpha A(\overline{z})+\frac{\kappa\left(\overline{z}\right)}{q_2^{(3)}\left(z\right)}-\kappa\left(z\right)\right]\right] \\
&+\left(1-p_1\right)\left[\textstyle\sum_{i\in\{3,5\}}\beta^i\ln\left[\alpha A(\overline{z})+\frac{\kappa\left(z\right)}{q_{i-2}^{(4)}\left(z\right)}-\kappa\left(\overline{z}\right)\right]\right. \\
&\left.\left.+\textstyle\sum_{i\in\mathbb{N}\setminus\{0,1,2,3,5\}}\beta^i\ln\left[\alpha A(\overline{z})+\frac{\kappa\left(\overline{z}\right)}{q_{i-2}^{(4)}\left(z\right)}-\kappa\left(\overline{z}\right)\right]\right]\right] \\
&+\left(1-p_B\right)\left[\beta^2\ln\left[\alpha A(\overline{z})+\frac{\kappa\left(\overline{z}\right)}{\beta^3}-\kappa\left(\overline{z}\right)\right]+p_{-1}\left[\beta^3\ln\left[\alpha A(z)+\frac{\kappa\left(z\right)}{q_1^{(5)}\left(z\right)}-\kappa\left(z\right)\right]\right.\right. \\
&\left.\textstyle\sum_{i=4}^{5}\beta^i\ln\left[\alpha A(\overline{z})+\frac{\kappa\left(\overline{z}\right)}{q_{i-2}^{(5)}\left(z\right)}-\kappa\left(z\right)\right]+\sum_{i=6}^{+\infty}\beta^i\ln\left[\alpha A(z)+\frac{\kappa\left(z\right)}{q_{i-2}^{(5)}\left(z\right)}-\kappa\left(z\right)\right]\right] \\
&+\left(1-p_{-1}\right)\left[\beta^3\ln\left[\alpha A(\overline{z})+\frac{\kappa\left(z\right)}{q_1^{(6)}\left(z\right)}-\kappa\left(\overline{z}\right)\right]\right. \\
&\left.\left.\left.\left.+\textstyle\sum_{i=4}^{+\infty}\beta^i\ln\left[\alpha A(\overline{z})+\frac{\kappa\left(\overline{z}\right)}{q_{i-2}^{(6)}\left(z\right)}-\kappa\left(\overline{z}\right)\right]\right]\right]\right]\right\},
\end{aligned}$$

where superscripts (1) to (6) correspond to the following cases:

Superscript	S_2	S_3	Technology
(1)	1	0 or 1	good
(2)	1	0 or 1	bad
(3)	0	1	good
(4)	0	1	bad
(5)	0	0	good
(6)	0	0	bad

Computations then lead to

$$q_1^{(1)}(z) = q(z,1,p_1,1),\ q_2^{(1)}(z) = q(z,\tau_2(z),p_2,1),\ q_3^{(1)}(z) = q(z,1,p_2,1),$$

$$q_i^{(1)}(z) = \beta^3 + \frac{\beta^3\kappa(z)}{\alpha A(z)-\kappa(z)}\left[\frac{1}{q_{i-3}^{(1)}(z)} - \frac{1}{\beta^3}\right] \text{ for } i \geq 4,$$

$$q_1^{(2)}(z) = q(z,1,p_1,1),\ q_2^{(2)}(z) = q(z,\tau_2(z),p_2,1),\ q_3^{(2)}(z) = q(z,1,p_2,1),$$

$$q_i^{(2)}(z) = \beta^3 + \frac{\beta^3\kappa(z)}{\alpha A(\overline{z})-\kappa(\overline{z})}\left[\frac{1}{q_{i-3}^{(2)}(z)} - \frac{1}{\beta^3}\right] + \frac{\kappa(z)-\kappa(\overline{z})}{\alpha A(\overline{z})-\kappa(\overline{z})} \text{ for } 4 \leq i \leq 6,$$

$$q_i^{(2)}(z) = \beta^3 + \frac{\beta^3\kappa(\overline{z})}{\alpha A(\overline{z})-\kappa(\overline{z})}\left[\frac{1}{q_{i-3}^{(2)}(z)} - \frac{1}{\beta^3}\right] \text{ for } i \geq 7,$$

$$q_1^{(3)}(z) = q(z,1,p_1,1),\ q_2^{(3)}(z) = q(z,\tau_2(z),p_0,0),\ q_3^{(3)}(z) = q(z,1,p_1,1),$$

$$q_i^{(3)}(z) = \beta^3 + \frac{\beta^3\kappa(z)}{\alpha A(z)-\kappa(z)}\left[\frac{1}{q_{i-3}^{(3)}(z)} - \frac{1}{\beta^3}\right] \text{ for } i = 4 \text{ and } i \geq 6,$$

$$q_5^{(3)}(z) = \beta^3 + \frac{\beta^3\kappa(\overline{z})}{\alpha A(z)-\kappa(z)}\left[\frac{1}{q_2^{(3)}(z)} - \frac{1}{\beta^3}\right] - \frac{\kappa(z)-\kappa(\overline{z})+\beta^3\left[\alpha A(z)-\alpha A(\overline{z})\right]}{\alpha A(z)-\kappa(z)},$$

$$q_1^{(4)}(z) = q(z,1,p_1,1),\ q_2^{(4)}(z) = q(z,\tau_2(z),p_0,0),\ q_3^{(4)}(z) = q(z,1,p_1,1),$$

$$q_i^{(4)}(z) = \beta^3 + \frac{\beta^3\kappa(z)}{\alpha A(\overline{z})-\kappa(\overline{z})}\left[\frac{1}{q_{i-3}^{(4)}(z)} - \frac{1}{\beta^3}\right] + \frac{\kappa(z)-\kappa(\overline{z})}{\alpha A(\overline{z})-\kappa(\overline{z})} \text{ for } i \in \{4,6\},$$

$$q_i^{(4)}(z) = \beta^3 + \frac{\beta^3\kappa(\overline{z})}{\alpha A(\overline{z})-\kappa(\overline{z})}\left[\frac{1}{q_{i-3}^{(4)}(z)} - \frac{1}{\beta^3}\right] \text{ for } i = 5 \text{ and } i \geq 7,$$

$$q_1^{(5)}(z) = q(z,1,p_1,1),\ q_2^{(5)}(z) = q(z,\tau_2(z),p_0,0),\ q_3^{(5)}(z) = q(z,1,p_{-1},0),$$

$$q_i^{(5)}(z) = \beta^3 + \frac{\beta^3\kappa(z)}{\alpha A(z)-\kappa(z)}\left[\frac{1}{q_{i-3}^{(5)}(z)} - \frac{1}{\beta^3}\right] \text{ for } i = 4 \text{ and } i \geq 7,$$

$$q_i^{(5)}(z) = \beta^3 + \frac{\beta^3\kappa(\overline{z})}{\alpha A(z)-\kappa(z)}\left[\frac{1}{q_{i-3}^{(5)}(z)} - \frac{1}{\beta^3}\right] - \frac{\kappa(z)-\kappa(\overline{z})+\beta^3\left[\alpha A(z)-\alpha A(\overline{z})\right]}{\alpha A(z)-\kappa(z)} \text{ for } i \in \{5,6\},$$

$$q_1^{(6)}(z) = q(z,1,p_1,1),\ q_2^{(6)}(z) = q(z,\tau_2(z),p_0,0),\ q_3^{(6)}(z) = q(z,1,p_{-1},0),$$

$$q_4^{(6)}(z) = \beta^3 + \frac{\beta^3\kappa(z)}{\alpha A(\overline{z})-\kappa(\overline{z})}\left[\frac{1}{q_1^{(6)}(z)} - \frac{1}{\beta^3}\right] + \frac{\kappa(z)-\kappa(\overline{z})}{\alpha A(\overline{z})-\kappa(\overline{z})},$$

$$q_i^{(6)}(z) = \beta^3 + \frac{\beta^3\kappa(\overline{z})}{\alpha A(\overline{z})-\kappa(\overline{z})}\left[\frac{1}{q_{i-3}^{(6)}(z)} - \frac{1}{\beta^3}\right] \text{ for } i \geq 5.$$

Using (15), (B.20) and (B.21), we get

$$\left.\frac{dq_1^{(k)}}{dz}\right|_{z=\overline{z}} = \frac{-1}{\alpha A\left(\overline{z}\right)-\kappa\left(\overline{z}\right)}\left[1+\beta^3(1-p_1)+\frac{\alpha p_1}{\left(1-\alpha\right)p_0}\right]\left.\frac{d\kappa}{dz}\right|_{z=\overline{z}} \quad \text{for } k\in\{1,...,6\}\,,$$

$$\left.\frac{dq_2^{(k)}}{dz}\right|_{z=\overline{z}} = \frac{-1}{\alpha A(\overline{z})-\kappa\left(\overline{z}\right)}\left[1+\beta^3\left(1+p_0-p_2\right)+\frac{\alpha}{1-\alpha}\frac{p_2}{p_0}\right]\left.\frac{d\kappa}{dz}\right|_{z=\overline{z}}\,,$$
$$+\frac{\beta^3}{2\left.\frac{d\kappa}{dz}\right|_{z=\overline{z}}}\left[\left.\frac{d^2\kappa}{dz^2}\right|_{z=\overline{z}}-\left(1-\alpha\right)\beta^3 p_0\left.\frac{d^2A}{dz^2}\right|_{z=\overline{z}}\right] \quad \text{for } k\in\{1,2\}\,,$$

$$\left.\frac{dq_2^{(k)}}{dz}\right|_{z=\overline{z}} = \frac{\beta^3}{2\left.\frac{d\kappa}{dz}\right|_{z=\overline{z}}}\left[\left.\frac{d^2\kappa}{dz^2}\right|_{z=\overline{z}}-\left(1-\alpha\right)\beta^3 p_0\left.\frac{d^2A}{dz^2}\right|_{z=\overline{z}}\right] \quad \text{for } k\in\{3,...,6\}\,,$$

$$\left.\frac{dq_3^{(k)}}{dz}\right|_{z=\overline{z}} = \frac{-1}{\alpha A\left(\overline{z}\right)-\kappa\left(\overline{z}\right)}\left[1+\beta^3(1-p_2)+\frac{\alpha p_2}{\left(1-\alpha\right)p_0}\right]\left.\frac{d\kappa}{dz}\right|_{z=\overline{z}} \quad \text{for } k\in\{1,2\}\,,$$

$$\left.\frac{dq_3^{(k)}}{dz}\right|_{z=\overline{z}} = \frac{-1}{\alpha A\left(\overline{z}\right)-\kappa\left(\overline{z}\right)}\left[1+\beta^3(1-p_1)+\frac{\alpha p_1}{\left(1-\alpha\right)p_0}\right]\left.\frac{d\kappa}{dz}\right|_{z=\overline{z}} \quad \text{for } k\in\{3,4\}\,,$$

$$\left.\frac{dq_3^{(k)}}{dz}\right|_{z=\overline{z}} = \frac{\beta^3 p_{-1}}{\alpha A\left(\overline{z}\right)-\kappa\left(\overline{z}\right)}\left.\frac{d\kappa}{dz}\right|_{z=\overline{z}} \quad \text{for } k\in\{5,6\}\,,$$

$$\left.\frac{dq_{3i+j}^{(1)}}{dz}\right|_{z=\overline{z}} = \left[\frac{-\kappa(\overline{z})}{\beta^3\left[\alpha A\left(\overline{z}\right)-\kappa\left(\overline{z}\right)\right]}\right]^i\left.\frac{dq_j^{(1)}}{dz}\right|_{z=\overline{z}} \quad \text{for } i\geq 1 \text{ and } j\in\{1,2,3\}\,,$$

$$\left.\frac{dq_{3i+j}^{(2)}}{dz}\right|_{z=\overline{z}} = \left[\frac{-\kappa(\overline{z})}{\beta^3\left[\alpha A\left(\overline{z}\right)-\kappa\left(\overline{z}\right)\right]}\right]^i\left[\left.\frac{dq_j^{(2)}}{dz}\right|_{z=\overline{z}}-\frac{\beta^3}{\kappa(\overline{z})}\left.\frac{d\kappa}{dz}\right|_{z=\overline{z}}\right] \quad \text{for } i\geq 1 \text{ and } j\in\{1,2,3\}\,,$$

$$\left.\frac{dq_{3i+j}^{(3)}}{dz}\right|_{z=\overline{z}} = \left[\frac{-\kappa(\overline{z})}{\beta^3\left[\alpha A\left(\overline{z}\right)-\kappa\left(\overline{z}\right)\right]}\right]^i\left.\frac{dq_j^{(3)}}{dz}\right|_{z=\overline{z}} \quad \text{for } i\geq 1 \text{ and } j\in\{1,3\}\,,$$

$$\left.\frac{dq_{3i+2}^{(3)}}{dz}\right|_{z=\overline{z}} = \left[\frac{-\kappa(\overline{z})}{\beta^3\left[\alpha A\left(\overline{z}\right)-\kappa\left(\overline{z}\right)\right]}\right]^i\left\{\left.\frac{dq_2^{(3)}}{dz}\right|_{z=\overline{z}}+\frac{\beta^3}{\kappa\left(\overline{z}\right)}\left[1+\frac{\alpha}{\left(1-\alpha\right)p_0}\right]\left.\frac{d\kappa}{dz}\right|_{z=\overline{z}}\right\} \quad \text{for } i\geq 1,$$

$$\left.\frac{dq_{3i+j}^{(4)}}{dz}\right|_{z=\overline{z}} = \left[\frac{-\kappa(\overline{z})}{\beta^3\left[\alpha A\left(\overline{z}\right)-\kappa\left(\overline{z}\right)\right]}\right]^i\left[\left.\frac{dq_j^{(4)}}{dz}\right|_{z=\overline{z}}-\frac{\beta^3}{\kappa(\overline{z})}\left.\frac{d\kappa}{dz}\right|_{z=\overline{z}}\right] \quad \text{for } i\geq 1 \text{ and } j\in\{1,3\}\,,$$

$$\left.\frac{dq_{3i+2}^{(4)}}{dz}\right|_{z=\overline{z}} = \left[\frac{-\kappa(\overline{z})}{\beta^3\left[\alpha A\left(\overline{z}\right)-\kappa\left(\overline{z}\right)\right]}\right]^i\left.\frac{dq_2^{(4)}}{dz}\right|_{z=\overline{z}} \quad \text{for } i\geq 1,$$

$$\left.\frac{dq_{3i+1}^{(5)}}{dz}\right|_{z=\overline{z}} = \left[\frac{-\kappa(\overline{z})}{\beta^3\left[\alpha A\left(\overline{z}\right)-\kappa\left(\overline{z}\right)\right]}\right]^i\left.\frac{dq_1^{(5)}}{dz}\right|_{z=\overline{z}} \quad \text{for } i\geq 1,$$

$$\left.\frac{dq_{3i+j}^{(5)}}{dz}\right|_{z=\overline{z}} = \left[\frac{-\kappa(\overline{z})}{\beta^3\left[\alpha A\left(\overline{z}\right)-\kappa\left(\overline{z}\right)\right]}\right]^i\left\{\left.\frac{dq_j^{(5)}}{dz}\right|_{z=\overline{z}}+\frac{\beta^3}{\kappa(\overline{z})}\left[1+\frac{\alpha}{\left(1-\alpha\right)p_0}\right]\left.\frac{d\kappa}{dz}\right|_{z=\overline{z}}\right\}$$
$$\text{for } i\geq 1 \text{ and } j\in\{2,3\}\,,$$

$$\left.\frac{dq_{3i+1}^{(6)}}{dz}\right|_{z=\overline{z}} = \left[\frac{-\kappa(\overline{z})}{\beta^3\left[\alpha A\left(\overline{z}\right)-\kappa\left(\overline{z}\right)\right]}\right]^i\left\{\left.\frac{dq_1^{(6)}}{dz}\right|_{z=\overline{z}}-\frac{\beta^3}{\kappa(\overline{z})}\left.\frac{d\kappa}{dz}\right|_{z=\overline{z}}\right\} \quad \text{for } i\geq 1,$$

$$\left.\frac{dq_{3i+j}^{(6)}}{dz}\right|_{z=\overline{z}} = \left[\frac{-\kappa(\overline{z})}{\beta^3\left[\alpha A\left(\overline{z}\right)-\kappa\left(\overline{z}\right)\right]}\right]^i\left.\frac{dq_j^{(6)}}{dz}\right|_{z=\overline{z}} \quad \text{for } i\geq 1 \text{ and } j\in\{2,3\}\,.$$

Using $p_1 = p_A p_2 + (1-p_A)\, p_0$, we end up with

$$\begin{aligned}
\left.\frac{dU_1^I}{dz}\right|_{z=\overline{z}} &= \frac{\left.\frac{d\kappa}{dz}\right|_{z=\overline{z}}}{\kappa(\overline{z}) + \beta^3\left[\alpha A(\overline{z}) - \kappa(\overline{z})\right]} \left\{ \frac{\kappa(\overline{z})}{\alpha A(\overline{z})}\left[1 + p_0\beta^4 + p_A\beta(1+\beta) + (1-p_A)\, p_B\beta^2\right] \right. \\
&+ \frac{\kappa(\overline{z})}{(1-\alpha)\, p_0 A(\overline{z})}\left[\left(1+\beta^4+\beta^5\right) p_1 + p_A p_2 \beta (1+\beta)\left(1-\beta^3\right) + (1-p_A)\, p_B p_1 \beta^2 \left(1-\beta^3\right)\right] \\
&+ \frac{\beta^3\alpha}{(1-\beta)(1-\alpha)\, p_0}\left[p_1\left(1-\beta+\beta^3\right) + p_A p_2 \beta\left(1-\beta^2\right) + (1-p_A)\, p_B p_1 \beta^2 (1-\beta)\right] \\
&\left. + \frac{\beta^4\kappa(\overline{z})\left[\alpha A(\overline{z}) - \kappa(\overline{z})\right]}{2\alpha A(\overline{z})}\left[\frac{(1-\alpha)\,\beta^3 p_0 \left.\frac{d^2A}{dz^2}\right|_{z=\overline{z}} - \left.\frac{d^2\kappa}{dz^2}\right|_{z=\overline{z}}}{\left(\left.\frac{d\kappa}{dz}\right|_{z=\overline{z}}\right)^2}\right]\right\} \\
&> 0
\end{aligned}$$

given (14).

Concerning entrepreneurs, we have

$$\begin{aligned}
\widehat{V}_1^I(z) &= p_A\beta^3\left\{p_2 \sum_{i=0}^{+\infty}\beta^i\left[(1-\alpha)\, A(z) - \frac{\kappa(z)}{q_{i+1}^{(1)}(z)}\right]\right. \\
&\left. + (1-p_2)\left[\sum_{i=0}^{2}\beta^i\left[(1-\alpha)\, A(\overline{z}) - \frac{\kappa(z)}{q_{i+1}^{(2)}(z)}\right] + \sum_{i=3}^{+\infty}\beta^i\left[(1-\alpha)\, A(\overline{z}) - \frac{\kappa(\overline{z})}{q_{i+1}^{(2)}(z)}\right]\right]\right\} \\
&+ (1-p_A)\,\beta^3\left\{p_B\left[p_1\left[\sum_{i\in\mathbb{N}\smallsetminus\{1\}}\beta^i\left[(1-\alpha)\, A(z) - \frac{\kappa(z)}{q_{i+1}^{(3)}(z)}\right]\right.\right.\right. \\
&\left.\left. + \beta\left[(1-\alpha)\, A(\overline{z}) - \frac{\kappa(\overline{z})}{q_2^{(3)}(z)}\right]\right] + (1-p_1)\left[\sum_{i\in\{0,2\}}\beta^i\left[(1-\alpha)\, A(\overline{z}) - \frac{\kappa(z)}{q_{i+1}^{(4)}(z)}\right]\right.\right. \\
&\left.\left.\left. + \sum_{i\in\mathbb{N}\smallsetminus\{0,2\}}\beta^i\left[(1-\alpha)\, A(\overline{z}) - \frac{\kappa(\overline{z})}{q_{i+1}^{(4)}(z)}\right]\right]\right]\right. \\
&+ (1-p_B)\left[p_{-1}\left[\sum_{i\in\mathbb{N}\smallsetminus\{1,2\}}\beta^i\left[(1-\alpha)\, A(z) - \frac{\kappa(z)}{q_{i+1}^{(5)}(z)}\right]\right.\right. \\
&\left. + \sum_{i=1}^{2}\beta^i\left[(1-\alpha)\, A(\overline{z}) - \frac{\kappa(\overline{z})}{q_{i+1}^{(5)}(z)}\right]\right] \\
&\left.\left. + (1-p_{-1})\left[(1-\alpha)\, A(\overline{z}) - \frac{\kappa(z)}{q_1^{(6)}(z)} + \sum_{i=1}^{+\infty}\beta^i\left[(1-\alpha)\, A(\overline{z}) - \frac{\kappa(\overline{z})}{q_{i+1}^{(6)}(z)}\right]\right]\right]\right\}.
\end{aligned}$$

Using (15) and $p_1 = p_A p_2 + (1 - p_A) p_0$, we end up with

$$\begin{aligned}
\left.\frac{d\widehat{V}_1^I}{dz}\right|_{z=\overline{z}} &= \left.\frac{d\kappa}{dz}\right|_{z=\overline{z}} \left\{ \frac{-1}{1-\beta} \left[(1-\beta) + p_1\beta^3 + (1-p_A) p_B \beta^2 (1-\beta) + p_A \beta \left(1-\beta^2\right) \right] \right. \\
&+ \frac{1}{p_0 (1-\beta)} \left[p_1 \left(1 - \beta + \beta^3\right) + p_A p_2 \beta \left(1-\beta^2\right) + (1-p_A) p_B p_1 \beta^2 (1-\beta) \right] \\
&- \frac{\kappa(\overline{z})}{A(\overline{z}) \alpha \beta^3} \left[1 + p_0 \beta^4 + p_A \beta (1+\beta) + (1-p_A) p_B \beta^2 \right] \\
&- \frac{\kappa(\overline{z})}{A(\overline{z})(1-\alpha) p_0 \beta^3} \left[\left(1+\beta^4+\beta^5\right) p_1 + p_A p_2 \beta (1+\beta) \left(1-\beta^3\right) \right. \\
&+ \left. (1-p_A) p_B p_1 \beta^2 \left(1-\beta^3\right) \right] \\
&- \left. \frac{\beta\kappa(\overline{z}) \left[\alpha A(\overline{z}) - \kappa(\overline{z})\right]}{2\alpha A(\overline{z})} \left[\frac{(1-\alpha)\beta^3 p_0 \left.\frac{d^2A}{dz^2}\right|_{z=\overline{z}} - \left.\frac{d^2\kappa}{dz^2}\right|_{z=\overline{z}}}{\left(\left.\frac{d\kappa}{dz}\right|_{z=\overline{z}}\right)^2} \right] \right\}.
\end{aligned}$$

The coefficient of $\frac{p_1}{p_0}$ in this expression linear in $\frac{p_1}{p_0}$ is

$$\begin{aligned}
& \left.\frac{d\kappa}{dz}\right|_{z=\overline{z}} \left\{ \frac{1}{(1-\beta)} \left[\left(1-\beta+\beta^3\right) + \lambda\beta\left(1-\beta^2\right) + (1-p_A) p_B \beta^2 (1-\beta) \right] \right. \\
& \left. - \frac{\kappa(\overline{z})}{A(\overline{z})(1-\alpha)\beta^3} \left[\left(1+\beta^4+\beta^5\right) + \lambda\beta(1+\beta)\left(1-\beta^3\right) + (1-p_A) p_B \beta^2 \left(1-\beta^3\right) \right] \right\} \\
> & \left.\frac{d\kappa}{dz}\right|_{z=\overline{z}} \left\{ \frac{1}{(1-\beta)} \left[\left(1-\beta+\beta^3\right) + \lambda\beta\left(1-\beta^2\right) + (1-p_A) p_B \beta^2 (1-\beta) \right] \right. \\
& \left. - \left[\left(1+\beta^4+\beta^5\right) + \lambda\beta(1+\beta)\left(1-\beta^3\right) + (1-p_A) p_B \beta^2 \left(1-\beta^3\right) \right] \right\} \\
= & \left.\frac{d\kappa}{dz}\right|_{z=\overline{z}} \frac{\beta^3}{(1-\beta)} \left[\left(1-\beta+\beta^3\right) + \lambda\beta\left(1-\beta^2\right) + (1-p_A) p_B \beta^2 (1-\beta) \right] \\
> & \; 0,
\end{aligned}$$

where the first inequality comes from the conditions $\alpha A(\overline{z}) - \kappa(\overline{z}) > 0$ and (16), so that we get

$$\lim_{(p_0,\lambda)\longrightarrow(0,1)} \left.\frac{d\widehat{V}_1^I}{dz}\right|_{z=\overline{z}} = \lim_{\frac{p_1}{p_0}\longrightarrow+\infty} \left.\frac{d\widehat{V}_1^I}{dz}\right|_{z=\overline{z}} = +\infty.$$

Moreover,

$$\begin{aligned}
\lim_{p_0\longrightarrow 1} \left.\frac{d\widehat{V}_1^I}{dz}\right|_{z=\overline{z}} &= -\left.\frac{d\kappa}{dz}\right|_{z=\overline{z}} \left\{ \frac{\kappa(\overline{z})}{A(\overline{z})\alpha\beta^3} \left[1 + \beta^4 + \lambda\beta(1+\beta) + (1-\lambda)\lambda\beta^2 \right] \right. \\
&+ \frac{\kappa(\overline{z})}{A(\overline{z})(1-\alpha)\beta^3} \left[\left(1+\beta^4+\beta^5\right) + \lambda\beta(1+\beta)\left(1-\beta^3\right) + (1-\lambda)\lambda\beta^2\left(1-\beta^3\right) \right] \\
&+ \left. \frac{\beta\kappa(\overline{z})\left[\alpha A(\overline{z}) - \kappa(\overline{z})\right]}{2\alpha A(\overline{z})} \left[\frac{(1-\alpha)\beta^3 \left.\frac{d^2A}{dz^2}\right|_{z=\overline{z}} - \left.\frac{d^2\kappa}{dz^2}\right|_{z=\overline{z}}}{\left(\left.\frac{d\kappa}{dz}\right|_{z=\overline{z}}\right)^2} \right] \right\} \\
&< 0,
\end{aligned}$$

given the conditions $\alpha A(\overline{z}) - \kappa(\overline{z}) > 0$ and (14). Lemma 3 follows.