The Implementability and Implementation of Feasible Paths by Stabilization Policy

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Abstract: In locally linearized dynamic stochastic rational-expectations models, I introduce the concepts of *feasible* paths (paths on which the policy instrument can be expressed as a function of the policymaker's observation set) and *implementable* paths (paths that can be obtained, in a minimally robust way, as the unique local equilibrium under a policy-instrument rule consistent with the policymaker's observation set). I show that, for relevant observation sets, the optimal feasible path under monetary policy can be non-implementable in the New Keynesian model, while constant-debt feasible paths under tax policy are always implementable in the Real Business Cycle model. The first result sounds a note of caution about one of the main lessons of the New Keynesian literature, namely the importance for central banks to track some key unobserved exogenous rates of interest, while the second one restores the role of income or labor-income taxes in safely stabilizing public debt. In a general framework, I develop a method of designing arithmetically a policy-instrument rule consistent with the policymaker's observation set and implementing any given implementable path as the robustly unique local equilibrium.

Keywords: stabilization policy, local-equilibrium determinacy, observation set, feasible path, implementable path, optimal monetary policy, debt-stabilizing tax policy.

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1 Introduction

Two traditions stand alongside each other in the literature on macroeconomic stabilization policy in locally linearized dynamic stochastic rational-expectations models. One tradition studies some specific exogenous-shock-contingent paths of interest for the endogenous variables (e.g., the path that they follow under Ramsey-optimal policy), without asking whether and how these paths could be implemented as the unique local equilibrium given the policymaker's observation set. The other tradition considers some specific policy-instrument rules ensuring local-equilibrium determinacy and involving only observed variables (e.g., the interest-rate rule proposed by Taylor, 1993), without requiring these rules to implement a given exogenous-shock-contingent path of interest.

My aim, in this paper, is to build a bridge between these two separate traditions. The starting point of my analysis consists of three inputs: (i) a given model of the private sector's behavior, i.e. a given system of equilibrium conditions excluding the policy-instrument rule; (ii) a given targeted path for all the endogenous variables as functions of current and past exogenous shocks, which the policymaker would like to implement; and (iii) a given observation set for the policymaker, made of the history of some endogenous variables and/or exogenous shocks until some current or past date. Given these inputs, I ask two questions: does there exist a policy-instrument rule consistent with the policymaker's observation set and implementing the targeted path as the unique local equilibrium in the model? And, if there exists such a rule, how to design it? The first question is the question of the *implementability* of the targeted path; the second one is the question of its *implementation*.

There are, of course, two trivial ways in which a given path may not be implementable. First, the path may be inconsistent with at least one structural equation, i.e. one equilibrium condition describing the private sector's behavior. Second, it may be inconsistent with the policymaker's observation set, in the sense that the policy instrument cannot be expressed, on this path, as a function of only elements of this observation set. (Such is the case, for instance, when the path makes the policy instrument depend on current exogenous shocks, while the observation set includes only past exogenous shocks or past endogenous variables.) To rule out these two uninteresting cases, I focus on the paths that are consistent both with the structural equations and with the observation set. I call them "feasible paths." So the questions I ask in this paper are, more specifically, those of the implementability and implementation of *feasible* paths.

I make two contributions. First, I show, through two case studies, that the (non-) implementability of feasible paths can be an issue in textbook models, for standard policy instruments, relevant observation sets, and interesting feasible paths − with important policy implications. Second, in a general framework, I develop a method of designing arithmetically a policy-instrument rule consistent with a given observation set and implementing a given implementable path as the unique local equilibrium in a given model.

In the first case study, I consider: (i) the New Keynesian (NK) model, with the central bank as the policymaker and the interest rate as the policy instrument; (ii) the central bank's observation set made of all past endogenous variables; and (iii) the optimal feasible path, i.e. the path that maximizes welfare subject to the structural equations and the central bank's observation-set constraint. In this context, there is an infinity of policy-instrument rules consistent with the feasible path and the observation set. As I show, however, whether the system made of the structural equations and one of these rules satisfies Blanchard and Kahn's (1980) determinacy conditions does not depend on the specific rule considered, but only on the values of the structural parameters.

For some values of the structural parameters, the system has as many "unstable eigenvalues" (i.e., eigenvalues outside the unit circle of the complex plane) as non-predetermined variables; therefore, the system has a unique local equilibrium, which coincides with the optimal feasible path; so, this path is implementable. For other values, the system has fewer unstable eigenvalues than non-predetermined variables; therefore, it has an infinity of local equilibria, one of which coincides with the optimal feasible path; so, this path is not implementable. And for still other values, the system has more unstable eigenvalues than non-predetermined variables. In this last case, because of a stochastic singularity, the system still has a unique local equilibrium, which coincides with the optimal feasible path. However, if an exogenous policy shock of arbitrarily small variance (capturing, e.g., the policymaker's "trembling hand" or round-off errors) is added to the rule, then the system has no local equilibrium anymore. So, to rule out such a knife-edge and practically useless determinacy result, I say that the path is not implementable in this case.

These non-implementability results sound a note of caution about one of the main lessons of the NK literature, namely the importance for central banks to track some key unobserved exogenous rates of interest such as, for instance, the counterfactual "natural rate of interest" (as emphasized by, e.g., Galí, 2015, Chapter 9, and Woodford, 2003, Chapter 4). From a normative perspective, the most important of these rates of interest is, ultimately, the exogenous-shock-contingent value taken by the interest rate on the optimal feasible path. As my results show, however, even when this value can be inferred in an infinity of alternative ways, on the optimal feasible path, from the variables observed by the central bank, there may be no way of setting the interest rate as a function of these variables that implements this path as the (robustly) unique local equilibrium. In this case, any attempt to track this rate of interest and implement the optimal feasible path will inevitably result, in the presence of exogenous policy shocks of arbitrarily small variance, in either local-equilibrium multiplicity or non-existence of a local equilibrium.

The second case study is about debt-stabilizing tax policy in the Real Business Cycle (RBC) model. Schmitt-Grohé and Uribe (1997) consider, in this model, a labor-income-tax-rate or income-tax-rate rule that stabilizes the current stock of public debt (in the absence of policy-implementation lags) or the expected future stock of public debt (in the presence of such lags). They find that this rule leads to local-equilibrium multiplicity for many empirically relevant values of the structural parameters. This finding has largely been interpreted as an argument against the use of labor-income or income taxes to stabilize public debt. I show however that, in the same model, for the same alternative tax instruments, and for a reasonable observation set of the tax authority, all feasible paths along which the current or expected future stock of public debt is stabilized are implementable for all structural-parameter values, even in the presence of policy-implementation lags of any length.

This implementability result implies that Schmitt-Grohé and Uribe's (1997) finding should be interpreted not as an argument against debt-stabilizing (labor-)income-tax policy *per se*, but instead as an argument against one specific way of conducting this policy: one that achieves the policy goal not only in equilibrium, but also, unnecessarily, out of equilibrium. Requiring that debt be stabilized also out of equilibrium may prevent the tax authority from putting the economy on an explosive path following a given deviation from the targeted feasible path, simply because all explosive paths may involve an explosive debt. However, as long as it is not required to achieve its goal also out of equilibrium, debt-stabilizing tax policy does not *inherently* generate local-equilibrium multiplicity. This result, thus, restores the role of income or labor-income taxes in safely stabilizing public debt.

In this second case study, I also show how to design an income-tax-rate or labor-income-tax-rate rule that is consistent with the tax authority's observation set and implements any given constant-debt or constant-expected-debt feasible path as the (robustly) unique local equilibrium. The method that I use directly transforms the polynomials characterizing the structural equations and the targeted feasible path into the polynomials characterizing the tax-rate rule, through a finite number of arithmetic operations (addition, subtraction, multiplication, and division). The coefficients of the rule are thus explicitly expressed as *rational functions* of the structural and feasible-path parameters, i.e. as fractions of polynomial functions of these parameters. These functions are particularly easy to manipulate analytically. For instance, their derivatives can be easily computed to determine how the coefficients of the policy-instrument rule respond to an arbitrarily small change in the value of the structural or feasible-path parameters.

The implementability and implementation results obtained in the second case study can be generalized along four dimensions: in terms of model, policy instrument, observation set, and feasible path. To generalize them, I consider a broad class of models with a scalar policy instrument (allowing for policy-implementation lags of any length), and a broad class of observation sets for the policymaker (allowing for observation lags of any length). I provide a sufficient condition for all feasible paths to be implementable, and, under this condition, I show how to design *arithmetically*, i.e. with a finite number of arithmetic operations, a policy-instrument rule that is consistent with the observation set and implements any given feasible path as the (robustly) unique local equilibrium.

The sufficient condition for all feasible paths to be implementable does not involve, in particular, the length of policy-implementation or observation lags (if any). Thus, even though these lags put the policymaker behind the curve by preventing her from reacting out of equilibrium to current or recent endogenous variables, they are irrelevant for feasible-path implementability under this condition.[1] In essence, however late, the policymaker will eventually be able to detect any deviation from the feasible path considered and react to this deviation in such a way that the economy embarks on a non-local (explosive) path.

This paper is the first one to raise and study the issue of feasible-path (non-)implementability. In the literature, the only result that can be interpreted as a feasible-path non-implementability result is that no feasible path may be implementable when the policy maker observes only exogenous shocks (as first shown by Sargent and Wallace, 1975). However, the case in which the policy maker observes only exogenous shocks seems practically irrelevant. Bassetto (2002, 2004, 2005) is a precursor in the study of implementability problems in a broader sense. But the constraints faced out of equilibrium by the policymaker are of a different nature in his papers − e.g. physical (impossibility of spending resources that do not exist), not informational as in my paper (impossibility of setting the policy instrument as a function of unobserved variables).

This paper is also the first one to propose a general method to design analytically a policy-instrument rule consistent with the policymaker's observation set and implementing a given implementable path as the (robustly) unique local equilibrium. Evans and Honkapohja (2003), Svensson and Woodford (2005), and Woodford (2003, Chapter 7) design analytically a policy-instrument rule implementing a specific path as the unique local equilibrium in a specific model. But these rules are not required to be consistent with a given observation set for the policymaker; and the method used to design them can be applied only to simple paths in simple models, as it requires to check whether the system made of the structural equations and a candidate rule satisfies Blanchard and Kahn's (1980) conditions for all structural-parameter values. Giannoni and Woodford (2017), building on their earlier work reported in Woodford (2003, Chapter 8), design analytically, in a general framework, "target criteria" that are consistent with a given path and ensure local-equilibrium determinacy. However, these target criteria do not address the issue of (operational) implementation, as they are typically not formulated as policy-instrument rules, let alone as policy-instrument rules consistent with a given observation set for the policymaker.

[1] For convenience, throughout the paper I refer to the policymaker with the female pronoun "she."

The rest of the paper is organized as follows. Section 2 illustrates the concept of feasible-path (non-)implementability in a simple framework. Sections 3 and 4 are devoted to the two case studies $-$ one about optimal monetary policy in the NK model, the other about debt-stabilizing tax policy in the RBC model. Section 5 derives the general implementability and implementation results. I then conclude and provide a technical appendix.

2 Simple Illustration

This section uses a simple monetary-policy model to introduce the concepts of feasible path and implementable path, and illustrate the two ways in which a feasible path may not be implementable.

2.1 Structural Equation and Observation Set

I consider an endowment economy whose agents are a private sector ($\mathcal{PS}$) and a central bank ($\mathcal{CB}$). At each date $t \in \mathbb{Z}$, $\mathcal{PS}$ sets the inflation rate π_t, and $\mathcal{CB}$ the nominal interest rate i_t. The behavior of $\mathcal{PS}$ obeys the following (locally log-linearized) structural equation:

$$i_t = \mathbb{E}_t \{\pi_{t+1}\} + \xi_t, \tag{1}$$

which is a Fisher equation derived from the consumption Euler equation, the goods-market-clearing condition, and the endowment assumption. The exogenous term ξ_t can be interpreted as a preference disturbance. It is assumed to follow a first-order moving-average process:

$$\xi_t = \varepsilon_t + \theta \varepsilon_{t-1}, \tag{2}$$

where $\theta \in \mathbb{R}$ and ε_t is an i.i.d. exogenous shock of mean zero realized at date t.[2] The operator $\mathbb{E}_t\{.\}$ denotes the rational-expectations operator conditionally on the observation set of $\mathcal{PS}$ when it sets π_t. For simplicity, this observation set is assumed to be made of all current and past endogenous variables and exogenous shocks (including π_t itself,

[2]In this simple model, a non-degenerate moving-average process for the exogenous disturbance (i.e. $\theta \neq 0$) is not needed to illustrate implementable and non-implementable feasible paths, but it is needed to illustrate the two different ways in which a feasible path may not be implementable.

following the standard convention). I thus abstract from any observation constraint for $\mathcal{PS}$, in order to focus on the implications of $\mathcal{CB}$'s observation constraints.

The observation set of $\mathcal{CB}$ when she sets i_t is assumed to be

$$O_t \equiv \{\pi^t, i^{t-1}\},$$

where, for any variable z and any date t, $z^t \equiv \{z_{t-k} | k \in \mathbb{N}\}$ denotes the history of variable z until date t included. Thus, $\mathcal{CB}$ observes current and past inflation rates and past interest rates, but no current or past shock (arguably a reasonable assumption for preference shocks). The behavior of $\mathcal{CB}$ is described by a rule that expresses i_t as a (locally log-linearized) function of elements of O_t. For the sake of practical relevance, I impose the constraint that this function should have a finite (but unbounded) number of arguments.

2.2 Implementable Paths

At this stage of the exposition, a well-established tradition in the literature would specify a given interest-rate rule for $\mathcal{CB}$ – a rule that is consistent with its observation set O_t and ensures local-equilibrium determinacy (i.e., existence and uniqueness of a local equilibrium). Consider, for instance, the Taylor rule

$$i_t = \phi \pi_t \tag{3}$$

with $\phi > 1$. This rule is consistent with O_t (since it expresses i_t as a function of π_t, which belongs to O_t) and delivers a unique local equilibrium (as is well known and shown in, e.g., Woodford, 2003, Chapter 2). It is straightforward to check that, in this unique local equilibrium, the endogenous variables follow the exogenous-shock-contingent path

$$\begin{bmatrix} \pi_t \\ i_t \end{bmatrix} = \begin{bmatrix} \frac{\theta+\phi}{\phi^2}\varepsilon_t + \frac{\theta}{\phi}\varepsilon_{t-1} \\ \frac{\theta+\phi}{\phi}\varepsilon_t + \theta\varepsilon_{t-1} \end{bmatrix}. \tag{4}$$

In this paper, I do not follow this tradition. I do not start from a given rule (as the input) to get a local path (as the output). Instead, I start from a given local path (as the input), which I assume the policymaker would like the economy to follow. And I ask the question of whether there exists a policy-instrument rule that is consistent with the policymaker's observation set and implements, in a minimally robust way, this given local path as the unique local equilibrium. The "minimal-robustness" requirement that I impose is that the addition of an exogenous policy shock of arbitrarily small variance to the policy-instrument rule in question (capturing, e.g., the policymaker's "trembling hand" or round-off errors) should still result in a unique local equilibrium, arbitrarily close to the path considered, rather than no local equilibrium at all. In Subsection 2.5 below, I explain why this minimal-robustness requirement matters and why I impose it.

If there exists such a rule, I say that the path is *implementable*; otherwise, I say it is *not implementable*. In my simple setup, for instance, the path (4) is implementable because there exists an interest-rate rule, namely (3), that has the following three properties: (i) it is is consistent with O_t; (ii) it implements this path as the unique local equilibrium; and (iii) when added an exogenous policy shock, it still delivers a unique local equilibrium (which converges to the path considered as the variance of the policy shock goes to zero).

2.3 Feasible Paths

There are two trivial ways in which a given local path may not be implementable: it may be inconsistent with at least one structural equation, or with the policymaker's observation set. To illustrate these two cases, consider the following two paths:

$$\begin{bmatrix} \pi_t \\ i_t \end{bmatrix} = \begin{bmatrix} \varepsilon_{t-1} \\ \theta\varepsilon_{t-1} \end{bmatrix}, \tag{5}$$

$$\begin{bmatrix} \pi_t \\ i_t \end{bmatrix} = \begin{bmatrix} 0 \\ \varepsilon_t + \theta\varepsilon_{t-1} \end{bmatrix}. \tag{6}$$

The path (5) is consistent with $\mathcal{CB}$'s observation set O_t, since (5) implies $i_t = \theta\pi_t$ and $\pi_t \in O_t$. However, it is not consistent with the structural equation (1), since (5) implies $i_t - \mathbb{E}_t\{\pi_{t+1}\} = -\varepsilon_t + \theta\varepsilon_{t-1} \neq \xi_t$. Alternatively, the path (6) is consistent with the structural equation (1), since (6) implies (1). However, it is not consistent with $\mathcal{CB}$'s observation set O_t, since, on the path (6), i_t depends on ε_t, but no element of O_t does, so that i_t cannot be expressed as a function of only elements of O_t.

To rule out these two uninteresting cases, I focus in the paper on the local paths that are consistent both with the structural equations and with the policymaker's observation set. I say that these paths are *feasible*, and that the other paths are *not feasible*. So, the implementability question that I ask in the paper is, more specifically, the question of the implementability of *feasible* paths.

2.4 Feasible-Path (Non-)Implementability

To illustrate the two different ways in which a feasible path may not be implementable, consider the following path:

$$\begin{bmatrix} \pi_t \\ i_t \end{bmatrix} = \begin{bmatrix} \psi\varepsilon_t \\ \varepsilon_t + \theta\varepsilon_{t-1} \end{bmatrix}, \tag{7}$$

where $\psi \in \mathbb{R} \smallsetminus \{0\}$. This path is consistent with the structural equation (1), since (7) implies (1). Moreover, it is also consistent with the observation set O_t, since (7) implies $i_t = (1/\psi)\pi_t + (\theta/\psi)\pi_{t-1}$ and $\{\pi_t, \pi_{t-1}\} \subset O_t$. Therefore, the path (7) is feasible. The question I now ask is whether it is implementable.

To answer this question, I start by noting that the (locally log-linearized) interest-rate rules consistent with the observation set O_t are the rules of type

$$\mathcal{P}(L)i_t + \mathcal{Q}(L)\pi_t = 0 \tag{8}$$

with $\mathcal{P}(X) \in \mathbb{R}[X]$, $\mathcal{P}(0) \neq 0$, and $\mathcal{Q}(X) \in \mathbb{R}[X]$, where L denotes the lag operator and $\mathbb{R}[X]$ the set of polynomials in X with real-number coefficients. The restriction $\mathcal{P}(0) \neq 0$ ensures that the rule involves i_t. If $\mathcal{P}(X)$ has no root inside the unit circle of the complex plane, then the operator $\mathcal{P}(L)$ is invertible, and the rule (8) can be equivalently rewritten as $i_t = -\mathcal{P}(L)^{-1}\mathcal{Q}(L)\pi_t$ (where $\mathcal{P}(X)^{-1}\mathcal{Q}(X)$ is typically a power series, not a polynomial). But I also allow for polynomials $\mathcal{P}(X)$ that have some roots inside the unit circle, implying that the operator $\mathcal{P}(L)$ is not invertible.[3]

Among the rules consistent with O_t, i.e. among the rules of type (8), the rules that are also consistent with the path (7) are those such that $\mathcal{P}(X)(1 + \theta X) + \mathcal{Q}(X)\psi = 0$. Replacing $\mathcal{Q}(L)$ by $-\mathcal{P}(L)(1+\theta L)/\psi$ in (8), I can rewrite these rules as the rules of type

$$\mathcal{P}(L)\left(i_t - \frac{1}{\psi}\pi_t - \frac{\theta}{\psi}\pi_{t-1}\right) = 0. \tag{9}$$

So the question of whether the path (7) is implementable can be equivalently restated as the question of whether there exists $\mathcal{P}(X) \in \mathbb{R}[X]$ with $\mathcal{P}(0) \neq 0$ such that the rule

[3] In this case, the rule (8) is said to be "superinertial" in the terminology used in the literature (e.g., Woodford, 2003, Chapter 8).

(9) robustly ensures local-equilibrium determinacy. By "robustly," I mean again that the addition of an exogenous policy shock to the rule in question (capturing, e.g., the central bank's "trembling hand" or round-off errors) should still result in a unique local equilibrium, rather than no local equilibrium at all.

To answer the restated question, I add an exogenous policy shock e_t to the rule (9) and ask whether there exists $\mathcal{P}(X) \in \mathbb{R}[X]$ with $\mathcal{P}(0) \neq 0$ such that the resulting rule,

$$\mathcal{P}(L)\left(i_t - \frac{1}{\psi}\pi_t - \frac{\theta}{\psi}\pi_{t-1}\right) = e_t, \tag{10}$$

ensures local-equilibrium determinacy. Clearly, if $\mathcal{P}(X)$ has (at least) one root inside the unit circle, then the equation (10) has no stationary solution in the variable $i_t - (1/\psi)\pi_t + (\theta/\psi)\pi_{t-1}$, and hence no stationary solution in the variables π_t and i_t. I can therefore restrict my search to the polynomials $\mathcal{P}(X)$ that have no root inside the unit circle. In this case, $\mathcal{P}(L)^{-1}$ exists and (10) can be equivalently rewritten as

$$i_t = \frac{1}{\psi}\pi_t + \frac{\theta}{\psi}\pi_{t-1} + \mathcal{P}(L)^{-1}e_t. \tag{11}$$

Replacing i_t in (1) by the right-hand side of (11), I then get the following dynamic equation in π_t:

$$\mathbb{E}_t\left\{\left(\psi - L - \theta L^2\right)\left(\pi_{t+1} - \psi\varepsilon_{t+1}\right)\right\} = \psi\mathcal{P}(L)^{-1}e_t. \tag{12}$$

It is straightforward to solve (12) for π_t by applying Blanchard and Kahn's (1980) analysis. Whether (12) has a unique stationary solution in π_t does not depend on the polynomial $\mathcal{P}(X)$. It depends, however, on the structural parameter θ and the feasible-path parameter ψ. The following three alternative cases are possible.

First, θ and ψ may be such that one root of the characteristic polynomial $\mathcal{Z}(X) \equiv \psi X^2 - X - \theta$ lies *inside* the unit circle, and the other *outside*. In this case, with one outside root for one non-predetermined variable, (12) has a unique stationary solution in π_t, whatever $\mathcal{P}(X)$. So, there exists an infinity of rules consistent with the observation set O_t and the path (7) and robustly ensuring local-equilibrium determinacy. Therefore, the path (7) is implementable.

Second, θ and ψ may be such that both roots of $\mathcal{Z}(X)$ lie *inside* the unit circle. In that case, with zero outside root for one non-predetermined variable, (12) has an infinity of stationary solutions in π_t, whatever $\mathcal{P}(X)$. So, there does not exist any rule consistent with the observation set O_t and the path (7) and robustly ensuring local-equilibrium determinacy. Therefore, the path (7) is not implementable.

Finally, θ and ψ may be such that both roots of $\mathcal{Z}(X)$ lie *outside* the unit circle. In this last case, with two outside roots for only one non-predetermined variable, (12) has no stationary solution in π_t, whatever $\mathcal{P}(X)$. So, there does not exist any rule consistent with the observation set O_t and the path (7) and robustly ensuring local-equilibrium determinacy. Therefore, the path (7) is not implementable.

The last two cases illustrate two different ways in which a feasible path may be non-implementable. One is that all the rules consistent with the observation set and this path may lead to local-equilibrium multiplicity. The other is that all these rules, when added an exogenous policy shock e_t, may lead to non-existence of a local equilibrium.

2.5 Discussion

The minimal-robustness requirement that I impose for implementability does matter. If I did not impose it, the set of rules consistent with the observation set O_t and the path (7) would still be the set of rules of type (9) with $\mathcal{P}(X) \in \mathbb{R}[X]$ and $\mathcal{P}(0) \neq 0$. But the path (7) would then be implementable if and only if there exists one rule in this set that ensures local-equilibrium determinacy (instead of "that *robustly* ensures local-equilibrium determinacy"). As a consequence, the path (7) would be implementable in the third case, i.e. when both roots of $\mathcal{Z}(X)$ lie outside the unit circle. Indeed, by construction, all rules of type (9) are consistent with the path (7), so that the system made of the structural equation (1) and any one of these rules has at least one stationary solution − namely, the path (7). In the third case, with two outside roots for only one non-predetermined variable, this system has no other stationary solution. Therefore, the path (7) would be implementable, if I did not impose the minimal-robustness requirement for implementability.

Such an "implementability" result would, however, be knife-edge and practically useless. In the third case, indeed, the system made of the structural equation (1) and any rule of type (9) has no stationary solution anymore as soon as an exogenous policy shock e_t (capturing, e.g., the policymaker's "trembling hand" or round-off errors) is added to the rule, even if this policy shock is of arbitrarily small variance. It is to avoid such knife-edge and practically useless results that I impose the minimal-robustness requirement.[4] To be clear, this minimal-robustness requirement is nothing more than a re-expression, in a stochastic-singularity context, of (part of) Blanchard and Kahn's (1980) conditions. In other words, a path is implementable if and only if there exists a policy-instrument rule consistent with the observation set and this path and such that the system made of the structural equation(s) and this rule satisfies Blanchard and Kahn's (1980) conditions.

The feasible path (7) considered in this section was chosen for the sake of the illustration. It is, of course, arbitrary: in this flexible-price endowment economy, there is no particular reason why the central bank would want to implement this specific path rather than another one, since inflation and the interest rate do not matter for welfare. In the next section, I show that the issue of feasible-path non-implementability also arises for interesting paths (the optimal feasible path) in textbook models (the NK model).

3 Optimal Monetary Policy in the NK Model

In this section, I study the implementability of the welfare-maximizing feasible path in the NK model, with the central bank as the policymaker and the interest rate as the policy instrument. Rotemberg and Woodford (1999, p. 103) once wrote that "the construction of a feedback rule for the funds rate that implements the optimal allocation − that is not only consistent with it but also renders it the unique stationary equilibrium consistent with the proposed policy rule − remains a nontrivial problem." In essence, this statement is still valid today when the optimal allocation and the feedback rule are required to be consistent with a given observation set of the central bank. In fact, as I show in this section, such a feedback rule may simply not exist for a reasonable observation set of the central bank.

In most of the section, I focus on the *basic* NK model, presented in detail in Woodford (2003, Chapters 2, 4, and 6) and Galí (2015, Chapter 3), with two endogenous variables set

[4] Alternatively and equivalently, I could consider a finite starting date and some initial conditions prior to that date, and require that the rule (without policy shock) ensure local-equilibrium determinacy whatever these initial conditions. However, paths and rules would then have to take transitional dynamics into account, which would substantially and fruitlessly burden the exposition.

by the private sector and two exogenous disturbances. At the end of the section, I extend the analysis to additional endogenous variables and additional exogenous disturbances in the same model, and to Svensson and Woodford's (2005) NK model with monetary-policy-transmission lags.

3.1 Structural Equations and Observation Set

I consider an economy described by the basic NK model and hit by two exogenous disturbances, one affecting the discount factor and the other the elasticity of substitution between differentiated goods. In this economy, at each date $t \in \mathbb{Z}$, the private sector ($\mathcal{PS}$) sets the inflation rate π_t and the output level y_t according to the following (locally log-linearized) IS equation and Phillips curve:

$$y_t = \mathbb{E}_t \{y_{t+1}\} - \frac{1}{\sigma} (i_t - \mathbb{E}_t \{\pi_{t+1}\}) + \eta_t, \tag{13}$$

$$\pi_t = \beta \mathbb{E}_t \{\pi_{t+1}\} + \kappa y_t + u_t, \tag{14}$$

where i_t denotes the interest rate set by the central bank ($\mathcal{CB}$) at date t. I assume that the exogenous disturbances η_t and u_t follow stationary ARMA(1,1) processes:

$$\eta_t = \rho_\eta \eta_{t-1} + \varepsilon_t^\eta + \theta_\eta \varepsilon_{t-1}^\eta, \tag{15}$$

$$u_t = \rho_u u_{t-1} + \varepsilon_t^u + \theta_u \varepsilon_{t-1}^u, \tag{16}$$

where ε_t^η and ε_t^u are two orthogonal i.i.d. exogenous shocks of mean zero. The structural parameters satisfy $0 < \beta < 1$, $\sigma > 0$, $\kappa > 0$, $-1 < \rho_\eta < 1$, and $-1 < \rho_u < 1$ (while θ_η and θ_u may take any real-number value).

The observation set that I consider for $\mathcal{CB}$ is

$$O_t \equiv \left\{\pi^{t-1}, y^{t-1}, i^{t-1}\right\}.$$

This observation set contains no exogenous shock, which seems reasonable given the nature of the two shocks considered. Moreover, it contains no current endogenous variable. This last feature can be justified on the following two grounds.

First, this feature can be viewed as a consequence of the timing in which $\mathcal{CB}$ plays before $\mathcal{PS}$ within each period. This timing is arguably better suited than the reverse timing to capture the fact that, due to information-collecting, information-processing, and decision-making frictions, central banks take their decisions at a lower frequency than the private sector considered as a whole (though not necessarily than each individual private agent). Many papers studying the conduct of monetary policy, from Poole (1970) and Sargent and Wallace (1975) to Svensson and Woodford (2005) and Atkeson, Chari, and Kehoe (2010), explicitly assume that the central bank plays before the private sector at each date and hence does not observe current endogenous variables when setting the interest rate. I follow them.

Second, this feature is necessary for the existence of an optimal feasible path. The optimal feasible path is the path that maximizes welfare subject to the structural equations (13)-(14) and to $\mathcal{CB}$'s observation-set constraint. I assume for simplicity that the steady state of the model is efficient, due to an employment or production subsidy offsetting the monopolistic-competition distortion. Therefore, the welfare-loss function, i.e. the opposite of the second-order approximation of households' intertemporal utility function in the neighborhood of the steady state, can be written as $L_t = \mathbb{E}_t\{\sum_{k=0}^{+\infty}\beta^k[(\pi_{t+k})^2 + \lambda(y_{t+k})^2]\}$, where $\lambda > 0$.

To understand why the optimal feasible path would not exist if $\mathcal{CB}$ also observed current endogenous variables, suppose for a moment that $\mathcal{CB}$'s observation set is $\widehat{O}_t \equiv \{\pi^t, y^t, i^{t-1}\}$, instead of $O_t \equiv \{\pi^{t-1}, y^{t-1}, i^{t-1}\}$. For the sake of the argument, assume for simplicity that the exogenous disturbance η_t is i.i.d. (i.e. $\rho_\eta = \theta_\eta = 0$ and $\eta_t = \varepsilon_t^\eta$), and that there is no disturbance u_t (i.e. $u_t = 0$). Consider first the optimal path, i.e. the path that minimizes L_t subject only to the structural equations (13) and (14), in the absence of any observation-set constraint. This path, denoted by $P^\star$, is trivially $[\,\pi_t \quad y_t \quad i_t\,] = [\,0 \quad 0 \quad \sigma\varepsilon_t^\eta\,]$: on this path, i_t reacts to ε_t^η so as to insulate π_t and y_t from ε_t^η and get $L_t = 0$. Thus, on this path, i_t depends on ε_t^η, but no element of $\widehat{O}_t$ does, so that i_t cannot be expressed as a function of only elements of $\widehat{O}_t$. Therefore, the path $P^\star$ is not feasible.

Now turn to the path $[\,\pi_t \quad y_t \quad i_t\,] = [\,\epsilon\kappa\varepsilon_t^\eta \quad \epsilon\varepsilon_t^\eta \quad (1-\epsilon)\sigma\varepsilon_t^\eta\,]$, where $\epsilon \in \mathbb{R} \setminus \{0\}$. This path, denoted by P_ϵ, is consistent with the structural equations (13) and (14) (since it implies them), and also with the observation set $\widehat{O}_t$ (since it implies $i_t = (1-\epsilon)\sigma\pi_t/(\kappa\epsilon)$ and $\pi_t \in \widehat{O}_t$). Therefore, P_ϵ is feasible. As ϵ is shrunk to zero, the feasible path P_ϵ

converges to the non-feasible path $P^{\star}$, and the value taken by L_t on P_ϵ goes to zero. Thus, the space of feasible paths is not closed, and the optimal path $P^{\star}$ lies at the boundary of this space. Therefore, there is no optimal feasible path when the central bank observes current endogenous variables, and this is the second reason why I consider O_t rather than $\widehat{O}_t$.

3.2 Optimal Feasible Path

The optimal feasible path that I consider is, more specifically, the optimal feasible path under Woodford's (1999) *timeless perspective*. This path can be defined as the limit of the date-t_0 Ramsey-optimal feasible path as $t_0 \to -\infty$. I consider the timeless-perspective optimal feasible path, rather than the date-t_0 Ramsey-optimal feasible path, to avoid having to deal with initial conditions (as explained in Footnote 4).

To determine this path, denoted by P, I proceed in two steps. First, I determine the timeless-perspective optimal feasible path when $\mathcal{CB}$'s observation set is $\widetilde{O}_t \equiv \{\varepsilon^{\eta,t-1}, \varepsilon^{u,t-1}\}$, instead of $O_t \equiv \{\pi^{t-1}, y^{t-1}, i^{t-1}\}$. That path, denoted by $\widetilde{P}$, weakly dominates the path P in the sense that the value taken by L_t on $\widetilde{P}$ cannot be strictly higher that the value taken by L_t on P, since the set of feasible paths under O_t is included into the set of feasible paths under $\widetilde{O}_t$. Second, I show that $\widetilde{P}$ is feasible under O_t. I conclude that the two paths P and $\widetilde{P}$ coincide with each other.

The advantage of this two-step procedure is that $\widetilde{P}$, contrary to P, can be easily obtained with the undetermined-coefficients method. I specify the interest rate i_t as a linear function of the elements of $\{\varepsilon^{\eta,t-1}, \varepsilon^{u,t-1}\}$, and the inflation rate π_t and output y_t as linear functions of the elements of $\{\varepsilon^{\eta,t}, \varepsilon^{u,t}\}$. I look for the values of the coefficients of these linear functions that minimize L_t subject to the structural equations (13) and (14). The computations are standard and of no particular interest, so I relegate them to the Supplementary Appendix S.1. Defining $\mu \equiv (2\beta\lambda)^{-1}[\lambda + \beta\lambda + \kappa^2 - \sqrt{(\lambda + \beta\lambda + \kappa^2)^2 - 4\beta\lambda^2}] \in (0,1)$ and focusing on the generic case $\rho_u \neq \mu$, I get the following minimal-orders ARMA representation for $\widetilde{P}$:

$$(1-\rho_u L)(1-\mu L)\begin{bmatrix} \mathbf{I}_2 & \mathbf{0} \\ \mathbf{0} & (1-\rho_\eta L) \end{bmatrix}\begin{bmatrix} \mathbf{Y}_t \\ i_t \end{bmatrix} = \begin{bmatrix} \mathbf{T}_Y(L) \\ \mathbf{T}_i(L)L \end{bmatrix}\boldsymbol{\varepsilon}_t, \qquad (17)$$

where $\mathbf{Y}_t \equiv [\pi_t \quad y_t]^T$; $\boldsymbol{\varepsilon}_t \equiv [\varepsilon_t^\eta \quad \varepsilon_t^u]^T$; $\mathbf{I}_2$ denotes the 2×2 identity matrix; $\mathbf{T}_Y(X) \in$

$\mathbb{R}^{2\times 2}[X]$, with $\det[\mathbf{T}_Y(0)] \neq 0$; and $\mathbf{T}_i(X) \in \mathbb{R}^{1\times 2}[X]$.[5] In turn, for any $(m,n) \in (\mathbb{N} \setminus \{0\})^2$, $\mathbb{R}^{m\times n}[X]$ denotes the set of polynomials in X whose coefficients are $m \times n$ matrices with real-number elements.

I use the ARMA representation (17) of the exogenous-shock-contingent path $\widetilde{P}$ for commodity reasons. This representation does not directly express the endogenous variables as functions of only current and past exogenous shocks, unlike the MA(1) representations (4), (5), (6), and (7) of the various exogenous-shock-contingent paths considered so far. But it is of course straightforward to invert (17) and get a MA(∞) representation of $\widetilde{P}$, since $|\rho_\eta| < 1$, $|\rho_u| < 1$, and $|\mu| < 1$.

The path $\widetilde{P}$, characterized by (17), is the optimal feasible path under the observation set $\widetilde{O}_t \equiv \{\varepsilon^{\eta,t-1}, \varepsilon^{u,t-1}\}$. I now show that this path is feasible also under the observation set $O_t \equiv \{\pi^{t-1}, y^{t-1}, i^{t-1}\}$. To do so, I rewrite the first two lines of (17) as

$$\det\left[\mathbf{T}_Y\left(L\right)\right] \boldsymbol{\varepsilon}_t = \left(1-\rho_u L\right)\left(1-\mu L\right) adj\left[\mathbf{T}_Y\left(L\right)\right]\mathbf{Y}_t \tag{18}$$

by using Laplace's expansion $adj[\mathbf{T}_Y(X)]\mathbf{T}_Y(X) = \det[\mathbf{T}_Y(X)]\mathbf{I}_2$, where $adj[\mathbf{T}_Y(X)] \in \mathbb{R}^{2\times 2}[X]$ denotes the adjugate of $\mathbf{T}_Y(X)$ (i.e. the transpose of its cofactor matrix). I then multiply the left- and right-hand sides of the last line of (17) by $(1-\rho_u L)^{-1}(1-\mu L)^{-1}\det[\mathbf{T}_Y(L)]$ and use (18) to get

$$\left(1-\rho_\eta L\right)\det\left[\mathbf{T}_Y\left(L\right)\right] i_t = \mathbf{T}_i(L) adj\left[\mathbf{T}_Y\left(L\right)\right]\mathbf{Y}_{t-1}. \tag{19}$$

Since $\det[\mathbf{T}_Y(0)] \neq 0$, the equation (19) expresses i_t as a function of only elements of $O_t \equiv \{\pi^{t-1}, y^{t-1}, i^{t-1}\}$. Therefore, the path $\widetilde{P}$ is feasible under O_t. Now, any path that is feasible under O_t is also feasible under $\widetilde{O}_t$, and $\widetilde{P}$ is the optimal feasible path under $\widetilde{O}_t$. I conclude that P, the optimal feasible path under O_t, coincides with $\widetilde{P}$.

Note that on the path P, characterized by (17), the response of the endogenous variables to both disturbances η_t and u_t is persistent, even when these disturbances are i.i.d. ($\rho_\eta = \rho_u = \theta_\eta = \theta_u = 0$, $\eta_t = \varepsilon_t^\eta$, and $u_t = \varepsilon_t^u$). In the i.i.d. case, indeed, $\mathbf{Y}_t$ follows on this path an ARMA process of autoregressive order 1 with autoregressive parameter

[5] Throughout the paper, I use the superscript T to denote the transpose operator, I use det[.] to denote the determinant operator, and I use letters in bold to denote vectors and matrices that have (at least potentially) more than one element. In particular, $\mathbf{0}$ denotes a vector or a matrix whose elements are all equal to zero and whose dimension depends on the specific context in which it is used.

μ, driven both by ε_t^η and by ε_t^u (since $\det[\mathbf{T}_Y(0)] \neq 0$). As is well known since Clarida, Galí, and Gertler (1999) and Woodford (1999), a persistent response to u_t is optimal because, by making $\mathbb{E}_t\{\pi_{t+1}\}$ depend negatively on u_t, it relaxes the constraint imposed on (π_t, y_t) by the Phillips curve (14). In my setup, similarly, a persistent response to η_t is optimal because, by making $\mathbb{E}_t\{y_{t+1}\} + (1/\sigma)\mathbb{E}_t\{\pi_{t+1}\}$ depend negatively on η_t, it relaxes the constraint imposed on y_t by the IS equation (13) and $\eta_t \notin O_t$.

3.3 (Non-)Implementability of the Optimal Feasible Path

I now turn to the question of whether the optimal feasible path P is *implementable* under O_t. To answer this question, I proceed in the same way as in Subsection 2.4. I start by noting that the (locally log-linearized) interest-rate rules consistent with the observation set O_t are the rules of type

$$\mathcal{P}(L)i_t + \boldsymbol{\mathcal{Q}}(L)\mathbf{Y}_{t-1} = 0 \tag{20}$$

with $\mathcal{P}(X) \in \mathbb{R}[X]$, $\mathcal{P}(0) \neq 0$, and $\boldsymbol{\mathcal{Q}}(X) \in \mathbb{R}^{1\times 2}[X]$. Among these rules, the rules that are also consistent with the path P − characterized by (17) − are those such that

$$\mathcal{P}(X)\,\mathbf{T}_i(X) + (1 - \rho_\eta X)\,\boldsymbol{\mathcal{Q}}(X)\,\mathbf{T}_Y(X) = 0. \tag{21}$$

Multiplying (21) by $adj[\mathbf{T}_Y(X)]$ and using Laplace's expansion $\mathbf{T}_Y(X)adj[\mathbf{T}_Y(X)] = \det[\mathbf{T}_Y(X)]\mathbf{I}_2$ leads to

$$\mathcal{P}(X)\,\mathbf{T}_i(X)\,adj\,[\mathbf{T}_Y(X)] + (1 - \rho_\eta X)\det[\mathbf{T}_Y(X)]\,\boldsymbol{\mathcal{Q}}(X) = 0. \tag{22}$$

Multiplying (20) by $(1 - \rho_\eta L)\det[\mathbf{T}_Y(L)]$ and using (22) then leads to

$$\mathcal{P}(L)\left\{(1 - \rho_\eta L)\det[\mathbf{T}_Y(L)]\,i_t - \mathbf{T}_i(L)\,adj\,[\mathbf{T}_Y(L)]\,\mathbf{Y}_{t-1}\right\} = 0. \tag{23}$$

Given that $\det[\mathbf{T}_Y(0)] \neq 0$, all the equations of type (23) with $\mathcal{P}(X) \in \mathbb{R}[X]$ and $\mathcal{P}(0) \neq 0$ are interest-rate rules consistent with O_t and P. Conversely, all the interest-rate rules consistent with O_t and P are (generically) of type (23) with $\mathcal{P}(X) \in \mathbb{R}[X]$ and $\mathcal{P}(0) \neq 0$, given that $(1 - \rho_\eta X)\det[\mathbf{T}_Y(X)]$ and $\mathbf{T}_i(X)adj[\mathbf{T}_Y(X)]$ have no common root (except in zero-measure cases).[6]

[6] If they had a common root, say a real number r, then the equation $(1-\rho_\eta L)\det[\mathbf{T}_Y(L)](r-L)^{-1}i_t - \mathbf{T}_i(L)adj[\mathbf{T}_Y(L)](r-L)^{-1}\mathbf{Y}_{t-1} = 0$ would also express i_t as a function of a *finite* number of elements of O_t, and therefore would also be an interest-rate rule consistent with O_t and P.

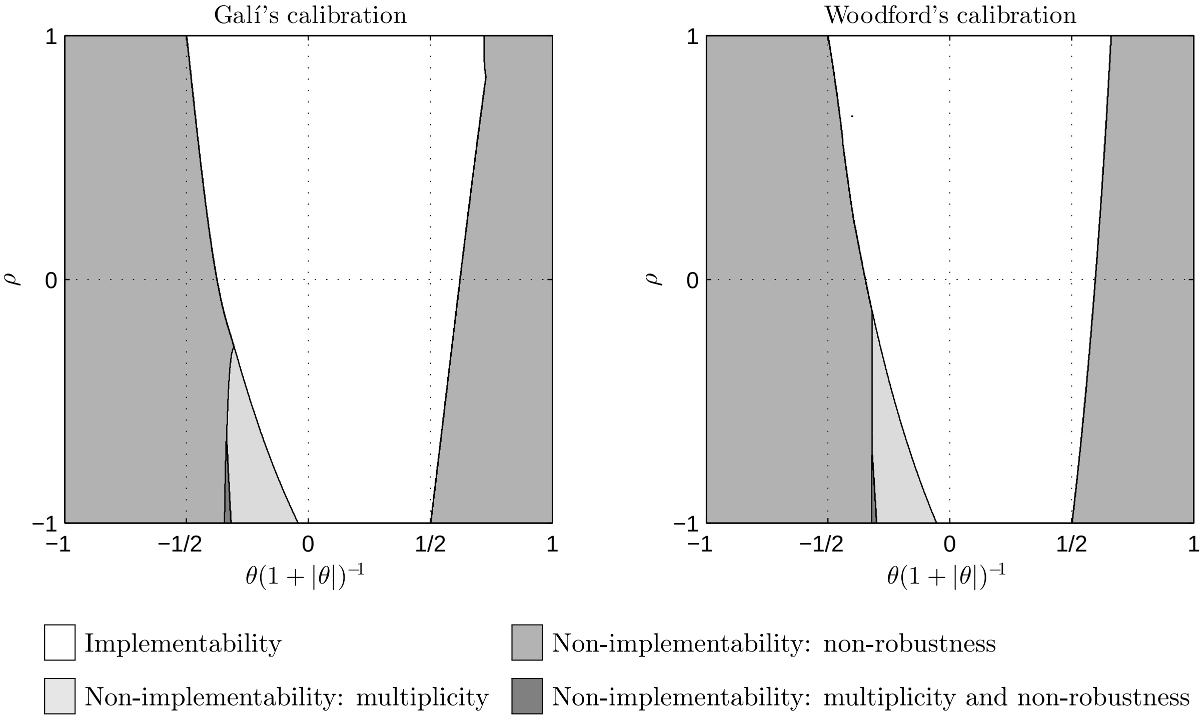
So, the question of whether the path P is implementable can be equivalently restated as the question of whether there exists $\mathcal{P}(X) \in \mathbb{R}[X]$ with $\mathcal{P}(0) \neq 0$ such that the rule (23) robustly ensures local-equilibrium determinacy. I can then conduct exactly the same reasoning as in Subsection 2.4 and conclude that there exists a rule of type (23) robustly ensuring local-equilibrium determinacy if and only if the specific rule (19), corresponding to $\mathcal{P}(X) = 1$, robustly ensures local-equilibrium determinacy in the first place. So, the question of whether the path P is implementable eventually boils down to the question of whether the specific rule (19) robustly ensures local-equilibrium determinacy, i.e. whether the system made of the structural equations (13)-(14) and the specific rule (19) satisfies Blanchard and Kahn's (1980) conditions.

This question can be straightforwardly answered numerically. For instance, let me consider Galí's (2015, Chapter 3) and Woodford's (2003, Chapter 4) calibrations of the basic NK model, respectively characterized by $(\beta, \sigma, \kappa, \lambda) = (0.99, 1.00, 0.125, 0.021)$ and $(\beta, \sigma, \kappa, \lambda) = (0.99, 0.16, 0.022, 0.003)$, and let me focus on the values of ρ_η, ρ_u, θ_η, and θ_u such that $\rho_\eta = \rho_u \equiv \rho$ and $\theta_\eta = \theta_u \equiv \theta$ (so that the two disturbances follow identical stochastic processes). As shown in Figure 1, I then obtain that P is not implementable for many values of ρ and θ, broadly the same values under both calibrations.

For some values of ρ and θ (light-gray areas in Figure 1), P is not implementable because all the interest-rate rules consistent with O_t and P lead to local-equilibrium multiplicity. For other values of ρ and θ (dark-gray areas in Figure 1), it is not implementable because adding an exogenous monetary-policy shock (even of arbitrarily small variance) to any interest-rate rule consistent with O_t and P leads to non-existence of a local equilibrium. For still other values of ρ and θ (very-dark-gray areas in Figure 1), P is not implementable because all the interest-rate rules consistent with O_t and P lead to local-equilibrium multiplicity in the absence of exogenous monetary-policy shocks and to non-existence of a local equilibrium in the presence of such shocks.

In the first two cases, the system made of the structural equations (13)-(14) and the specific rule (19) does not meet Blanchard and Kahn's (1980) *root-counting* condition because it has strictly fewer (in the first case) or strictly more (in the second case) eigenvalues outside the unit circle than non-predetermined variables. In the third case, this system meets Blanchard and Kahn's (1980) *root-counting* condition but not their *no-*

Figure 1 – Implementability of the optimal feasible path,
in the basic NK model, when $\mathcal{CB}$ observes only past endogenous variables



decoupling condition.[7] Sims (2007) has claimed that systems meeting the root-counting condition but not the no-decoupling condition "can easily arise in economic research." This third case provides a concrete economic example in support of that claim.[8]

For positive values of ρ and θ, in particular, P is not implementable provided that θ is sufficiently large, the value $\theta = 3$ (respectively $\theta = 2$) being enough to make P non-implementable for all positive values of ρ under Galí's (respectively Woodford's) calibration. In other words, P is not implementable provided that the responses of the disturbances to the shocks are sufficiently *hump-shaped.* As θ tends towards infinity, P remains non-implementable. This limit case can be interpreted as a situation in which *news shocks* perfectly inform $\mathcal{PS}$ about one-period-ahead disturbances.[9]

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[7]The "no-decoupling condition" requires that the system should not be "decoupled" in the sense of Sims (2007). It is formulated as a matrix-rank condition in Blanchard and Kahn (1980, p. 1308), and is often called the "rank condition" in the literature. Sims' (2007) bare-bones example of a system meeting the root-counting condition but not the no-decoupling condition is $x_t = 1.1x_{t-1} + \varepsilon_t$ and $\mathbb{E}_t\{y_{t+1}\} = 0.9y_t + \nu_t$.

[8]I obtain this numerical result using indifferently my own Matlab code, or Sims' (2001) gensys.m code, or Dynare.

[9]Indeed, as θ tends towards infinity, and the variance of ε_t^η and ε_t^u towards zero at speed θ^2 (so that the variances of $\widetilde{\varepsilon}_t^\eta \equiv \theta\varepsilon_t^\eta$ and $\widetilde{\varepsilon}_t^u \equiv \theta\varepsilon_t^u$ are constant), the stochastic processes of η_t and u_t converge towards $\eta_t = \rho\eta_{t-1} + \widetilde{\varepsilon}_{t-1}^\eta$ and $u_t = \rho u_{t-1} + \widetilde{\varepsilon}_{t-1}^u$, so that $\widetilde{\varepsilon}_t^\eta$ and $\widetilde{\varepsilon}_t^u$ can be interpreted as news shocks perfectly informing $\mathcal{PS}$ about one-period-ahead disturbances.

3.4 Policy Implications

This non-implementability result sounds a note of caution about one of the main lessons of the NK literature, namely the importance for central banks to track some key unobserved exogenous rates of interest such as, for instance, the counterfactual "natural rate of interest" (as emphasized by, e.g., Galí, 2015, Chapter 9, and Woodford, 2003, Chapter 4).[10] This lesson is drawn from analyses that implicitly assume that the central bank can produce an *exogenous estimate* of these exogenous rates of interest, that is to say, in effect, that it can infer them from its observation of exogenous shocks only. Under this assumption, the interest-rate rule can involve that estimate as an exogenous term, which is neutral for robust local-equilibrium determinacy.[11] Add to this term a suitably chosen out-of-equilibrium reaction, and the resulting rule implements, as the robustly unique local equilibrium, a path along which the interest rate is equal to that estimate.

In my framework, for instance, if $\mathcal{CB}$'s observation set were $\{\varepsilon^{\eta,t-1}, \varepsilon^{u,t-1}, \pi^{t-1}, y^{t-1}, i^{t-1}\}$ instead of $O_t \equiv \{\pi^{t-1}, y^{t-1}, i^{t-1}\}$, then $\mathcal{CB}$ could follow the rule

$$i_t = i_t^* + \phi\left(\pi_{t-1} - \pi_{t-1}^*\right) \tag{24}$$

with $1 < \phi < 1 + 2(1+\beta)\sigma/\kappa$, where

$$\begin{aligned} i_t^* &\equiv (1-\rho_\eta L)^{-1}(1-\rho_u L)^{-1}(1-\mu L)^{-1}\mathbf{T}_i(L)\boldsymbol{\varepsilon}_{t-1} \\ \pi_t^* &\equiv (1-\rho_u L)^{-1}(1-\mu L)^{-1}\left[\begin{array}{cc} 1 & 0 \end{array}\right]\mathbf{T}_Y(L)\boldsymbol{\varepsilon}_t \end{aligned}$$

denote the exogenous-shock-contingent values taken by i_t and π_t on the path P. This rule is consistent with the path P, in the sense that it is satisfied on this path. Moreover, it robustly ensures local-equilibrium determinacy, as I show in Appendix A.1. Therefore, this rule robustly implements the path P as the unique local equilibrium. Thus, producing an (exact) exogenous estimate of i_t^* (and of π_{t-1}^*), that is to say inferring i_t^* (and π_{t-1}^*) from the observed past exogenous shocks $\{\varepsilon^{\eta,t-1}, \varepsilon^{u,t-1}\}$ only, enables $\mathcal{CB}$ to robustly implement P as the unique local equilibrium.

[10] In Galí's (2015, Chapter 9) words: "these new models identify tracking the natural equilibrium of the economy, which is not directly observable, as an important challenge for central banks." And in Woodford's (2003, Chapter 4): "keeping track of its current value would be an important (and far from trivial) task of central-bank staff."

[11] Interest-rate rules involving such exogenous rates of interest can be found in, e.g., Woodford (2003, Chapters 4, 5, 7, and 8), and, more recently, Barsky, Justiniano, and Melosi (2014), Cúrdia, Ferrero, Ng, and Tambalotti (2015), and Galí (2015, Chapters 4, 5, and 8).

As my analysis shows, however, things are different when the central bank can only produce *endogenous estimates* of these exogenous rates of interest, as seems more reasonable to assume, i.e. when it has to infer them from its observation of endogenous variables. The result that I obtain is that even when i_t^* (and π_{t-1}^*) can be inferred in many alternative ways, on the optimal feasible path P, from the endogenous variables $\{\pi^{t-1}, y^{t-1}, i^{t-1}\}$ observed by the central bank, there may be no way of setting the interest rate i_t as a function of these variables that implements this path as the robustly unique local equilibrium. In this case, any attempt to track the rate of interest i_t^* and implement the optimal feasible path will inevitably result, in the presence of exogenous policy shocks of arbitrarily small variance, in either local-equilibrium multiplicity or non-existence of a local equilibrium (depending on the values of the structural parameters). Central banks should, therefore, make a cautious use of the key unobserved rates of interest that the NK literature recommends them to track.

3.5 Robustness Analysis: Additional Variables

I have so far focused on two key endogenous variables set by $\mathcal{PS}$: the inflation rate π_t and output y_t. But $\mathcal{PS}$ sets three other endogenous variables in the basic NK model: the consumption level c_t, hours worked h_t, and the real wage w_t. Taking these additional variables into account, and assuming that $\mathcal{CB}$ does not observe them, would leave my non-implementability results unchanged. However, under the alternative assumption that $\mathcal{CB}$ does observe these additional variables, the set of rules consistent with the optimal feasible path and $\mathcal{CB}$'s observation set is broader, raising the question of whether the optimal feasible path is still non-implementable for some structural-parameter values.

In Appendix A.2, I answer this question and show that my non-implementability results, and in particular Figure 1, are unchanged also under this alternative assumption. In essence, the observation of the additional variables by $\mathcal{CB}$ brings additional degrees of freedom in the choice of a rule consistent with $\mathcal{CB}$'s observation set and the optimal feasible path, but these additional degrees of freedom are *neutral* for robust local-equilibrium determinacy. The reason is that any "new rule" can be rewritten as a linear combination of an "old rule" and the (new) structural equations; so, the system made of the structural equations and the new rule is equivalent to the system made of the structural equations and the old rule, and the former system satisfies Blanchard and Kahn's (1980) conditions if and only if the latter system does.

3.6 Robustness Analysis: Additional Disturbances

I have so far assumed that the economy is hit by only two exogenous disturbances (affecting the discount factor and the elasticity of substitution between differentiated goods). In Appendix A.3, I show that my non-implementability results are robust to the relaxation of this assumption. More specifically, I introduce three additional disturbances into the model. These disturbances affect productivity, government purchases, and consumption utility or labor disutility. They follow stationary ARMA processes of arbitrary orders, which are fundamental in the sense of Hansen and Sargent (1981, 1991). I assume that the central bank does not observe them, so that its observation set is the same as previously, namely $\{\pi^{t-1}, y^{t-1}, c^{t-1}, n^{t-1}, w^{t-1}, i^{t-1}\}$. I show that the set of structural-parameter values for which the optimal feasible path is implementable is unaffected by the introduction of these additional disturbances into the model. So, in particular, Figure 1 is still valid in the presence of these additional disturbances.

This robustness is essentially due to the fact that these additional disturbances appear in the *intra-temporal* structural equations, so that their realizations can be inferred from $\mathcal{CB}$'s observation set using only these structural equations, in a way that is neutral for robust local-equilibrium determinacy. By contrast, the realizations of the two disturbances considered so far, η_t and u_t, could not be inferred from $\mathcal{CB}$'s observation set using only the structural equations, because the only structural equations involving them, (13) and (14), are *inter-temporal* and thus also involve unobserved expectations.

3.7 Policy-Transmission Lags

I have so far considered the *basic* NK model, and found that the optimal feasible path is non-implementable only when the stochastic process of the exogenous disturbances has a (sufficiently strong) moving-average component. In this subsection, I introduce monetary-policy-transmission lags into the basic NK model, and I show that the optimal feasible path can then be non-implementable even for a zero moving-average parameter.

More specifically, I consider Svensson and Woodford's (2005) model, which amounts to the basic NK model with one-period monetary-policy-transmission lags and two AR(1) exogenous disturbances with non-negative autoregressive parameters. When these disturbances are interpreted as affecting the discount factor for one and the elasticity of substitution between differentiated goods for the other (as previously), the only changes to be brought to the setup described in Subsection 3.1 are that the IS equation (13) and Phillips curve (14) should be replaced respectively by

$$y_t = \mathbb{E}_{t-1}\{y_{t+1}\} - \frac{1}{\sigma}\left(\mathbb{E}_{t-1}\{i_t\} - \mathbb{E}_{t-1}\{\pi_{t+1}\}\right) + \eta_t, \tag{25}$$

$$\pi_t = \beta\mathbb{E}_{t-1}\{\pi_{t+1}\} + \kappa\mathbb{E}_{t-1}\{y_t\} + u_t, \tag{26}$$

and that the parameters in (15) and (16) now satisfy $(\rho_\eta, \rho_u) \in [0,1)^2$ and $\theta_\eta = \theta_u = 0$.[12] The structural equations (25) and (26) involve expectations formed at date $t-1$ because $\mathcal{PS}$ makes its decisions one period in advance in this model. This feature generates one-period monetary-policy-transmission lags, in the sense that an unexpected announcement made by $\mathcal{CB}$ at date t can affect (π_{t+1}, y_{t+1}), but not (π_t, y_t).

I consider the same observation set for $\mathcal{CB}$ as previously, namely $O_t \equiv \{\pi^{t-1}, y^{t-1}, i^{t-1}\}$. Svensson and Woodford (2005) compute the timeless-perspective optimal feasible path when $\mathcal{CB}$'s observation set is $\{\varepsilon^{\eta,t-1}, \varepsilon^{u,t-1}\}$, instead of O_t. Using their results, it is easy to get the following minimal-orders ARMA representation for this path:

$$(1-\rho_u L)(1-\mu L)\begin{bmatrix} \mathbf{I}_2 & \mathbf{0} \\ \mathbf{0} & (1-\rho_\eta L) \end{bmatrix}\begin{bmatrix} \mathbf{Y}_t \\ i_t \end{bmatrix} = \begin{bmatrix} \mathbf{T}_Y^{SW}(L) \\ \mathbf{T}_i^{SW}(L)L \end{bmatrix}\boldsymbol{\varepsilon}_t, \tag{27}$$

where $\mathbf{T}_Y^{SW}(X) \in \mathbb{R}^{2\times 2}[X]$ and $\mathbf{T}_i^{SW}(X) \in \mathbb{R}^{1\times 2}[X]$, with $\det[\mathbf{T}_Y^{SW}(0)] \neq 0$.[13] This representation is the same as (17), except for the replacement of $\mathbf{T}_Y(X)$ and $\mathbf{T}_i(X)$ by $\mathbf{T}_Y^{SW}(X)$ and $\mathbf{T}_i^{SW}(X)$. So, I can directly infer two results from my analysis in Subsections 3.2 and 3.3. First, the path (27) is the timeless-perspective optimal feasible path not only under the observation set $\{\varepsilon^{\eta,t-1}, \varepsilon^{u,t-1}\}$, but also under the observation set O_t. Second, this path is implementable under O_t if and only if the specific rule

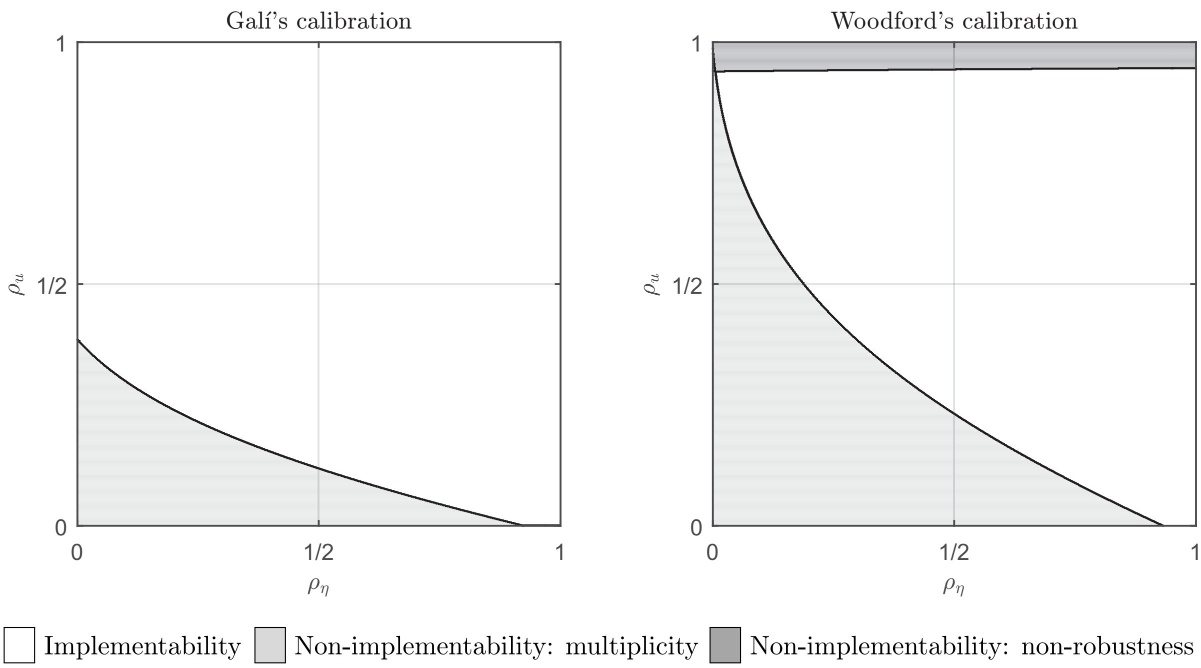
$$D(L)^{-1}(1-\rho_\eta L)\det\left[\mathbf{T}_Y^{SW}(L)\right] i_t = D(L)^{-1}\mathbf{T}_i^{SW}(L) adj\left[\mathbf{T}_Y^{SW}(L)\right]\mathbf{Y}_{t-1} \tag{28}$$

[12] Svensson and Woodford (2005) allow the mean of η_t to be non-zero; for simplicity and without any loss in generality, I set it to zero. They also write the IS equation and the Phillips curve in terms of the welfare-relevant output gap, rather than the output level; under my interpretation of the exogenous disturbances, these two variables coincide with each other.

[13] The superscript "SW" stands for "Svensson-Woodford." The computations are straightforward and of no particular interest, so I relegate them to the Supplementary Appendix S.2.

robustly ensures local-equilibrium determinacy. This rule is the same as (19) except for: (i) the replacement of $\mathbf{T}_Y(X)$ and $\mathbf{T}_i(X)$ by $\mathbf{T}_Y^{SW}(X)$ and $\mathbf{T}_i^{SW}(X)$, and (ii) the division of the left- and right-hand sides by $D(L)$, where $D(X) \equiv (1-\rho_u X)(1-\mu X)$ is the greatest common divisor, defined up to a multiplicative non-zero real-number scalar, of $(1-\rho_\eta X)\det[\mathbf{T}_Y^{SW}(X)]$ and $\mathbf{T}_i^{SW}(X) adj[\mathbf{T}_Y^{SW}(X)]$. So, the optimal feasible path (27) is implementable if and only if the system made of the structural equations (25)-(26) and the specific rule (28) satisfies Blanchard and Kahn's (1980) conditions. Under the same two alternative calibrations as previously, I then obtain that this path is not implementable for many values of ρ_η and ρ_u, as shown in Figure 2.

Figure 2 – Implementability of the optimal feasible path, in Svensson and Woodford's (2005) model, when $\mathcal{CB}$ observes only past endogenous variables



For some values of ρ_η and ρ_u (light-gray areas in Figure 2), and in particular for $\rho_\eta = \rho_u = 0$ (i.i.d. disturbances), the path (27) is not implementable because all the interest-rate rules consistent with O_t and this path lead to local-equilibrium multiplicity. For some other values of ρ_η and ρ_u (dark-gray area in Figure 2), it is not implementable because adding an exogenous monetary-policy shock (even of arbitrarily small variance) to any interest-rate rule consistent with O_t and this path leads to non-existence of a local equilibrium. This result, thus, shows that moving-average disturbances are not necessary for feasible-path non-implementability of either kind.

4 Debt-Stabilizing Tax Policy in the RBC Model

In the previous section, I have illustrated how feasible-path *non-implementability* can be an issue in a textbook model (the NK model), for a standard policy instrument (the interest rate), a relevant observation set (made of past endogenous variables), and an interesting feasible path (the optimal feasible path). In the current section, I illustrate this time how feasible-path *implementability* can obtain, against conventional wisdom, in a textbook model (the RBC model), for a standard policy instrument (the labor-income-tax or income-tax rate), a relevant observation set (made of endogenous variables), and interesting feasible paths (constant-debt or constant-expected-debt feasible paths). I also address the issue of feasible-path *implementation* in this context.

Schmitt-Grohé and Uribe (1997) consider, in the RBC model, a labor-income-tax-rate or income-tax-rate rule that stabilizes, both in and out of equilibrium, the current stock of public debt (in the absence of tax-policy-implementation lags) or the expected future stock of public debt (in the presence of such lags).[14] They find that this rule leads to local-equilibrium multiplicity for many empirically relevant values of the structural parameters. Their finding has largely been interpreted as an argument against the use of labor-income or income taxes to stabilize the current or expected future stock of public debt. However, the fact that *this* (labor-)income-tax-rate rule fails to ensure local-equilibrium determinacy does not imply that *all* the (labor-)income-tax-rate rules that stabilize the current or expected future stock of public debt in equilibrium fail to ensure local-equilibrium determinacy.

In this section, I challenge the interpretation commonly made of their finding by showing that, in the same model, for the same alternative tax instruments, and for a reasonable observation set of the tax authority, all feasible paths along which the current or expected future stock of public debt is stabilized are implementable for all structural-parameter values, even in the presence of tax-policy-implementation lags of any length.[15]

[14]Most of Schmitt-Grohé and Uribe's (1997) analysis is conducted in continuous time. I refer here to the discrete-time analysis conducted at the end of Section 3, in Section 4, and in the Appendix of their paper.

[15]Since Schmitt-Grohé and Uribe (1997), several papers have shown that policy instruments other than labor-income or income taxes (e.g., government purchases in Guo and Harrison, 2004) can be used in the same model to stabilize the current or expected future stock of public debt without generating local-equilibrium multiplicity. My point is that this can be done even with labor-income or income taxes.

I show the implementability of each of these feasible paths by designing *arithmetically*, i.e. with a finite number of arithmetic operations (addition, subtraction, multiplication, and division), a (labor-)income-tax-rate rule that is consistent with the tax authority's observation set and implements this path as the robustly unique local equilibrium. For pedagogical reasons, I design this rule first in the absence of policy-implementation lags, then in the presence of such lags − and for i.i.d. disturbances in both cases. At the end of the section, I extend the analysis to ARMA disturbances.

4.1 Structural Equations and Observation Set

I thus start with the version of Schmitt-Grohé and Uribe's (1997) model *without* tax-policy-implementation lags − and will turn to the version *with* such lags in Subsection 4.4. In this version of their model, at each date $t \in \mathbb{Z}$, the private sector ($\mathcal{PS}$) sets output y_t, the capital stock k_t, investment x_t, hours worked h_t, consumption c_t, the (after-tax) rental price of capital r_t, the (after-tax) wage w_t, and the stock of public debt b_t, according to the following structural equations, log-linearized in the neighborhood of the zero-debt steady state:

$$y_t = a_t + (1-\alpha)k_t + \alpha h_t, \tag{29}$$

$$k_t = (1-\delta)k_{t-1} + \delta x_{t-1}, \tag{30}$$

$$y_t = s_g g_t + s_x x_t + (1 - s_g - s_x)c_t \tag{31}$$

$$w_t = \sigma c_t + \chi h_t, \tag{32}$$

$$c_t = \mathbb{E}_t\{c_{t+1}\} - [1-\beta(1-\delta)]\sigma^{-1}\mathbb{E}_t\{r_{t+1}\}, \tag{33}$$

$$r_t = a_t - \alpha(k_t - h_t) - \omega\tau(1-\tau)^{-1}\tau_t, \tag{34}$$

$$w_t = a_t + (1-\alpha)(k_t - h_t) - \tau(1-\tau)^{-1}\tau_t, \tag{35}$$

$$b_t = \beta^{-1}b_{t-1} + s_g g_t - [\alpha + (1-\alpha)\omega]\tau(y_t + \tau_t), \tag{36}$$

where τ_t denotes the labor-income-tax rate (when $\omega = 0$) or the income-tax rate (when $\omega = 1$) set by the tax authority ($\mathcal{TA}$) at date t.[16] The structural parameters satisfy $0 < \alpha < 1$, $0 < \beta < 1$, $0 < \delta < 1$, $\chi > 0$, $\sigma > 0$, $0 < s_g < 1$, $0 < s_x < 1$, $0 < s_g + s_x < 1$, $0 < \tau < 1$, and $\omega \in \{0, 1\}$. To lighten the exposition, I assume for now that the exogenous

[16]All variables are expressed in percentage deviation from their steady-state value − except public debt b_t, which is expressed as a fraction of steady-state output (since steady-state public debt is zero).

productivity and government-purchases disturbances a_t and g_t are not only orthogonal and of mean zero, but also i.i.d., and will relax this assumption in Subsection 4.6.

The observation set that I consider for $\mathcal{TA}$ is $O_t \equiv \{y^t, x^t, h^t, c^t, u^t, w^t, b^t, \tau^{t-1}\}$. I thus assume that $\mathcal{TA}$ observes all endogenous variables, except the capital stock, and no exogenous disturbance.[17] Moreover, I assume that she observes these variables without any lag − a simplifying assumption that I will also relax in Subsection 4.4.

4.2 Constant-Debt Feasible Paths

To characterize the set of constant-debt feasible paths, I replace b_t and b_{t-1} in (36) by 0 and get

$$\tau_t = -y_t + \frac{s_g}{[\alpha + (1-\alpha)\omega]\,\tau} g_t, \tag{37}$$

which corresponds to the balanced-budget tax-rate rule considered by Schmitt-Grohé and Uribe (1997). I can rewrite (29)-(32), (34)-(35), and (37) as

$$\begin{bmatrix} y_t & x_t & h_t & c_t & r_t & w_t & \tau_t \end{bmatrix}^T = \mathbf{A} \begin{bmatrix} \kappa_t & \kappa_{t-1} & a_t & g_t \end{bmatrix}^T, \tag{38}$$

where $\kappa_t \equiv k_{t+1}$ is determined at t, and $\mathbf{A} \in \mathbb{R}^{7\times 4}$ is defined in the Supplementary Appendix S.3. Using (38), I can in turn rewrite (33) as

$$\mathbb{E}_t \{S(L)\,\kappa_{t+1}\} = A_{43} a_t + A_{44} g_t, \tag{39}$$

where

$$S(X) \equiv \left[A_{41} - \frac{1-\beta(1-\delta)}{\sigma} A_{51}\right] + \left[A_{42} - A_{41} - \frac{1-\beta(1-\delta)}{\sigma} A_{52}\right] X - A_{42} X^2$$

and, for any matrix $\mathbf{M}$, M_{ij} denotes its row-i column-j element. I focus on the case in which the two roots of $S(X)$ lie outside the unit circle, implying that (39) has multiple stationary solutions for κ_t, as it is the case in which Schmitt-Grohé and Uribe's (1997) balanced-budget tax-rate rule (37) leads to indeterminacy. In this case, the class of stationary constant-debt paths that do not involve sunspot shocks is characterized block-recursively by (38) and the following stationary ARMA(2,1) process for κ_t:

$$S(L)\,\kappa_t = \psi_a a_t + A_{43} a_{t-1} + \psi_g g_t + A_{44} g_{t-1}, \tag{40}$$

[17] The results would be identical if $\mathcal{TA}$ were assumed to observe the capital stock and/or one or the two exogenous disturbances.

which is parametrized by $(\psi_a, \psi_g) \in \mathbb{R}^2$.[18] On any of these paths, (31) and (37) are satisfied and together imply the relationship

$$\tau_t = -y_t + \frac{y_t - s_x x_t - (1 - s_g - s_x)c_t}{[\alpha + (1-\alpha)\omega]\,\tau},$$

which expresses τ_t as a function of only elements of O_t. As a consequence, any of these paths is feasible, and the class of constant-debt *feasible* paths is also characterized by (38) and (40). Note finally, for later use, that (38) and (40) imply the following stationary ARMA(2,2) process for τ_t:

$$S(L)\,\tau_t = T_a(L)\,a_t + T_g(L)\,g_t, \tag{41}$$

where $T_a(X) \equiv (A_{71} + A_{72}X)(\psi_a + A_{43}X) + A_{73}S(X)$ and $T_g(X) \equiv (A_{71} + A_{72}X)(\psi_g + A_{44}X) + A_{74}S(X)$.

4.3 Implementability and Implementation of These Paths

Consider an arbitrarily given constant-debt feasible path, characterized by (38) and (40) for some $(\psi_a, \psi_g) \in \mathbb{R}^2$. This path, denoted by P, is implementable if and only if there exists a tax-rate rule consistent with O_t and such that P is the robustly unique stationary solution of the system made of the structural equations (29)-(36) and that rule. In this subsection, I show that there exists such a rule, and I design it arithmetically.

I proceed in four steps. In the first step, I point out that under any tax-rate rule that does not involve the debt level, $(y_t, x_t, h_t, c_t, r_t, w_t, k_t, \tau_t)$ will be (uniquely or not uniquely) determined by the system made of (29)-(35) and that rule, while b_t will be residually determined by (36). Now, in the presence of a tax-policy shock (even of arbitrarily small variance), (36) will generate explosive dynamics for b_t, since $\beta^{-1} > 1$. Therefore, any tax-rate rule that does not involve the debt level will fail to *robustly* ensure local-equilibrium determinacy. In essence, the system made of the structural equations and any such rule will fail to meet Blanchard and Kahn's (1980) *no-decoupling* condition: β^{-1} will be an "unstable eigenvalue" of this system, and the associated eigenvector will be the

[18] I exclude constant-debt paths involving sunspot shocks from the analysis because the tax authority is unlikely to be interested in implementing such paths.

predetermined variable b_t.[19] Thus, the rule that I am looking for necessarily involves the debt level.

In the second step, I rewrite the system of structural equations (29)-(36) in a block-recursive way. More specifically, I rewrite (29)-(32) and (34)-(35) as

$$\left[\, y_t \quad x_t \quad h_t \quad c_t \quad r_t \quad w_t \,\right]^T = \mathbf{B} \left[\, \kappa_t \quad \kappa_{t-1} \quad \tau_t \quad a_t \quad g_t \,\right]^T, \tag{42}$$

where $\mathbf{B} \in \mathbb{R}^{6\times 5}$ is defined in the Supplementary Appendix S.3. In turn, using (42), I rewrite (33) and (36) as

$$\mathbb{E}_t \left\{ P_k\left(L\right) \kappa_{t+1} + P_\tau\left(L\right) \tau_{t+1} \right\} + P_a a_t + P_g g_t = 0, \tag{43}$$

$$Q_b\left(L\right) b_t + Q_k\left(L\right) \kappa_t + Q_\tau \tau_t + Q_a a_t + Q_g g_t = 0, \tag{44}$$

where $(P_k(X), P_\tau(X), Q_b(X), Q_k(X)) \in \mathbb{R}[X]^4$ and $(P_a, P_g, Q_\tau, Q_a, Q_g) \in \mathbb{R}^5$ are defined in the Supplementary Appendix S.3.

In the third step, I design a rule that implements P as the robustly unique local equilibrium, but is not consistent with O_t. Consider the class of rules of type

$$S\left(L\right) \left[\mathcal{R}_b\left(L\right) b_t + \mathcal{R}_\tau\left(L\right) \tau_t\right] + \mathcal{R}_a\left(L\right) a_t + \mathcal{R}_g\left(L\right) g_t = 0, \tag{45}$$

which is parametrized by $(\mathcal{R}_b(X), \mathcal{R}_\tau(X), \mathcal{R}_a(X), \mathcal{R}_g(X)) \in \mathbb{R}[X]^4$ such that $\mathcal{R}_\tau(0) \neq 0$. Since the two roots of $S(X)$ lie outside the unit circle, any rule of type (45) is equivalent to

$$\mathcal{R}_b\left(L\right) b_t + \mathcal{R}_\tau\left(L\right) \tau_t + S\left(L\right)^{-1} \left[\mathcal{R}_a\left(L\right) a_t + \mathcal{R}_g\left(L\right) g_t\right] = 0. \tag{46}$$

Under any rule of this type, (b_t, k_t, τ_t) is (uniquely or not uniquely) determined by the system made of (43)-(44) and (46), while $(y_t, x_t, h_t, c_t, r_t, w_t)$ is residually determined by (42). The system made of (43)-(44) and (46) has exactly one non-predetermined variable, corresponding to the term $\mathbb{E}_t\{P_k(0)\kappa_{t+1} + P_\tau(0)\tau_{t+1}\}$ in (43). It is easy to check that its reciprocical characteristic polynomial is $Z_b(X)\mathcal{R}_b(X) + Z_\tau(X)\mathcal{R}_\tau(X)$, where $Z_b(X) \equiv P_\tau(X)Q_k(X) - P_k(X)Q_\tau$ is of degree 2 and $Z_\tau(X) \equiv P_k(X)Q_b(X)$ of degree 3.

[19] In particular, Schmitt-Grohé and Uribe's (1997) balanced-budget tax-rate rule (37) does not *robustly* ensure local-equilibrium determinacy: add a tax-policy shock to this rule, and the debt level will necessarily explode over time. The case in which the system made of the structural equations and this rule meets Blanchard and Kahn's (1980) *root-counting* condition can be seen as another concrete economic example in support of Sims' (2007) claim − although this example is somewhat trivial in comparison with the one provided in Subsection 3.3.

Now consider an arbitrarily given real number $\phi \in (-1,1) \setminus \{0\}$. In Appendix B.1, I use the Sylvester matrix of $Z_b(X)$ and $Z_\tau(X)$ to arithmetically design $\mathcal{R}_b^*(X) \in \mathbb{R}[X]$ of degree 2 and $\mathcal{R}_\tau^*(X) \in \mathbb{R}[X]$ of degree 1 such that $Z_b(X)\mathcal{R}_b^*(X) + Z_\tau(X)\mathcal{R}_\tau^*(X) = X - \phi$, $\mathcal{R}_b^*(\phi)\mathcal{R}_\tau^*(\phi) \neq 0$, and $\mathcal{R}_\tau^*(0) \neq 0$, except possibly for a zero-measure set of structural-parameter values. Therefore, the system made of (43)-(44) and any rule of type (45) with $(\mathcal{R}_b(X), \mathcal{R}_\tau(X)) = (\mathcal{R}_b^*(X), \mathcal{R}_\tau^*(X))$ has one "unstable eigenvalue," namely ϕ^{-1}, for one non-predetermined variable. In other words, the system made of the structural equations and any such rule meets Blanchard and Kahn's (1980) *root-counting* condition. It also meets their *no-decoupling* condition, since (i) such a rule involves the debt level (as explained in the first step of my analysis), and (ii) $\mathcal{R}_b^*(\phi)\mathcal{R}_\tau^*(\phi) \neq 0$ implies that $\mathcal{R}_b^*(X)$ and $\mathcal{R}_\tau^*(X)$ have no common root inside the unit circle. Therefore, all rules of type (45) with $(\mathcal{R}_b(X), \mathcal{R}_\tau(X)) = (\mathcal{R}_b^*(X), \mathcal{R}_\tau^*(X))$ robustly ensure local-equilibrium determinacy. Among them, the rule

$$S(L)\left[\mathcal{R}_b^*(L)\, b_t + \mathcal{R}_\tau^*(L)\, \tau_t\right] + \mathcal{R}_a^*(L)\, a_t + \mathcal{R}_g^*(L)\, g_t = 0, \tag{47}$$

where $\mathcal{R}_a^*(X) \equiv -\mathcal{R}_\tau^*(X)T_a(X)$ and $\mathcal{R}_g^*(X) \equiv -\mathcal{R}_\tau^*(X)T_g(X)$, is satisfied on the path P, since it is satisfied when both $b_t = 0$ and (41) hold. This rule, therefore, implements P as the robustly unique local equilibrium.

In the fourth and last step, I transform the rule (47) designed in the previous step into a rule consistent with O_t in a way that is neutral for robust local-equilibrium determinacy. Note that (29) and (30) together imply

$$[1-(1-\delta)L]\, a_t = [1-(1-\delta)L]\,(y_t - \alpha h_t) - (1-\alpha)\,\delta x_{t-1}. \tag{48}$$

Multiplying the left- and right-hand sides of (47) by $1-(1-\delta)L$, using (48) to replace $[1-(1-\delta)L]a_t$, and using (31) to replace g_t, I get

$$\begin{aligned} &[1-(1-\delta)L]\, S(L)\left[\mathcal{R}_b^*(L)\, b_t + \mathcal{R}_\tau^*(L)\, \tau_t\right] \\ &+\mathcal{R}_a^*(L)\left\{[1-(1-\delta)L]\,(y_t - \alpha h_t) - (1-\alpha)\,\delta x_{t-1}\right\} \\ &+s_g^{-1}\,[1-(1-\delta)L]\,\mathcal{R}_g^*(L)\left[y_t - s_x x_t - (1-s_g-s_x)\, c_t\right] = 0. \end{aligned} \tag{49}$$

The rule (49) expresses τ_t as a function of only elements of O_t. Moreover, because the only equations used to transform (47) into (49) are the structural equations (29)-(31), the system made of all structural equations and the rule (47) is equivalent to the system made of all structural equations and the rule (49). Since P is the unique stationary solution of the former system, it is therefore also the unique stationary solution of the latter system. Finally, because the root of the polynomial $1-(1-\delta)X$ lies outside the unit circle, (49) still ensures local-equilibrium determinacy when added an exogenous tax-policy shock, like (47). So, to sum up, (49) is consistent with O_t and implements P as the robustly unique local equilibrium. As a consequence, P is implementable, and the rule (49) that I have arithmetically designed implements it.

4.4 Policy-Implementation (or Observation) Lags

I now turn to the version of Schmitt-Grohé and Uribe's (1997) model with tax-policy-implementation lags of length $\ell \in \mathbb{N} \setminus \{0\}$. In this (arguably more realistic) version, $\mathcal{TA}$ has to set unconditionally its tax rate ℓ periods in advance. Introducing such lags into the model amounts to replace τ_t by $\tau_{t-\ell}$ in the structural equations (29)-(36) without changing $\mathcal{TA}$'s observation set O_t, so that the tax rate that matters for date t is now $\tau_{t-\ell}$, which was set at date $t-\ell$ as a function of $O_{t-\ell}$. Note that policy-implementation lags are equivalent to observation lags, since replacing τ_t by $\tau_{t-\ell}$ in the structural equations without changing O_t is equivalent to replacing O_t by $O_{t-\ell}$ without changing the structural equations. Thus, the following analysis can be straightforwardly re-interpreted in terms of observation lags.

Under policy-implementation lags, constant-debt paths are no longer feasible. Indeed, in the structural equation

$$b_t = \beta^{-1} b_{t-1} + s_g g_t - [\alpha + (1-\alpha)\omega]\tau(y_t + \tau_{t-\ell}), \tag{50}$$

which replaces (36), $\tau_{t-\ell}$ cannot offset the effects of disturbances occurring between date $t-\ell+1$ and date t on b_t. Therefore, under policy-implementation lags, Schmitt-Grohé and Uribe (1997) consider the tax-rate rule

$$\tau_t = \mathbb{E}_t \left\{ -y_{t+\ell} + \frac{1}{[\alpha + (1-\alpha)\omega]\,\tau} \left[\beta^{-1} b_{t+\ell-1} + s_g g_{t+\ell}\right] \right\}, \tag{51}$$

instead of the tax-rate rule (37). The rule (51), together with (50), ensures that the expected future debt level is constantly zero: $\mathbb{E}_t\{b_{t+\ell}\} = 0$. Schmitt-Grohé and Uribe

(1997) find that this rule leads to local-equilibrium indeterminacy for many empirically relevant structural-parameter values.

So, the question that I now want to address is whether constant-*expected*-debt feasible paths are implementable or not. The answer may be thought to be negative, or at least not always positive, because policy-implementation lags put $\mathcal{TA}$ behind the curve by preventing the date-t-relevant tax rate (i.e., $\tau_{t-\ell}$) from reacting out of equilibrium to current or recent endogenous variables (i.e., date-$(t-j)$ variables for $j \in \{0, ..., \ell-1\}$). In Appendix B.2, I show that the answer is, in fact, always positive: all constant-expected-debt feasible paths are implementable in the presence of such lags, whatever the length ℓ of these lags, for all structural-parameter values. To establish this result, I proceed in the same way as in Subsections 4.2 and 4.3 and design arithmetically a (labor-)income-tax-rate rule consistent with O_t and implementing any of these paths as the robustly unique local equilibrium. In essence, however late, the tax rate in force can eventually react to any deviation from the targeted constant-expected-debt feasible path in such a way that the economy embarks on a non-local (explosive) path.

4.5 Policy and Methodological Implications

Thus, in the context of Schmitt-Grohé and Uribe's (1997) model, their two alternative tax instruments, and the observation set that I consider, a tax authority can always conduct a tax policy that stabilizes the current or expected future stock of public debt in equilibrium without generating local-equilibrium multiplicity, even in the presence of policy-implementation or observation lags of any length. This result implies that Schmitt-Grohé and Uribe's (1997) finding should be interpreted not as an argument against debt-stabilizing (labor-)income-tax policy *per se*, but instead as an argument against one specific way of conducting this policy: one that achieves the policy goal not only in equilibrium, but also, unnecessarily, out of equilibrium. Requiring that debt be stabilized also out of equilibrium may prevent the tax authority from putting the economy on an explosive path following a deviation from the targeted feasible path, simply because all explosive paths may involve an explosive debt. In this case, it is only by threatening to put the economy on an explosive-debt path that the tax authority can implement the targeted constant-debt or constant-expected-debt feasible path as the robustly unique local equilibrium.[20] This result – that debt-stabilizing tax policy does not *inherently* generate local-equilibrium multiplicity, as long as it is not required to achieve its goal also out of equilibrium – thus restores the role of income or labor-income taxes in safely stabilizing public debt.

On the methodological front, for each constant-debt or constant-expected-debt feasible path, I have shown how to design *arithmetically*, i.e. with a finite number of arithmetic operations (addition, subtraction, multiplication, and division), a (labor-)income-tax-rate rule that is consistent with the tax authority's observation set and implements this path as the robustly unique local equilibrium. This method of designing rules does not require, in particular, to determine any polynomial roots (except trivially roots of polynomials of degree one), nor any inequality condition for determinacy. Instead, it directly transforms: (i) the polynomials and parameters characterizing the structural equations ($P_k(X)$, $P_\tau(X)$, $Q_b(X)$, $Q_k(X)$, and Q_τ), (ii) the polynomials characterizing the targeted feasible path ($S(X)$, $T_a(X)$, and $T_g(X)$), and (iii) the "unstable eigenvalue" chosen to match the non-predetermined variable (ϕ^{-1}), into the polynomials characterizing the rule ($\mathcal{R}_j^*(X)$ for $j \in \{b, \tau, a, g\}$). Because the rule is arithmetically designed, its coefficients are explicitly expressed as *rational functions* of the structural and feasible-path parameters, i.e. as fractions of polynomial functions of these parameters. Such functions are particularly easy to manipulate analytically. For instance, their derivatives can be easily computed – with the help of a symbolic-computation software – to determine how the coefficients of the rule respond to an arbitrarily small change in the value of the structural or feasible-path parameters.

4.6 Robustness Analysis: ARMA Disturbances

I have so far assumed for simplicity that the productivity and government-purchases disturbances a_t and g_t were i.i.d. In Appendix B.4, I show that my implementability and arithmetic-implementation results are robust to the relaxation of this assumption. More specifically, I allow these disturbances to follow stationary ARMA processes of arbitrary orders, which are fundamental in the sense of Hansen and Sargent (1981, 1991).

[20] Despite making debt explode out of equilibrium, this tax policy is "locally Ricardian" in the sense of Woodford (2003, Chapter 4), because it makes debt explode *only if* other endogenous variables explode as well.

I then show that all constant-debt feasible paths (in the absence of policy-implementation lags) and all constant-expected-debt feasible paths (in the presence of such lags) are still implementable under O_t, and I design arithmetically a (labor-)income-tax-rate rule consistent with O_t and implementing any of these paths as the robustly unique local equilibrium. This robustness is essentially due to the facts that: (i) the unobserved innovations of a_t and g_t can be inferred from O_t using the structural equations (29)-(31) and the stochastic processes of a_t and g_t; and (ii) using these structural equations and stochastic processes to replace, in a tax-rate rule, these unobserved innovations by some functions of O_t is *neutral* for robust local-equilibrium determinacy.

5 Stabilization Policy in a General Framework

In this section, I generalize the implementability and implementation results obtained in the previous section along four dimensions: in terms of model, policy instrument, observation set, and feasible path. I first present the classes of structural equations, observation sets, and feasible paths that I consider. I then establish the generalized implementability and implementation results in two cases in turn: the case in which all shocks are observed, and the case in which all unobserved shocks can be inferred from the observation set using only the structural equations.

5.1 Structural Equations and Observation Set

The agents are a private sector ($\mathcal{PS}$) and a policymaker ($\mathcal{PM}$). The behavior of $\mathcal{PS}$ consists in setting, at each date $t \in \mathbb{Z}$, an N-dimension vector of endogenous variables $\mathbf{Y}_t$ according to the following (locally log-linearized) structural equations:

$$\mathbb{E}_t \left\{ \mathbf{\Delta} \left(L^{-1} \right) \left[\mathbf{A} \left(L \right) \mathbf{Y}_t + L^{-z} \mathbf{B} \left(L \right) i_t \right] \right\} + \mathbf{C}(L) \boldsymbol{\xi}_t = \mathbf{0}, \tag{52}$$

where i_t denotes the policy instrument set by $\mathcal{PM}$ at date t, $\boldsymbol{\xi}_t$ a N_ξ-dimension vector of exogenous disturbances, and $\mathbb{E}_t\{.\}$ the rational-expectations operator conditionally on the observation set of $\mathcal{PS}$ when it sets $\mathbf{Y}_t$. As previously, I assume for simplicity that $\mathcal{PS}$'s observation set is made of all current and past endogenous variables and exogenous shocks, thus abstracting from any observation constraint for $\mathcal{PS}$, in order to focus on the implications of $\mathcal{PM}$'s observation constraints.

The structural equations (52) are parametrized by $(N, N_\xi) \in (\mathbb{N} \setminus \{0\})^2$, $z \in \mathbb{Z}$, $\mathbf{A}(X) \in \mathbb{R}^{N\times N}[X]$, $\mathbf{B}(X) \in \mathbb{R}^{N\times 1}[X]$, $\mathbf{C}(X) \in \mathbb{R}^{N\times N_\xi}[X]$, and the diagonal matrix $\boldsymbol{\Delta}(X) \in \mathbb{R}^{N\times N}[X]$ whose j^{th} diagonal element is X^{δ_j}, where $\delta_j \in \mathbb{N}$. Let $\Psi_j(X) \in \mathbb{R}[X]$, for each $j \in \{1, ..., N+1\}$, denote the determinant of the $N \times N$ matrix obtained by removing the j^{th} column of the $N\times(N+1)$ matrix $X^{z^+}[\,\mathbf{A}(X) \quad X^{-z}\mathbf{B}(X)\,]$, where $z^+ \equiv \max(0, z) \geq 0$. I make the following three non-restrictive assumptions on $\mathbf{A}(X)$ and $\mathbf{B}(X)$:

Assumption 1: $\det[\mathbf{A}(0)] \neq 0$.

Assumption 2: $\mathbf{B}(0) \neq \mathbf{0}$.

Assumption 3: $\forall j \in \{1, ..., N\}$, $\Psi_j(X) \neq 0$.

These assumptions are made without any loss in generality for the following reasons. First, any system of *independent* structural equations of type (52) that does not satisfy Assumption 1 can be equivalently rewritten as a system of type (52) that satisfies this assumption. Second, any system of type (52) needs to satisfy $\mathbf{B}(X) \neq \mathbf{0}$ for $\mathcal{PM}$'s policy instrument to have an effect on the endogenous variables set by $\mathcal{PS}$, and any system of type (52) satisfying $\mathbf{B}(X) \neq \mathbf{0}$ but not Assumption 2 can be equivalently rewritten as a system of type (52) satisfying Assumption 2 (simply by changing the value of z). And third, if Assumption 3 were not satisfied, i.e. if there existed $j \in \{1, ..., N\}$ such that $\Psi_j(X) = 0$, then there would exist a linear combination of the structural equations that would involve only elements of $\{\mathbb{E}_t\{Y^j_{t+k}\}|k \in \mathbb{Z}\}$ and $\{\mathbb{E}_t\{\boldsymbol{\xi}_{t+k}\}|k \in \mathbb{Z}\}$, where Y^j_t denotes the j^{th} element of $\mathbf{Y}_t$, so that the variable Y^j_t should then be considered as exogenous, not endogenous.

The vector of exogenous disturbances $\boldsymbol{\xi}_t$ follows a stationary VARMA process:

$$\mathbf{D}(L)\boldsymbol{\xi}_t = \mathbf{E}(L)\boldsymbol{\varepsilon}_t, \tag{53}$$

where $\boldsymbol{\varepsilon}_t$ is a N_ε-dimension vector of orthogonal i.i.d. exogenous shocks of mean zero. This process is parametrized by $N_\varepsilon \in \mathbb{N} \setminus \{0\}$, $\mathbf{D}(X) \in \mathbb{R}^{N_\xi\times N_\xi}[X]$, and $\mathbf{E}(X) \in \mathbb{R}^{N_\xi\times N_\varepsilon}[X]$, where $\det[\mathbf{D}(0)] \neq 0$ and $\det[\mathbf{D}(X)]$ has no root inside the unit circle.

Overall, the specification (52)-(53) encompasses, arguably, most dynamic stochastic general-equilibrium (DSGE) models commonly used for policy analysis. In particular, by allowing z to take any negative value, this specification allows for policy-implementation lags of any length (as in Subsection 4.4).

Let O_t denote the observation set of $\mathcal{PM}$ when she sets i_t. I consider the class of alternative observation sets of type

$$O_t = \{\mathbf{Y}^{\mathcal{J},t-\ell}, i^{t-1}, \boldsymbol{\varepsilon}^{\mathcal{K},t-\ell}\} \tag{54}$$

with $\mathcal{J} \subseteq \{1, ..., N\}$, $\mathcal{K} \subseteq \{1, ..., N_\varepsilon\}$, and $\ell \in \mathbb{N}$, where $\mathbf{Y}_t^{\mathcal{J}}$ and $\boldsymbol{\varepsilon}_t^{\mathcal{K}}$ denote the vectors whose elements are, respectively, the j^{th} elements of $\mathbf{Y}_t$ for $j \in \mathcal{J}$ and the k^{th} elements of $\boldsymbol{\varepsilon}_t$ for $k \in \mathcal{K}$.

This class of observation sets allows for the non-observation of some endogenous variables (i.e. for $\mathcal{J} \subsetneq \{1, ..., N\}$), for the observation of some exogenous shocks (i.e. for $\mathcal{K} \neq \varnothing$), and for observation lags of any length (i.e. for $\ell \geq 1$). One may think of Lagrange multipliers of $\mathcal{PS}$'s optimization problems as an example of unobserved endogenous variables; and of exogenous policy measures or foreign macroeconomic developments (considered as exogenous from the point of view of a small open economy) as examples of observed exogenous shocks. Observation lags may be due to macroeconomic-data-publication lags (as emphasized by, e.g., McCallum, 1999). Observation lags of length $\ell = 1$, more specifically, may also be due to the timing assumption that $\mathcal{PM}$ plays before $\mathcal{PS}$ within each period (as in Subsection 3.1).

Note, for later use, that since $O_t \subseteq \{\mathbf{Y}^{t-\ell}, i^{t-1}, \boldsymbol{\varepsilon}^{t-\ell}\}$, the set of (locally log-linearized) policy-instrument rules consistent with O_t is included in the set of rules of type

$$\boldsymbol{\mathcal{F}}(L)\,\mathbf{Y}_{t-\ell} + \mathcal{G}(L)\,i_t + \boldsymbol{\mathcal{H}}(L)\,\boldsymbol{\varepsilon}_{t-\ell} = 0 \tag{55}$$

with $\boldsymbol{\mathcal{F}}(X) \in \mathbb{R}^{1\times N}[X]$, $\mathcal{G}(X) \in \mathbb{R}[X]$, $\mathcal{G}(0) \neq 0$, and $\boldsymbol{\mathcal{H}}(X) \in \mathbb{R}^{1\times N_\varepsilon}[X]$. As previously, the fact that $\boldsymbol{\mathcal{F}}(X)$, $\mathcal{G}(X)$, and $\boldsymbol{\mathcal{H}}(X)$ are polynomials (rather than power series) reflects the constraint − imposed for the sake of practical relevance − that a rule should express the policy instrument as a function of a finite (but unbounded) number of arguments.

5.2 Technical Preliminaries

I first establish a useful preliminary result. To state this preliminary result, let me denote by $\mathcal{J}^\star$ the set of integers $j \in \{1, ..., N\}$ such that the structural equations (52) feature at least one term of type $\mathbb{E}_t\{Y^j_{t+k}\}$ with $k \geq 1$. Following standard practice, let me also call "eigenvalues" of a rational-expectations system that can be written in Blanchard and Kahn's (1980) form the eigenvalues of the matrix that characterizes the deterministic part of this form.[21] Finally, let me denote by $\mathcal{F}_j(X)$, for any $j \in \{1, ..., N\}$, the j^{th} element of $\boldsymbol{\mathcal{F}}(X)$. The preliminary result, which I prove in Appendix C.1, can then be stated as follows:

Lemma 1: *for any rule of type (55) such that $X^{\ell-z-1}\boldsymbol{\mathcal{F}}(X) \in \mathbb{R}^{1\times N}[X]$ and $\mathcal{F}_j(X) \neq 0$ for all $j \in \mathcal{J}^\star$, the system made of the structural equations (52) and this rule can be written in Blanchard and Kahn's (1980) form with $\delta \equiv \sum_{j=1}^N \delta_j$ non-predetermined variables, and its non-zero eigenvalues are the inverses of the non-zero roots of the polynomial*

$$\sum\nolimits_{j=1}^{N} (-1)^{N+1-j} X^\ell \mathcal{F}_j(X) \Psi_j(X) + \mathcal{G}(X) \Psi_{N+1}(X). \qquad (56)$$

The condition $X^{\ell-z-1}\boldsymbol{\mathcal{F}}(X) \in \mathbb{R}^{1\times N}[X]$ stated in Lemma 1 just means that $X^{\ell-z-1}\boldsymbol{\mathcal{F}}(X)$ has no negative powers of X. This condition ensures that the system made of the structural equations and the policy-instrument rule has the same number of non-predetermined variables as the system made of the structural equations with an exogenous policy instrument. The latter system has exactly δ non-predetermined variables, given Assumption 1. The other condition stated in Lemma 1, namely $\mathcal{F}_j(X) \neq 0$ for all $j \in \mathcal{J}^\star$, ensures that the system mentioned in the lemma can be written in Blanchard and Kahn's (1980) form.[22] The last part of the lemma, finally, is essentially a consequence of Laplace's expansion.

5.3 Case in Which Shocks Are Observed

I start with the case in which $\mathcal{PM}$ observes all exogenous shocks (i.e. $\mathcal{K} = \{1, ..., N_\varepsilon\}$). This case is admittedly restrictive, but the results obtained will also serve, in the next

[21] In other words, the eigenvalues of a rational-expectations system that can be written in a form of type $\mathbb{E}_t\{\mathbf{Z}_{t+1}\} = \mathbf{M}\mathbf{Z}_t + \boldsymbol{\eta}_t$, where $\mathbf{Z}_t$ is a vector of endogenous variables set at date t or earlier and $\boldsymbol{\eta}_t$ a vector of stochastic exogenous terms realized at date t or earlier, are the eigenvalues of the matrix $\mathbf{M}$.

[22] To illustrate the role of this condition, consider the model of Section 2, with the single structural equation (1). If this condition were not met, then the interest-rate rule would not involve any inflation term, so that the only inflation term featuring in the system made of the structural equation and the rule would be the term $\mathbb{E}_t\{\pi_{t+1}\}$ in the structural equation, which would imply that π_t is indeterminate and that the system cannot be written in Blanchard and Kahn's (1980) form.

subsection, as a useful starting point to investigate an alternative case in which some (possibly all) exogenous shocks are unobserved.

I provide a sufficient condition for all feasible paths to be *arithmetically implementable* in this case. By an "arithmetically implementable" feasible path, I mean a feasible path that is not only implementable, but also such that one can design *arithmetically* a policy-instrument rule consistent with the observation set and implementing this path as the robustly unique local equilibrium (as in Subsection 4.3). And by "all feasible paths," I mean more specifically all feasible paths for the endogenous variables that can be written as stationary VARMA processes driven by the vector of exogenous shocks $\boldsymbol{\varepsilon}_t$, i.e. all feasible paths of type

$$\mathbf{S}(L)\begin{bmatrix}\mathbf{Y}_t \\ i_t\end{bmatrix} = \mathbf{T}(L)\,\boldsymbol{\varepsilon}_t \tag{57}$$

with $\mathbf{S}(X) \in \mathbb{R}^{(N+1)\times(N+1)}[X]$, $\det[\mathbf{S}(0)] \neq 0$, and $\mathbf{T}(X) \in \mathbb{R}^{(N+1)\times N_\varepsilon}[X]$, where $\det[\mathbf{S}(X)]$ has no root inside the unit circle. Note, for later use, that one can use Cramer's rule to rewrite (57) as

$$\det[\mathbf{S}(L)]\begin{bmatrix}\mathbf{Y}_t \\ i_t\end{bmatrix} = \begin{bmatrix}\mathbf{T}_Y(L) \\ \mathbf{T}_i(L)\,L^\ell\end{bmatrix}\boldsymbol{\varepsilon}_t, \tag{58}$$

where $\mathbf{T}_Y(X) \in \mathbb{R}^{N\times N_\varepsilon}[X]$ and $\mathbf{T}_i(X) \in \mathbb{R}^{1\times N_\varepsilon}[X]$. The presence of L^ℓ in factor of $\mathbf{T}_i(L)$ in (58) comes from the fact that the path (57) is feasible under $O_t \equiv \{\mathbf{Y}^{\mathcal{J},t-\ell}, i^{t-1}, \boldsymbol{\varepsilon}^{\mathcal{K},t-\ell}\}$, implying that i_t cannot depend on $\boldsymbol{\varepsilon}_{t-k}$ for $k \in \{0, ..., \ell-1\}$ on this path (when $\ell \geq 1$).

The sufficient condition for all feasible paths to be arithmetically implementable is that $D_{\mathcal{J}}(X)$ should have no root inside the unit circle, where $D_{\mathcal{J}}(X) \equiv \gcd[\Psi_j(X)]_{j\in\mathcal{J}\cup\{N+1\}} \in \mathbb{R}[X]$ denotes the greatest common divisor, defined up to a multiplicative non-zero real-number scalar, of all the polynomials $\Psi_j(X)$ for $j \in \mathcal{J}\cup\{N+1\}$ (none of which is zero, given Assumptions 1 and 3). The following proposition formalizes this result:

Proposition 1 (Arithmetic Implementability of Feasible Paths When Shocks Are Observed): *for any set $\mathcal{J}$ such that $\mathcal{J}^\star \subseteq \mathcal{J} \subseteq \{1, ..., N\}$ and any $\ell \in \mathbb{N}$, if $D_{\mathcal{J}}(X)$ has no root inside the unit circle, then all feasible paths of type (57) are arithmetically implementable when $O_t = \{\mathbf{Y}^{\mathcal{J},t-\ell}, i^{t-1}, \boldsymbol{\varepsilon}^{t-\ell}\}$.*

The proof of Proposition 1 below can be viewed as a generalization of the first three steps of the reasoning conducted in Subsection 4.3. This proof rests on the arithmetic design, in Appendix C.2, of some polynomials $\boldsymbol{\mathcal{F}}^*(X) \in \mathbb{R}^{1\times N}[X]$ and $\mathcal{G}^*(X) \in \mathbb{R}[X]$ with five key properties. These polynomials will serve as $\boldsymbol{\mathcal{F}}(X)$ and $\mathcal{G}(X)$ in the rule (55). In Subsection 4.3 (and Appendix B.1), I used a Sylvester matrix to design $\boldsymbol{\mathcal{F}}^*(X)$ and $\mathcal{G}^*(X)$.[23] In Appendix C.2, I cannot use a Sylvester matrix because $\boldsymbol{\mathcal{F}}(X)$ has generically more than one non-zero element, and Sylvester matrices are associated with only two scalar polynomials. Instead, I use Bézout's identity, which can be applied to an arbitrary finite number of scalar polynomials.[24]

Proof of Proposition 1: In Appendix C.2, I design arithmetically some polynomials $\boldsymbol{\mathcal{F}}^*(X)$ and $\mathcal{G}^*(X)$ with the following five properties: (i) $X^{\ell-z-1}\boldsymbol{\mathcal{F}}^*(X) \in \mathbb{R}^{1\times N}[X]$ and $\mathcal{F}_j^*(X) \neq 0$ for all $j \in \mathcal{J}^\star$; (ii) the polynomial (56) with $\boldsymbol{\mathcal{F}}(X) = \boldsymbol{\mathcal{F}}^*(X)$ and $\mathcal{G}(X) = \mathcal{G}^*(X)$ has exactly δ non-zero roots inside the unit circle; (iii) $\boldsymbol{\mathcal{F}}^*(X)$ and $\mathcal{G}^*(X)$ have no common root inside the unit circle; (iv) $\{\det[\mathbf{S}(X)]\}^{-1}\boldsymbol{\mathcal{F}}^*(X) \in \mathbb{R}^{1\times N}[X]$ and $\{\det[\mathbf{S}(X)]\}^{-1}\mathcal{G}^*(X) \in \mathbb{R}[X]$; and (v) $\mathcal{F}_j^*(X) = 0$ for all $j \in \{1, ..., N\} \setminus \mathcal{J}$. The role of the restriction $\mathcal{J}^\star \subseteq \mathcal{J}$ in Proposition 1 is to ensure that Properties (i) and (v) are consistent with each other.

Property (i) implies that the conditions stated in Lemma 1 are satisfied for $\boldsymbol{\mathcal{F}}(X) = \boldsymbol{\mathcal{F}}^*(X)$. Property (ii) and Lemma 1 imply that the system made of the structural equations (52) and any rule of type (55) with $\boldsymbol{\mathcal{F}}(X) = \boldsymbol{\mathcal{F}}^*(X)$ and $\mathcal{G}(X) = \mathcal{G}^*(X)$ meets Blanchard and Kahn's (1980) *root-counting* condition. Property (iii) and the assumption that $D_{\mathcal{J}}(X)$ has no root inside the unit circle imply that this system meets Blanchard and Kahn's (1980) *no-decoupling* condition too.[25] Therefore, this system has a unique stationary solution, even when an exogenous policy shock is added to the rule. Property (iv) means that $\boldsymbol{\mathcal{F}}^*(X)$ and $\mathcal{G}^*(X)$ are polynomial multiples of $\det[\mathbf{S}(X)]$. It implies

[23] In the third step of the reasoning conducted in Subsection 4.3, we had $\mathcal{G}^*(X) = S(X)\mathcal{R}_\tau^*(X)$, and the only non-zero element of $\boldsymbol{\mathcal{F}}^*(X)$ was $S(X)\mathcal{R}_b^*(X)$.

[24] Bézout's identity is named after Bézout (1767), who extended to polynomials a result first obtained for integers by Bachet de Méziriac (1624, Proposition 18). It is sometimes unnamed and presented as a corollary of the Euclidean algorithm (as in, e.g., Prasolov, 2004, Chapter 2, Theorem 2.1.1). It has already been used in economics by d'Autumne (1990) and Loisel (2009).

[25] If Property (iii) were not satisfied, i.e. if $\boldsymbol{\mathcal{F}}^*(X)$ and $\mathcal{G}^*(X)$ had a common root inside the unit circle, say a real number $r \in (-1, 1)$, then any rule of type (55) with $\boldsymbol{\mathcal{F}}(X) = \boldsymbol{\mathcal{F}}^*(X)$, $\mathcal{G}(X) = \mathcal{G}^*(X)$, and an exogenous policy shock would generate explosive dynamics for the variable $(r-L)^{-1}[\boldsymbol{\mathcal{F}}^*(L)\mathbf{Y}_t + \mathcal{G}^*(L)i_t]$. Similarly, if $D_{\mathcal{J}}(X)$ had a root inside the unit circle, then the polynomial (56) with $\boldsymbol{\mathcal{F}}(X) = \boldsymbol{\mathcal{F}}^*(X)$ and $\mathcal{G}(X) = \mathcal{G}^*(X)$ would have the same root, so that the system would have an "unstable eigenvalue" whose eigenvector would be a linear combination of the variables Y_t^j for $j \in \{1, ..., N\} \setminus \mathcal{J}$, all of which are predetermined given that $\mathcal{J}^\star \subseteq \mathcal{J}$. As an illustration, this case would arise in Section 4 if the debt level were unobserved (as discussed in Subsection 4.3).

that the value $\boldsymbol{\mathcal{H}}^*(X)$ of $\boldsymbol{\mathcal{H}}(X)$ that makes the rule (55) with $\boldsymbol{\mathcal{F}}(X) = \boldsymbol{\mathcal{F}}^*(X)$ and $\mathcal{G}(X) = \mathcal{G}^*(X)$ consistent with the targeted feasible path (57) is a polynomial, i.e. a power series of *finite* degree:

$$\boldsymbol{\mathcal{H}}^*(X) \equiv -\{\det[\mathbf{S}(X)]\}^{-1}[\boldsymbol{\mathcal{F}}^*(X)\mathbf{T}_Y(X) + \mathcal{G}^*(X)\mathbf{T}_i(X)] \in \mathbb{R}^{1\times N_\varepsilon}[X].$$

Finally, Property (v) implies that the rule (55) with $\boldsymbol{\mathcal{F}}(X) = \boldsymbol{\mathcal{F}}^*(X)$, $\mathcal{G}(X) = \mathcal{G}^*(X)$, and $\boldsymbol{\mathcal{H}}(X) = \boldsymbol{\mathcal{H}}^*(X)$ is consistent with the observation set O_t. To sum up, I have arithmetically designed a rule of type (55) that is consistent with the observation set O_t and implements the targeted feasible path (57) as the robustly unique local equilibrium. Proposition 1 follows. ■

The method of designing policy-instrument rules that I have just used generalizes the method that I used in Section 4. This method directly transforms the polynomials $\mathbf{A}(X)$ and $\mathbf{B}(X)$ characterizing the structural equations, the polynomials $\mathbf{S}(X)$ and $\mathbf{T}(X)$ characterizing the targeted feasible path, and the polynomial whose roots are the "unstable eigenvalues" chosen to match the non-predetermined variables, into the polynomials $\boldsymbol{\mathcal{F}}(X)$, $\mathcal{G}(X)$, and $\boldsymbol{\mathcal{H}}(X)$ characterizing the policy-instrument rule. This transformation involves only a finite number of arithmetic operations (addition, subtraction, multiplication, and division). Therefore, for any model, policy instrument, observation set, and feasible path, the coefficients of the corresponding policy-instrument rule can be explicitly expressed as *rational functions* of the structural and feasible-path parameters, i.e. as fractions of polynomial functions of these parameters − with obvious advantages in terms of analytical tractability.

The condition stated in Proposition 1 − that $D_{\mathcal{J}}(X)$ should have no root inside the unit circle − does not involve the observation-lags length ℓ. Therefore, observation lags are irrelevant for feasible-path implementability when shocks are observed. So are policy-implementation lags, given the equivalence, discussed in Subsection 4.4, between these two kinds of lags. This irrelevance result, which echoes a similar result in Subsection 4.4, may be viewed as surprising at first sight. Indeed, most of the policy-instrument rules considered in the literature, in particular Taylor's (1993) interest-rate rule in DSGE models, manage to ensure (robust) local-equilibrium determinacy by requiring the policy instrument to react out of equilibrium to *current* endogenous variables. Because they prevent $\mathcal{PM}$ from reacting out of equilibrium to *current* or even *recent* endogenous variables, observation and policy-implementation lags could be suspected to reduce her ability to ensure robust local-equilibrium determinacy and, therefore, her ability to implement any feasible path as the robustly unique local equilibrium. Proposition 1 shows that they do not. Whatever the length of these lags, $\mathcal{PM}$ will eventually be able to detect any off-equilibrium deviation from the targeted feasible path and react to this deviation in such a way that the economy embarks on an explosive path.

5.4 Case in Which Unobserved Shocks Are Inferable

I now turn to the case in which $\mathcal{PM}$ does not observe all exogenous shocks (i.e. $\mathcal{K} \subsetneq \{1, ..., N_\varepsilon\}$), but all the unobserved shocks can be inferred from the observed variables and shocks using *only* the structural equations (52) and the stochastic process (53). More specifically, I mean the case in which (52) and (53) alone imply a relationship of type

$$\mathbf{\Gamma}(L)\boldsymbol{\varepsilon}_t = \mathbf{\Lambda}(L) \begin{bmatrix} \mathbf{Y}_t^{\mathcal{J}} \\ i_{t-1} \\ \boldsymbol{\varepsilon}_t^{\mathcal{K}} \end{bmatrix} \tag{59}$$

with $\mathbf{\Gamma}(X) \in \mathbb{R}^{N_\varepsilon \times N_\varepsilon}[X]$, $\det[\mathbf{\Gamma}(0)] \neq 0$, $\det[\mathbf{\Gamma}(X)]$ having no root inside the unit circle, and $\mathbf{\Lambda}(X) \in \mathbb{R}^{N_\varepsilon \times (|\mathcal{J}|+|\mathcal{K}|+1)}[X]$, where $|.|$ denotes the cardinality operator (when applied to a set). In other words, (52) and (53) alone should make the unobserved shocks "fundamental," in the sense of Hansen and Sargent (1981, 1991), for the observed variables and shocks. This is the case in the RBC model considered in the previous section, in which the equations (31) and (48) play the role of (59) when the disturbances a_t and g_t are i.i.d.

I show that all feasible paths are arithmetically implementable in this case as well. The proof is essentially a generalization of the last step of the reasoning conducted in Subsection 4.3. More specifically, for any path P that would be feasible if all shocks were observed, i.e. if the observation set were $\widetilde{O}_t \equiv \{\mathbf{Y}^{\mathcal{J},t-\ell}, i^{t-1}, \boldsymbol{\varepsilon}^{t-\ell}\}$ instead of $O_t \equiv \{\mathbf{Y}^{\mathcal{J},t-\ell}, i^{t-1}, \boldsymbol{\varepsilon}^{\mathcal{K},t-\ell}\}$, I can use Proposition 1 to design arithmetically a rule

$$\boldsymbol{\mathcal{F}}^*(L)\,\mathbf{Y}_{t-\ell}^{\mathcal{J}} + \mathcal{G}^*(L)\,i_t + \boldsymbol{\mathcal{H}}^*(L)\,\boldsymbol{\varepsilon}_{t-\ell} = 0, \tag{60}$$

where $\boldsymbol{\mathcal{F}}^*(X) \in \mathbb{R}^{1\times|\mathcal{J}|}[X]$, $\mathcal{G}^*(X) \in \mathbb{R}[X]$, $\mathcal{G}^*(0) \neq 0$, and $\boldsymbol{\mathcal{H}}^*(X) \in \mathbb{R}^{1\times N_\varepsilon}[X]$, which is consistent with $\widetilde{O}_t$ and P and robustly ensures local-equilibrium determinacy, provided that $D_{\mathcal{J}}(X)$ has no root inside the unit circle. Moreover, I can also use Cramer's rule to rewrite (59) as

$$\det[\mathbf{\Gamma}(L)]\boldsymbol{\varepsilon}_t = \widetilde{\mathbf{\Lambda}}(L) \begin{bmatrix} \mathbf{Y}_t^{\mathcal{J}} \\ i_{t-1} \\ \boldsymbol{\varepsilon}_t^{\mathcal{K}} \end{bmatrix}, \tag{61}$$

where $\widetilde{\mathbf{\Lambda}}(X) \in \mathbb{R}^{N_\varepsilon \times (|\mathcal{J}|+|\mathcal{K}|+1)}[X]$. Multiplying the left- and right-hand sides of (60) by $\det[\mathbf{\Gamma}(L)]$ and using (61) gives

$$\det[\mathbf{\Gamma}(L)]\boldsymbol{\mathcal{F}}^*(L)\,\mathbf{Y}_{t-\ell}^{\mathcal{J}} + \det[\mathbf{\Gamma}(L)]\mathcal{G}^*(L)\,i_t + \boldsymbol{\mathcal{H}}^*(L)\,\widetilde{\mathbf{\Lambda}}(L) \begin{bmatrix} \mathbf{Y}_{t-\ell}^{\mathcal{J}} \\ i_{t-\ell-1} \\ \boldsymbol{\varepsilon}_{t-\ell}^{\mathcal{K}} \end{bmatrix} = 0. \tag{62}$$

The rule (62) is consistent with O_t and P. Therefore, the path P is feasible under O_t. More generally, the set of feasible paths is the same under O_t as under $\widetilde{O}_t$.

Moreover, because the only equation used to transform the rule (60) into the rule (62) is (59), and because (59) is implied by the structural equations (52) and the stochastic process (53), the system made of (52), (53), and (60) is equivalent to the system made of (52), (53), and (62). Since P is the unique stationary solution of the former system, it is therefore also the unique stationary solution of the latter system. Finally, because $\det[\mathbf{\Gamma}(X)]$ has no root inside the unit circle, (62) still ensures local-equilibrium determinacy when added an exogenous policy shock, like (60). So, to sum up, (62) is consistent with O_t and implements P as the robustly unique local equilibrium. As a consequence, P is implementable under O_t, and the rule (62) that I have arithmetically designed implements it. The following proposition formalizes this result:

Proposition 2 (Arithmetic Implementability of Feasible Paths When Unobserved Shocks Are Inferable): *for any set $\mathcal{J}$ such that $\mathcal{J}^\star \subseteq \mathcal{J} \subseteq \{1, ..., N\}$, any set $\mathcal{K} \subsetneq \{1, ..., N_\varepsilon\}$, and any $\ell \in \mathbb{N}$, if $D_{\mathcal{J}}(X)$ has no root inside the unit circle and if (52)-(53) imply a relationship of type (59), then all feasible paths are arithmetically implementable when $O_t = \{\mathbf{Y}^{\mathcal{J},t-\ell}, i^{t-1}, \boldsymbol{\varepsilon}^{\mathcal{K},t-\ell}\}$.*

Let me emphasize that what matters for Proposition 2 is that the unobserved shocks can be inferred from the observation set O_t using *only* the structural equations (52) and the stochastic process (53), i.e., without using the feasible path P considered. Indeed, using P to replace the unobserved shocks in the rule (60) by functions of observed variables and shocks would not necessarily be neutral for robust local-equilibrium determinacy.

Proposition 2 implies that observation lags (captured by ℓ), or equivalently policy-implementation lags (captured by z), do not matter for the arithmetic implementability of feasible paths in the case of unobserved but inferable shocks. The reason is the same as in the case of observed shocks (considered in the previous subsection), given how the proof of Proposition 2 builds on Proposition 1.

Proposition 2 can be readily applied to the setup of Section 4, about (labor-)income-tax policy in the RBC model. Indeed, the structural equations (29)-(36) can be easily rewritten in a form of type (52) satisfying Assumptions 1-3. The observation set considered for the tax authority is of type (54) with $\mathcal{J}^\star \subseteq \mathcal{J}$. It is easy to check that $D_{\mathcal{J}}(X)$ has no root inside the unit circle (simply because it has no root at all). And the structural equations (29)-(31) alone imply a relationship of type (59) − which consists of the equations (31) and (48) when the disturbances a_t and g_t are i.i.d.

Applying Proposition 2 to the setup of Section 4 enables one to extend the results obtained in Section 4 in two directions. First, not only all constant-debt or constant-expected-debt feasible paths, but more generally all feasible paths are arithmetically implementable in this setup. Second, all feasible paths would remain arithmetically implementable if an unobserved exogenous consumption-utility or labor-disutility disturbance, following a stationary and fundamental ARMA process of arbitrary orders, were added to Schmitt-Grohé and Uribe's (1997) model. The reason is that the shock driving this disturbance could then be inferred from the observation set using the structural equation (32) augmented with this disturbance.

6 Concluding Remarks

This paper has sought to dig deeper into the modeling of stabilization policy. The starting point for stabilization-policy modeling should be to specify the policymaker's observation set − a key feature of the environment that should be explicitly stated alongside other features such as preferences, technologies, and markets. Once this observation set is specified, two concepts naturally emerge, those of feasible paths and implementable paths. The goal of the paper has been to show, through two case studies, that feasible-path (non-)implementability can be an issue in textbook models, for standard policy instruments, relevant observation sets, and interesting feasible paths − with important policy implications; and to develop and illustrate a method of designing arithmetically, for each observation set and implementable path, a policy-instrument rule consistent with this observation set and implementing that path as the robustly unique local equilibrium.

As in most of the literature on stabilization policy, I have focused throughout the paper on *local*-equilibrium determinacy, that is to say that I have abstracted from the possible existence of *non-local* equilibria. In the context of interest-rate rules, as argued by Cochrane (2011), there is usually no solid economic reason to assume away the existence of non-local equilibria. The most common policy proposal to eliminate them, made initially by Christiano and Rostagno (2001) and Benhabib, Schmitt-Grohé, and Uribe (2002), discussed by Woodford (2003, Chapter 2), and used notably by Atkeson, Chari, and Kehoe (2010), consists in switching from an interest-rate rule ensuring local-equilibrium determinacy to a money-growth rule (possibly accompanied by a non-Ricardian fiscal policy) when the economy goes *outside* a specified neighborhood of the steady state considered. The interest-rate rules that I design (when the policy instrument is the interest rate) fit naturally into this proposal, insofar as they are followed *inside* the specified neighborhood.

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Appendix A: Optimal Monetary Policy in the NK Model

In this appendix, which complements Section 3, I derive the necessary and sufficient condition for the rule (24) to robustly ensure local-equilibrium determinacy, and I conduct two robustness analyses.

A.1 Determinacy Under a Taylor Rule with Lagged Inflation

In this subsection, I show that the rule

$$i_t = i_t^* + \phi \left(\pi_{t-1} - \pi_{t-1}^* \right) \tag{24}$$

robustly ensures local-equilibrium determinacy in the basic NK model if and only if

$$1 < \phi < 1 + \frac{2(1+\beta)\sigma}{\kappa}. \tag{63}$$

To do so, I use the Phillips curve (14) to replace y_t and y_{t+1} in the IS equation (13), I use the rule (24) to replace i_t, and I get the following dynamic equation in inflation:

$$\mathbb{E}_t \left\{ \pi_{t+2} - \left(1 + \frac{1}{\beta} + \frac{\kappa}{\beta\sigma} \right) \pi_{t+1} + \left(\frac{1}{\beta} \right) \pi_t + \left(\frac{\kappa\phi}{\beta\sigma} \right) \pi_{t-1} \right\} = 0,$$

where I have ignored all exogenous terms to lighten the exposition, as they do not matter for the analysis. This dynamic equation has two non-predetermined variables ($\mathbb{E}_t\{\pi_{t+2}\}$ and $\mathbb{E}_t\{\pi_{t+1}\}$), and its characteristic polynomial is

$$\mathcal{Z}(X) \equiv X^3 - \left(1 + \frac{1}{\beta} + \frac{\kappa}{\beta\sigma}\right) X^2 + \left(\frac{1}{\beta}\right) X + \left(\frac{\kappa\phi}{\beta\sigma}\right).$$

Therefore, the rule (24) robustly ensures local-equilibrium determinacy if and only if $\mathcal{Z}(X)$ has two roots outside the unit circle and one root inside. One *necessary* condition for that is that $\mathcal{Z}(-1)$ and $\mathcal{Z}(1)$ should be of opposite signs. Since

$$\mathcal{Z}(-1) = \frac{\kappa\phi}{\beta\sigma} - \left(2 + \frac{2}{\beta} + \frac{\kappa}{\beta\sigma}\right) < \frac{\kappa\phi}{\beta\sigma} - \frac{\kappa}{\beta\sigma} = \mathcal{Z}(1),$$

$\mathcal{Z}(-1)$ and $\mathcal{Z}(1)$ are of opposite signs if and only if $\mathcal{Z}(-1) < 0 < \mathcal{Z}(1)$, that is to say of and only if (63) holds.

Conversely, suppose that (63) holds. Then $\mathcal{Z}(-1) < 0 < \kappa\phi/(\beta\sigma) = \mathcal{Z}(0)$. So, $\mathcal{Z}(X)$ has at least one real root in $(-1, 0)$, which I denote by r_1. Since $\mathcal{Z}(X)$ is of type $X^3 - a_2X^2 + a_1X + a_0$, we have $r_1 + r_2 + r_3 = a_2 \equiv 1 + 1/\beta + \kappa/(\beta\sigma) > 2$, where r_2 and r_3 denote the other two roots of $\mathcal{Z}(X)$. Therefore, $r_2 + r_3 = a_2 - r_1 > 2$, which implies that r_2 and r_3 lie outside the unit circle. As a consequence, the rule (24) robustly ensures local-equilibrium determinacy.

A.2 Robustness Analysis: Additional Variables

I add the consumption level c_t, hours worked h_t, and the real wage w_t to the list of endogenous variables that I consider in the basic NK model of Subsections 3.1-3.4, and I assume that $\mathcal{CB}$'s observation set is now $\overline{O}_t \equiv \{\pi^{t-1}, y^{t-1}, c^{t-1}, n^{t-1}, w^{t-1}, i^{t-1}\}$. The three additional (locally log-linearized) structural equations are the goods-market-clearing condition $c_t = y_t$, the production function $y_t = \alpha h_t$, and the consumption/leisure trade-off condition $w_t = \sigma c_t + \chi h_t$, where $0 < \alpha < 1$ and $\chi > 0$. These additional structural equations make c_t, h_t, and w_t proportional to y_t, with proportionality coefficients 1, α^{-1}, and $\sigma + \chi\alpha^{-1}$ respectively. So, the optimal feasible path for all endogenous variables is now characterized block-recursively by (17) for $\mathbf{Y}_t$ and i_t and by

$$\mathbf{Z}_t = \mathbf{\Omega}\mathbf{Y}_t \tag{64}$$

for $\mathbf{Z}_t \equiv [\,c_t \quad h_t \quad w_t\,]^T$, where

$$\mathbf{\Omega} \equiv \begin{bmatrix} 0 & 1 \\ 0 & \alpha^{-1} \\ 0 & \sigma + \chi\alpha^{-1} \end{bmatrix}.$$

To characterize the set of rules consistent with the observation set $\overline{O}_t$ and this path, I start by noting that the rules consistent with the observation set $\overline{O}_t$ are the rules of type

$$\mathcal{P}(L)i_t + \boldsymbol{\mathcal{Q}}(L)\mathbf{Y}_{t-1} + \boldsymbol{\mathcal{R}}(L)\mathbf{Z}_{t-1} = 0$$

with $\mathcal{P}(X) \in \mathbb{R}[X]$, $\mathcal{P}(0) \neq 0$, $\boldsymbol{\mathcal{Q}}(X) \in \mathbb{R}^{1\times 2}[X]$, and $\boldsymbol{\mathcal{R}}(X) \in \mathbb{R}^{1\times 3}[X]$. For any given $\boldsymbol{\mathcal{R}}(X)$, choosing freely $\boldsymbol{\mathcal{Q}}(X)$ in $\mathbb{R}^{1\times 2}[X]$ amounts to choosing freely $\widetilde{\boldsymbol{\mathcal{Q}}}(X) \equiv \boldsymbol{\mathcal{Q}}(X) + \boldsymbol{\mathcal{R}}(X)\mathbf{\Omega}$ in $\mathbb{R}^{1\times 2}[X]$. So I can equivalently rewrite the rules consistent with $\overline{O}_t$ as the rules of type

$$\mathcal{P}(L)i_t + \widetilde{\boldsymbol{\mathcal{Q}}}(L)\mathbf{Y}_{t-1} + \boldsymbol{\mathcal{R}}(L)\left(\mathbf{Z}_{t-1} - \mathbf{\Omega}\mathbf{Y}_{t-1}\right) = 0 \tag{65}$$

with $\mathcal{P}(X) \in \mathbb{R}[X]$, $\mathcal{P}(0) \neq 0$, $\widetilde{\boldsymbol{\mathcal{Q}}}(X) \in \mathbb{R}^{1\times 2}[X]$, and $\boldsymbol{\mathcal{R}}(X) \in \mathbb{R}^{1\times 3}[X]$. Among these rules, the rules that are also consistent with the optimal feasible path − characterized by (17) and (64) − are those such that

$$\mathcal{P}\left(X\right)\mathbf{T}_i\left(X\right) + \left(1 - \rho_\eta X\right)\widetilde{\boldsymbol{\mathcal{Q}}}\left(X\right)\mathbf{T}_Y\left(X\right) = 0. \tag{66}$$

In the same way as I used (21) to replace $\boldsymbol{\mathcal{Q}}(L)$ in (20), I can use (66) to replace $\widetilde{\boldsymbol{\mathcal{Q}}}(L)$ in (65), and thus rewrite the rules consistent with $\overline{O}_t$ and the optimal feasible path as

$$\begin{aligned} \mathcal{P}(L)\left\{(1 - \rho_\eta L)\det\left[\mathbf{T}_Y\left(L\right)\right] i_t - \mathbf{T}_i\left(L\right) adj\left[\mathbf{T}_Y\left(L\right)\right]\mathbf{Y}_{t-1}\right\} & \\ + \boldsymbol{\mathcal{R}}\left(L\right)\left(\mathbf{Z}_{t-1} - \mathbf{\Omega}\mathbf{Y}_{t-1}\right) = 0 & \end{aligned} \tag{67}$$

with $\mathcal{P}(X) \in \mathbb{R}[X]$, $\mathcal{P}(0) \neq 0$, and $\boldsymbol{\mathcal{R}}(X) \in \mathbb{R}^{1\times 3}[X]$. The optimal feasible path is implementable under $\overline{O}_t$ if and only if there exists one such rule that robustly ensures local-equilibrium determinacy.

Now, the three additional structural equations imply $\mathbf{Z}_{t-1} - \mathbf{\Omega}\mathbf{Y}_{t-1} = \mathbf{0}$. Therefore, for any $\mathcal{P}(X)$ and $\boldsymbol{\mathcal{R}}(X)$, the system made of the structural equations and the rule (67) is equivalent to the system made of the structural equations and the rule (23). So, the former system satisfies Blanchard and Kahn's (1980) conditions if and only if the latter system does. As a consequence, the optimal feasible path is implementable under $\overline{O}_t$ if and only if it is implementable under O_t. Thus, my non-implementability results, and in particular Figure 1, are unchanged. The observation of the three additional variables by $\mathcal{CB}$ brings three additional degrees of freedom, captured by $\boldsymbol{\mathcal{R}}(X)$, in the choice of a rule consistent with $\mathcal{CB}$'s observation set and the optimal feasible path. But these additional degrees of freedom are *neutral* for robust local-equilibrium determinacy.

A.3 Robustness Analysis: Additional Disturbances

I introduce three additional disturbances into the basic NK model considered in Subsections 3.5 and A.2: a productivity disturbance a_t, a government-purchases disturbance g_t, and a consumption-utility or labor-disutility disturbance ν_t. To introduce a government-purchases disturbance, I need non-zero government purchases at the steady state. So let $s_g \in (0,1)$ denote the steady-state fraction of government purchases to output. The structural equations then become

$$y_t = \mathbb{E}_t\{y_{t+1}\} - \frac{1}{\sigma}(i_t - \mathbb{E}_t\{\pi_{t+1}\}) + \eta_t + s_g g_t - s_g \mathbb{E}_t\{g_{t+1}\}, \tag{68}$$

$$\pi_t = \beta \mathbb{E}_t\{\pi_{t+1}\} + \kappa(y_t - \psi_a a_t - \psi_g g_t - \psi_\nu \nu_t) + u_t, \tag{69}$$

$$y_t = (1-s_g)c_t + s_g g_t, \tag{70}$$

$$y_t = \alpha h_t + a_t, \tag{71}$$

$$w_t = (1-s_g)\,\sigma c_t + \chi h_t + \nu_t, \tag{72}$$

where $\psi_a a_t + \psi_g g_t + \psi_\nu \nu_t$ represents the flexible-price output level, with $(\psi_a, \psi_g, \psi_\nu) \in \mathbb{R}^3$. I assume that the three additional disturbances follow stationary ARMA processes of arbitrary orders, which are fundamental in the sense of Hansen and Sargent (1981, 1991):

$$\rho_a(L)a_t = \theta_a(L)\varepsilon_t^a, \tag{73}$$

$$\rho_g(L)g_t = \theta_g(L)\varepsilon_t^g, \tag{74}$$

$$\rho_\nu(L)\nu_t = \theta_\nu(L)\varepsilon_t^\nu, \tag{75}$$

where ε_t^a, ε_t^g, and ε_t^ν are three orthogonal i.i.d. exogenous shocks of mean zero, and, for all $j \in \{a, g, \nu\}$, $\rho_j(X) \in \mathbb{R}[X]$, $\theta_j(X) \in \mathbb{R}[X]$, $\rho_j(0) \neq 0$, $\theta_j(0) \neq 0$, and $\rho_j(X)$ and $\theta_j(X)$ have no root inside the unit circle.

I assume that $\mathcal{CB}$ does not observe these additional shocks, so that its observation set is the same as in Subsection 3.5, namely $\overline{O}_t \equiv \{\pi^{t-1}, y^{t-1}, c^{t-1}, n^{t-1}, w^{t-1}, i^{t-1}\}$. This assumption does not matter, in the sense that the results would be unchanged if I assumed instead that $\mathcal{CB}$ observes some or all of these additional shocks until date $t-1$.

The date-t welfare-loss function is now $L_t = \mathbb{E}_t\{\sum_{k=0}^{+\infty}\beta^k[(\pi_{t+k})^2 + \lambda(y_{t+k} - \psi_a a_{t+k} - \psi_g g_{t+k} - \psi_\nu \nu_{t+k})^2]\}$. If $\mathcal{CB}$'s observation set were $\{\varepsilon^{\eta,t-1}, \varepsilon^{u,t-1}, \varepsilon^{a,t-1}, \varepsilon^{g,t-1}, \varepsilon^{\nu,t-1}\}$, then the optimal feasible path for all endogenous variables would be characterized block-recursively by

$$\rho(L)\Pi(L)\begin{bmatrix} \mathbf{I}_2 & \mathbf{0} \\ \mathbf{0} & (1-\rho_\eta L) \end{bmatrix}\begin{bmatrix} \mathbf{Y}_t \\ i_t \end{bmatrix} = \rho(L)\begin{bmatrix} \mathbf{T}_Y(L) \\ \mathbf{T}_i(L)L \end{bmatrix}\boldsymbol{\varepsilon}_t + \begin{bmatrix} \mathbf{T}_Y^*(L) \\ \mathbf{T}_i^*(L)L \end{bmatrix}\boldsymbol{\varepsilon}_t^* \tag{76}$$

for $\mathbf{Y}_t$ and i_t, where $\rho(X) \equiv \rho_a(X)\rho_g(X)\rho_\nu(X)$, $\Pi(X) \equiv (1-\rho_u X)(1-\mu X)$, $\mathbf{T}_Y^*(X) \in \mathbb{R}^{2\times 3}[X]$, $\mathbf{T}_i^*(X) \in \mathbb{R}^{1\times 3}[X]$, and $\boldsymbol{\varepsilon}_t^* \equiv [\,\varepsilon_t^a \quad \varepsilon_t^g \quad \varepsilon_t^\nu\,]^T$; and by

$$\mathbf{Z}_t = \boldsymbol{\Omega}_g \mathbf{Y}_t + \boldsymbol{\Phi}\begin{bmatrix} a_t \\ g_t \\ \nu_t \end{bmatrix} \tag{77}$$

for $\mathbf{Z}_t$, where

$$\boldsymbol{\Omega}_g \equiv \begin{bmatrix} 0 & (1-s_g)^{-1} \\ 0 & \alpha^{-1} \\ 0 & \sigma + \chi\alpha^{-1} \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Phi} \equiv \begin{bmatrix} 0 & -s_g(1-s_g)^{-1} & 0 \\ -\alpha^{-1} & 0 & 0 \\ -\chi\alpha^{-1} & -s_g\sigma & 1 \end{bmatrix}.$$

To show that this path is also the optimal feasible path under $\overline{O}_t$, I just need to show that this path is feasible under $\overline{O}_t$. To do so, I first rewrite the first two lines of (76) as

$$\rho(L)\det\left[\mathbf{T}_Y(L)\right]\boldsymbol{\varepsilon}_t = \rho(L)\Pi(L)adj\left[\mathbf{T}_Y(L)\right]\mathbf{Y}_t - adj\left[\mathbf{T}_Y(L)\right]\mathbf{T}_Y^*(L)\boldsymbol{\varepsilon}_t^*. \tag{78}$$

I then multiply the left- and right-hand sides of the last line of (76) by $\det[\mathbf{T}_Y(L)]$ and use (78) to get

$$\begin{aligned}(1-\rho_\eta L)\,\rho(L)\Pi(L)\det\left[\mathbf{T}_Y(L)\right] i_t &= \rho(L)\Pi(L)\mathbf{T}_i(L)adj\left[\mathbf{T}_Y(L)\right]\mathbf{Y}_{t-1} \\ &+ \left\{\det\left[\mathbf{T}_Y(L)\right]\mathbf{T}_i^*(L) - \mathbf{T}_i(L)adj\left[\mathbf{T}_Y(L)\right]\mathbf{T}_Y^*(L)\right\}\boldsymbol{\varepsilon}_{t-1}^*. \end{aligned} \tag{79}$$

Next, I multiply the left- and right-hand sides of (79) by $\theta(L) \equiv \theta_a(L)\theta_g(L)\theta_\nu(L)$, I use (73)-(75) to replace $\theta_a(L)\varepsilon_t^a$, $\theta_g(L)\varepsilon_t^g$, $\theta_\nu(L)\varepsilon_t^\nu$, and then (77) to replace a_t, g_t, ν_t, and I get

$$\begin{aligned}(1-\rho_\eta L)\,\rho(L)\theta(L)\Pi(L)\det\left[\mathbf{T}_Y(L)\right] i_t &= \rho(L)\theta(L)\Pi(L)\mathbf{T}_i(L)adj\left[\mathbf{T}_Y(L)\right]\mathbf{Y}_{t-1} \\ + \left\{\det\left[\mathbf{T}_Y(L)\right]\mathbf{T}_i^*(L) - \mathbf{T}_i(L)adj\left[\mathbf{T}_Y(L)\right]\mathbf{T}_Y^*(L)\right\}&\mathbf{D}(L)\boldsymbol{\Phi}^{-1}\left(\mathbf{Z}_{t-1} - \boldsymbol{\Omega}_g\mathbf{Y}_{t-1}\right), \end{aligned} \tag{80}$$

where

$$\mathbf{D}(X) \equiv \left[\begin{array}{ccc} \rho_a(X)\theta_g(X)\theta_\nu(X) & 0 & 0 \\ 0 & \theta_a(X)\rho_g(X)\theta_\nu(X) & 0 \\ 0 & 0 & \theta_a(X)\theta_g(X)\rho_\nu(X) \end{array} \right].$$

Since $\det[\mathbf{T}_Y(0)] \neq 0$, the equation (80) expresses i_t as a function of only elements of $\overline{O}_t$. Therefore, the path characterized by (76) and (77) is feasible under $\overline{O}_t$. As a consequence, it is the optimal feasible path under $\overline{O}_t$.

To characterize the set of rules consistent with $\overline{O}_t$ and this path, I start by noting that the rules consistent with $\overline{O}_t$ are the rules of type

$$\mathcal{P}(L)i_t + \boldsymbol{\mathcal{Q}}(L)\mathbf{Y}_{t-1} + \boldsymbol{\mathcal{R}}(L)\mathbf{Z}_{t-1} = 0$$

with $\mathcal{P}(X) \in \mathbb{R}[X]$, $\mathcal{P}(0) \neq 0$, $\boldsymbol{\mathcal{Q}}(X) \in \mathbb{R}^{1\times2}[X]$, and $\boldsymbol{\mathcal{R}}(X) \in \mathbb{R}^{1\times3}[X]$. For any given $\boldsymbol{\mathcal{R}}(X)$, choosing freely $\boldsymbol{\mathcal{Q}}(X)$ in $\mathbb{R}^{1\times2}[X]$ amounts to choosing freely $\widetilde{\boldsymbol{\mathcal{Q}}}(X) \equiv \boldsymbol{\mathcal{Q}}(X) + \boldsymbol{\mathcal{R}}(X)\boldsymbol{\Omega}_g$ in $\mathbb{R}^{1\times2}[X]$. So I can equivalently rewrite the rules consistent with $\overline{O}_t$ as the rules of type

$$\mathcal{P}(L)i_t + \widetilde{\boldsymbol{\mathcal{Q}}}(L)\mathbf{Y}_{t-1} + \boldsymbol{\mathcal{R}}(L)\left(\mathbf{Z}_{t-1} - \boldsymbol{\Omega}_g\mathbf{Y}_{t-1}\right) = 0 \tag{81}$$

with $\mathcal{P}(X) \in \mathbb{R}[X]$, $\mathcal{P}(0) \neq 0$, $\widetilde{\boldsymbol{\mathcal{Q}}}(X) \in \mathbb{R}^{1\times2}[X]$, and $\boldsymbol{\mathcal{R}}(X) \in \mathbb{R}^{1\times3}[X]$. Among these rules, the rules that are also consistent with the optimal feasible path − characterized by (76) and (77) − are those such that

$$\mathcal{P}\left(X\right)\mathbf{T}_i\left(X\right) + \left(1-\rho_\eta X\right)\widetilde{\boldsymbol{\mathcal{Q}}}\left(X\right)\mathbf{T}_Y\left(X\right) = \mathbf{0}, \tag{82}$$

$$\mathcal{P}\left(X\right)\mathbf{T}_i^*\left(X\right) + \left(1-\rho_\eta X\right)\widetilde{\boldsymbol{\mathcal{Q}}}\left(X\right)\mathbf{T}_Y^*\left(X\right) + \left(1-\rho_\eta X\right)\Pi(X)\boldsymbol{\mathcal{R}}(X)\boldsymbol{\Phi}\widetilde{\mathbf{D}}(X) = \mathbf{0}, \tag{83}$$

where

$$\widetilde{\mathbf{D}}(X) \equiv \left[\begin{array}{ccc} \theta_a(X)\rho_g(X)\rho_\nu(X) & 0 & 0 \\ 0 & \rho_a(X)\theta_g(X)\rho_\nu(X) & 0 \\ 0 & 0 & \rho_a(X)\rho_g(X)\theta_\nu(X) \end{array} \right].$$

Using (83) to replace $\boldsymbol{\mathcal{R}}(L)$ in (81), and then (82) to replace $\widetilde{\boldsymbol{\mathcal{Q}}}(L)$ in the resulting equation, I can rewrite the rules consistent with $\overline{O}_t$ and the optimal feasible path as the rules of type

$$\mathcal{P}(L)R_t = 0 \tag{84}$$

with $\mathcal{P}(X) \in \mathbb{R}[X]$ and $\mathcal{P}(0) \neq 0$, where

$$\begin{aligned} R_t \equiv & \left(1-\rho_\eta L\right) \rho(L)\theta(L)\Pi(L) \det\left[\mathbf{T}_Y\left(L\right)\right] i_t - \rho(L)\theta(L)\Pi(L)\mathbf{T}_i(L) adj\left[\mathbf{T}_Y\left(L\right)\right] \mathbf{Y}_{t-1} \\ & - \left\{\det\left[\mathbf{T}_Y\left(L\right)\right] \mathbf{T}_i^*(L) - \mathbf{T}_i(L) adj\left[\mathbf{T}_Y\left(L\right)\right] \mathbf{T}_Y^*(L)\right\} \mathbf{D}(L) \mathbf{\Phi}^{-1} \left(\mathbf{Z}_{t-1} - \mathbf{\Omega}_g \mathbf{Y}_{t-1}\right) \end{aligned}$$

is the difference between the left-hand side and the right-hand side of the rule (80). The optimal feasible path is implementable under $\overline{O}_t$ if and only if at least one of these rules robustly ensures local-equilibrium determinacy. I can then conduct exactly the same reasoning as in Subsection 2.4 and conclude that there exists one such rule if and only if the specific rule (80) (that is to say the rule $R_t = 0$, corresponding to $\mathcal{P}(X) = 1$) robustly ensures local-equilibrium determinacy in the first place.

Now, the structural equations (70)-(72) imply $\mathbf{Z}_{t-1} - \mathbf{\Omega}_g \mathbf{Y}_{t-1} = \mathbf{\Phi}[\, a_{t-1} \quad g_{t-1} \quad \nu_{t-1}\,]^T$. Therefore, the system made of the structural equations and the rule (80) is equivalent to the system made of the structural equations and the rule

$$\left(1-\rho_\eta L\right) \rho(L)\theta(L)\Pi(L) \det\left[\mathbf{T}_Y\left(L\right)\right] i_t = \rho(L)\theta(L)\Pi(L)\mathbf{T}_i(L) adj\left[\mathbf{T}_Y\left(L\right)\right] \mathbf{Y}_{t-1} + \zeta_t, \tag{85}$$

where

$$\zeta_t \equiv \left\{\det\left[\mathbf{T}_Y\left(L\right)\right] \mathbf{T}_i^*(L) - \mathbf{T}_i(L) adj\left[\mathbf{T}_Y\left(L\right)\right] \mathbf{T}_Y^*(L)\right\} \mathbf{D}(L)[\, a_{t-1} \quad g_{t-1} \quad \nu_{t-1}\,]^T$$

is an exogenous term. So, the rule (80) robustly ensures local-equilibrium determinacy if and only if the rule (85) does. In turn, the rule (85) can be rewritten as

$$\left(1-\rho_\eta L\right) \det\left[\mathbf{T}_Y\left(L\right)\right] i_t = \mathbf{T}_i(L) adj\left[\mathbf{T}_Y\left(L\right)\right] \mathbf{Y}_{t-1} + \rho(L)^{-1}\theta(L)^{-1}\Pi(L)^{-1}\zeta_t, \tag{86}$$

since $\rho(X)$, $\theta(X)$, and $\Pi(X)$ have no root inside the unit circle; and the rule (86) robustly ensures local-equilibrium determinacy if and only if the rule (19) does. As a consequence, the optimal feasible path under $\overline{O}_t$ is implementable in the presence of the three additional disturbances a_t, g_t, and ν_t for exactly the same values of the structural parameters as in the absence of these disturbances. In particular, Figure 1 is still valid in the presence of these disturbances.

Appendix B: Debt-Stabilizing Tax Policy in the RBC Model

In this appendix, which complements Section 4, I design the polynomials $\mathcal{R}_b^*(X)$ and $\mathcal{R}_\tau^*(X)$, I derive the implementability and implementation results in the presence of policy-implementation lags, and I conduct a robustness analysis.

B.1 Design of $\mathcal{R}_b^*(X)$ and $\mathcal{R}_\tau^*(X)$

Let z_i^b, z_j^τ, r_i^{b*}, and $r_k^{\tau*}$ for $i \in \{0,1,2\}$, $j \in \{0,1,2,3\}$, and $k \in \{0,1\}$ denote the coefficients of the polynomials $Z_b(X)$ and $Z_\tau(X)$ (which are known) and those of the polynomials $\mathcal{R}_b^*(X)$ and $\mathcal{R}_\tau^*(X)$ (which are unknown):

$$\begin{aligned}
Z_b(X) &= z_0^b + z_1^b X + z_2^b X^2, \\
Z_\tau(X) &= z_0^\tau + z_1^\tau X + z_2^\tau X^2 + z_3^\tau X^3, \\
\mathcal{R}_b^*(X) &= r_0^{b*} + r_1^{b*} X + r_2^{b*} X^2, \\
\mathcal{R}_\tau^*(X) &= r_0^{\tau*} + r_1^{\tau*} X.
\end{aligned}$$

The equation $Z_b(X)\mathcal{R}_b^*(X) + Z_\tau(X)\mathcal{R}_\tau^*(X) = X - \phi$ can then be rewritten as

$$\mathbf{S}\begin{bmatrix} r_2^{b*} \\ r_1^{b*} \\ r_0^{b*} \\ r_1^{\tau*} \\ r_0^{\tau*} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -\phi \end{bmatrix}, \quad \text{where} \quad \mathbf{S} \equiv \left[\begin{array}{ccc|cc} z_2^b & 0 & 0 & z_3^\tau & 0 \\ z_1^b & z_2^b & 0 & z_2^\tau & z_3^\tau \\ z_0^b & z_1^b & z_2^b & z_1^\tau & z_2^\tau \\ 0 & z_0^b & z_1^b & z_0^\tau & z_1^\tau \\ 0 & 0 & z_0^b & 0 & z_0^\tau \end{array}\right]$$

is the transpose of the Sylvester matrix of the polynomials $Z_b(X)$ and $Z_\tau(X)$. A Sylvester matrix of two polynomials (with real-number coefficients) is invertible if and only if these polynomials have no common (real or complex) root. Now, the polynomials $Z_b(X)$ and $Z_\tau(X)$ have no common root, except possibly for a zero-measure set of structural-parameter values. Therefore, their Sylvester matrix is invertible, and so is its transpose $\mathbf{S}$, so that the coefficients of $\mathcal{R}_b^*(X)$ and $\mathcal{R}_\tau^*(X)$ can be arithmetically obtained as

$$\begin{bmatrix} r_2^{b*} & r_1^{b*} & r_0^{b*} & r_1^{\tau*} & r_0^{\tau*} \end{bmatrix}^T = \mathbf{S}^{-1}\begin{bmatrix} 0 & 0 & 0 & 1 & -\phi \end{bmatrix}^T.$$

In particular, $\mathcal{R}_\tau^*(0) \neq 0$, $\mathcal{R}_b^*(\phi) \neq 0$, and $\mathcal{R}_\tau^*(\phi) \neq 0$, except again possibly for a zero-measure set of structural-parameter values.

B.2 Policy-Implementation Lags

In the presence of tax-policy-implementation lags of length $\ell \in \mathbb{N} \setminus \{0\}$, the structural equations (29)-(33) are unchanged, while the structural equations (34)-(36) become

$$r_t = a_t - \alpha(k_t - h_t) - \omega\tau(1-\tau)^{-1}\tau_{t-\ell}, \tag{87}$$

$$w_t = a_t + (1-\alpha)(k_t - h_t) - \tau(1-\tau)^{-1}\tau_{t-\ell}, \tag{88}$$

$$b_t = \beta^{-1}b_{t-1} + s_g g_t - [\alpha + (1-\alpha)\omega]\tau(y_t + \tau_{t-\ell}). \tag{89}$$

I start by (partially) characterizing the set of constant-expected-debt feasible paths. On any feasible path along which $\mathbb{E}_t\{b_{t+\ell}\} = 0$, there exist $\widetilde{B}_a(X) \in \mathbb{R}[X]$ and $\widetilde{B}_g(X) \in \mathbb{R}[X]$, of degrees lower than ℓ, such that

$$b_t = \widetilde{B}_a(L)\, a_t + \widetilde{B}_g(L)\, g_t. \tag{90}$$

Using this expression for b_t, I can rewrite (29)-(32) and (87)-(89) in a form of type

$$\begin{aligned} &\begin{bmatrix} y_t & x_t & h_t & c_t & r_t & w_t & \tau_{t-\ell} \end{bmatrix}^T = \\ &\widetilde{\mathbf{A}} \left[\kappa_t \quad \kappa_{t-1} \quad a_t \quad g_t \quad (1-\beta^{-1}L)\widetilde{B}_a(L)a_t \quad (1-\beta^{-1}L)\widetilde{B}_g(L)g_t \right]^T, \end{aligned} \tag{91}$$

where $\widetilde{\mathbf{A}} \in \mathbb{R}^{7\times 6}$. Using (91), I can in turn rewrite (33) in a form of type

$$\mathbb{E}_t\{S(L)\,\kappa_{t+1}\} = K_a(L)\, a_t + K_g(L)\, g_t \tag{92}$$

with $(K_a(X), K_g(X)) \in \mathbb{R}[X]^2$, where $S(X) \in \mathbb{R}[X]$ is defined in Subsection 4.2. As I focus again on paths that do not involve sunspot shocks (since $\mathcal{TA}$ is unlikely to be interested in implementing a path that does), the equation (92) implies that κ_t follows an ARMA process of type

$$S(L)\,\kappa_t = \widetilde{K}_a(L)\, a_t + \widetilde{K}_g(L)\, g_t,$$

where $\widetilde{K}_a(X) \in \mathbb{R}[X]$ and $\widetilde{K}_g(X) \in \mathbb{R}[X]$ are such that $\widetilde{K}_a(X) = \widetilde{K}_a(0) + XK_a(X)$ and $\widetilde{K}_g(X) = \widetilde{K}_g(0) + XK_g(X)$. Using (91), I conclude that τ_t also follows an ARMA process of type

$$S(L)\,\tau_t = \widetilde{T}_a(L)\, a_t + \widetilde{T}_g(L)\, g_t$$

with $(\widetilde{T}_a(X), \widetilde{T}_g(X)) \in \mathbb{R}[X]^2$.

Let me now consider an arbitrarily given (non-sunspot-driven) constant-expected-debt feasible path, which I denote by $\widetilde{P}$. To show the implementability of $\widetilde{P}$ and design a tax-rate rule consistent with O_t and implementing $\widetilde{P}$ as the robustly unique local equilibrium, I follow the same four steps as in Subsection 4.3. The first two steps are unchanged, except that τ_t in (42), $P_\tau(L)$ in (43), and Q_τ in (44) should be respectively replaced by $\tau_{t-\ell}$, $P_\tau(L)L^\ell$, and $Q_\tau L^\ell$. In the third step, the resulting dynamic system still has exactly one non-predetermined variable, corresponding to $\mathbb{E}_t\{P_k(0)\kappa_{t+1}\}$, but its reciprocical characteristic polynomial is now $Z_b(X)X^\ell\mathcal{R}_b(X) + Z_\tau(X)\mathcal{R}_\tau(X)$. In Appendix B.3, I proceed in the same way as in Appendix B.1 to design arithmetically $\widetilde{\mathcal{R}}_b^*(X) \in \mathbb{R}[X]$ of degree 2 and $\widetilde{\mathcal{R}}_\tau^*(X) \in \mathbb{R}[X]$ of degree $\ell+1$ such that $Z_b(X)X^\ell\widetilde{\mathcal{R}}_b^*(X) + Z_\tau(X)\widetilde{\mathcal{R}}_\tau^*(X) = X - \phi$, $\widetilde{\mathcal{R}}_b^*(\phi)\widetilde{\mathcal{R}}_\tau^*(\phi) \neq 0$, and $\widetilde{\mathcal{R}}_\tau^*(0) \neq 0$, except possibly for a zero-measure set of structural-parameter values. Therefore, all rules of type (45) with $(\mathcal{R}_b(X), \mathcal{R}_\tau(X)) = (\widetilde{\mathcal{R}}_b^*(X), \widetilde{\mathcal{R}}_\tau^*(X))$ robustly ensure local-equilibrium determinacy. Among them, the rule with

$$\mathcal{R}_a(X) = \widetilde{\mathcal{R}}_a^*(X) \equiv -S(X)\widetilde{\mathcal{R}}_b^*(X)\widetilde{B}_a(X) - \widetilde{\mathcal{R}}_\tau^*(X)\widetilde{T}_a(X)$$
$$\mathcal{R}_g(X) = \widetilde{\mathcal{R}}_g^*(X) \equiv -S(X)\widetilde{\mathcal{R}}_b^*(X)\widetilde{B}_g(X) - \widetilde{\mathcal{R}}_\tau^*(X)\widetilde{T}_g(X)$$

is satisfied on the path $\widetilde{P}$, since it is satisfied when both (90) and (B.2) hold. This rule, therefore, implements $\widetilde{P}$ as the robustly unique local equilibrium. Finally, in the fourth step, I transform this rule into a rule consistent with O_t in the same way as previously. The latter rule, which corresponds to (49) in which $\mathcal{R}_j^*(X)$ is replaced by $\widetilde{\mathcal{R}}_j^*(X)$ for $j \in \{b, \tau, a, g\}$, is consistent with O_t and implements $\widetilde{P}$ as the robustly unique local equilibrium.

B.3 Design of $\widetilde{\mathcal{R}}_b^*(X)$ and $\widetilde{\mathcal{R}}_\tau^*(X)$

Let $\widetilde{r}_i^{b*}$ and $\widetilde{r}_j^{\tau*}$ for $i \in \{0, 1, 2\}$ and $j \in \{0, ..., \ell+1\}$ denote the coefficients of the polynomials $\widetilde{\mathcal{R}}_b^*(X)$ and $\widetilde{\mathcal{R}}_\tau^*(X)$ (which are unknown):

$$\widetilde{\mathcal{R}}_b^*(X) = \widetilde{r}_0^{b*} + \widetilde{r}_1^{b*}X + \widetilde{r}_2^{b*}X^2,$$
$$\widetilde{\mathcal{R}}_\tau^*(X) = \widetilde{r}_0^{\tau*} + \widetilde{r}_1^{\tau*}X + \cdots + \widetilde{r}_{\ell+1}^{\tau*}X^{\ell+1}.$$

The equation $Z_b(X)X^\ell \widetilde{\mathcal{R}}_b^*(X) + Z_\tau(X)\widetilde{\mathcal{R}}_\tau^*(X) = X - \phi$ can then be rewritten as

$$\widetilde{\mathbf{S}} \begin{bmatrix} \widetilde{r}_2^{b*} \\ \widetilde{r}_1^{b*} \\ \widetilde{r}_0^{b*} \\ \widetilde{r}_{\ell+1}^{\tau*} \\ \vdots \\ \widetilde{r}_0^{\tau*} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -\phi \end{bmatrix}, \quad \text{where} \quad \widetilde{\mathbf{S}} \equiv \left[\begin{array}{ccc|ccc} z_2^b & & & z_3^\tau & & \\ z_1^b & z_2^b & & z_2^\tau & \ddots & \\ z_0^b & z_1^b & z_2^b & z_1^\tau & \ddots & z_3^\tau \\ & z_0^b & z_1^b & z_0^\tau & \ddots & z_2^\tau \\ & & z_0^b & & \ddots & z_1^\tau \\ & & & & & z_0^\tau \end{array}\right]$$

is the transpose of the Sylvester matrix of the polynomials $Z_b(X)X^\ell$ and $Z_\tau(X)$.[26] Since the polynomials $Z_b(X)X^\ell$ and $Z_\tau(X)$ have no common root, except possibly for a zero-measure set of structural-parameter values, their Sylvester matrix is invertible, and so is its transpose $\widetilde{\mathbf{S}}$, so that the coefficients of $\widetilde{\mathcal{R}}_b^*(X)$ and $\widetilde{\mathcal{R}}_\tau^*(X)$ can be arithmetically obtained as

$$\begin{bmatrix} \widetilde{r}_2^{b*} & \widetilde{r}_1^{b*} & \widetilde{r}_0^{b*} & \widetilde{r}_{\ell+1}^{\tau*} & \dots & \widetilde{r}_0^{\tau*} \end{bmatrix}^T = \widetilde{\mathbf{S}}^{-1} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -\phi \end{bmatrix}^T.$$

In particular, $\widetilde{\mathcal{R}}_\tau^*(0) \neq 0$, $\widetilde{\mathcal{R}}_b^*(\phi) \neq 0$, and $\widetilde{\mathcal{R}}_\tau^*(\phi) \neq 0$, except again possibly for a zero-measure set of structural-parameter values.

B.4 Robustness Analysis: ARMA Disturbances

I consider the structural equations (29)-(33) and (87)-(89) with $\ell \in \mathbb{N}$, which allow both for the presence (when $\ell \geq 1$) and for the absence (when $\ell = 0$) of tax-policy-implementation lags. I assume that a_t and g_t follow stationary ARMA processes of arbitrary orders, which are fundamental in the sense of Hansen and Sargent (1981, 1991):

$$\rho_a(L)a_t = \theta_a(L)\varepsilon_t^a, \tag{93}$$

$$\rho_g(L)g_t = \theta_g(L)\varepsilon_t^g, \tag{94}$$

where ε_t^a and ε_t^g are two orthogonal i.i.d. exogenous shocks of mean zero, and, for all $j \in \{a, g\}$, $\rho_j(X) \in \mathbb{R}[X]$, $\theta_j(X) \in \mathbb{R}[X]$, $\rho_j(0) \neq 0$, $\theta_j(0) \neq 0$, and $\rho_j(X)$ and $\theta_j(X)$ have no root inside the unit circle.

I first (partially) characterize the constant-debt or constant-expected-debt feasible paths. On any feasible path along which $\mathbb{E}_t\{b_{t+\ell}\} = 0$, there exist $\widehat{B}_a(X) \in \mathbb{R}[X]$ and $\widehat{B}_g(X) \in$

[26] To lighten the exposition, I have displayed only the non-zero elements of $\widetilde{\mathbf{S}}$.

$\mathbb{R}[X]$, of degrees lower than ℓ, such that

$$b_t = \widehat{B}_a(L)\,\varepsilon_t^a + \widehat{B}_g(L)\,\varepsilon_t^g. \tag{95}$$

Using (95), I can rewrite (29)-(32) and (87)-(89) as

$$\begin{gathered}\begin{bmatrix} y_t & x_t & h_t & c_t & r_t & w_t & \tau_{t-\ell}\end{bmatrix}^T = \\ \widetilde{\mathbf{A}}\left[\kappa_t \quad \kappa_{t-1} \quad a_t \quad g_t \quad (1-\beta^{-1}L)\widehat{B}_a(L)\varepsilon_t^a \quad (1-\beta^{-1}L)\widehat{B}_g(L)\varepsilon_t^g\right]^T,\end{gathered} \tag{96}$$

where $\widetilde{\mathbf{A}}$ is the same matrix as in (91). Using (96), I can in turn rewrite (33) in a form of type

$$\mathbb{E}_t\left\{S(L)\,\kappa_{t+1} + K_a^1(L)\,a_{t+1} + K_g^1(L)\,g_{t+1}\right\} + K_a^2(L)\,\varepsilon_t^a + K_g^2(L)\,\varepsilon_t^g = 0 \tag{97}$$

with $(K_a^1(X), K_g^1(X), K_a^2(X), K_g^2(X)) \in \mathbb{R}[X]^4$, where $S(X) \in \mathbb{R}[X]$ is defined in Subsection 4.2. As I focus again on paths that do not involve sunspot shocks (since $\mathcal{TA}$ is unlikely to be interested in implementing a path that does), the equation (97) implies that κ_t follows a process of type

$$S(L)\,\kappa_t + \mathbb{E}_{t-1}\left\{K_a^1(L)\,a_t + K_g^1(L)\,g_t\right\} + \widehat{\psi}_a\varepsilon_t^a + \widehat{\psi}_g\varepsilon_t^g + K_a^2(L)\,\varepsilon_{t-1}^a + K_g^2(L)\,\varepsilon_{t-1}^g = 0$$

with $(\widehat{\psi}_a, \widehat{\psi}_g) \in \mathbb{R}^2$, which can be rewritten as

$$\begin{aligned} S(L)\,\kappa_t + K_a^1(L)\,a_t + K_g^1(L)\,g_t &+ \left[\widehat{\psi}_a - K_a^1(0) + K_a^2(L)\,L\right]\varepsilon_t^a \\ &+ \left[\widehat{\psi}_g - K_g^1(0) + K_g^2(L)\,L\right]\varepsilon_t^g = 0. \end{aligned}$$

Using (96), I conclude that τ_t follows a process of type

$$S(L)\,\tau_t = \mathbf{T}_a(L)\,[\,a_t \quad \varepsilon_t^a\,]^T + \mathbf{T}_g(L)\,[\,g_t \quad \varepsilon_t^g\,]^T$$

with $(\mathbf{T}_a(X), \mathbf{T}_g(X)) \in (\mathbb{R}^{1\times 2}[X])^2$, which can be rewritten as an ARMA process of type

$$\rho_a(L)\,\rho_g(L)\,S(L)\,\tau_t = \widehat{T}_a(L)\,\varepsilon_t^a + \widehat{T}_g(L)\,\varepsilon_t^g \tag{98}$$

with $(\widehat{T}_a(X), \widehat{T}_g(X)) \in \mathbb{R}[X]^2$.

I then consider an arbitrarily given (non-sunspot-driven) feasible path along which $\mathbb{E}_t\{b_{t+\ell}\}$ $= 0$, which I denote by $\widehat{P}$. To show the implementability of $\widehat{P}$ and design a tax-rate rule consistent with O_t and implementing $\widehat{P}$ as the robustly unique local equilibrium, I follow the same four steps as in Subsection 4.3. The first step is unchanged, and leads to the conclusion that the rule to design necessarily involves the debt level. In the second step, I rewrite the system of structural equations in a block-recursive way. More specifically, I rewrite (29)-(32) and (87)-(88) as

$$\left[y_t \quad x_t \quad h_t \quad c_t \quad r_t \quad w_t\right]^T = \mathbf{B}\left[\kappa_t \quad \kappa_{t-1} \quad \tau_{t-\ell} \quad a_t \quad g_t\right]^T, \tag{99}$$

where $\mathbf{B}$ is the same matrix as in (42). In turn, using (99), I rewrite (33) and (89) as

$$\mathbb{E}_t\left\{P_k(L)\,\kappa_{t+1} + P_\tau(L)\,L^\ell \tau_{t+1}\right\} + \mathbf{P}_a(L)\left[a_t \quad \varepsilon_t^a\right]^T + \mathbf{P}_g(L)\left[g_t \quad \varepsilon_t^g\right]^T = 0, \tag{100}$$

$$Q_b(L)\,b_t + Q_k(L)\,\kappa_t + Q_\tau L^\ell \tau_t + \mathbf{Q}_a(L)\left[a_t \quad \varepsilon_t^a\right]^T + \mathbf{Q}_g(L)\left[g_t \quad \varepsilon_t^g\right]^T = 0, \tag{101}$$

where $P_k(X)$, $P_\tau(X)$, $Q_b(X)$, $Q_k(X)$, and Q_τ are the same as in (43)-(44), while $(\mathbf{P}_a(X), \mathbf{P}_g(X), \mathbf{Q}_a(X), \mathbf{Q}_g(X)) \in (\mathbb{R}^{1\times 2}[X])^4$.

In the third step, I design a rule that implements $\widehat{P}$ as the robustly unique local equilibrium, but is not consistent with O_t. Consider the class of rules of type

$$\rho_a(L)\,\rho_g(L)\,S(L)\left[\mathcal{R}_b(L)\,b_t + \mathcal{R}_\tau(L)\,\tau_t\right] + \mathcal{R}_a(L)\,\varepsilon_t^a + \mathcal{R}_g(L)\,\varepsilon_t^g = 0, \tag{102}$$

which is parametrized by $(\mathcal{R}_b(X), \mathcal{R}_\tau(X), \mathcal{R}_a(X), \mathcal{R}_g(X)) \in \mathbb{R}[X]^4$ such that $\mathcal{R}_\tau(0) \neq 0$. Since $\rho_a(X)$, $\rho_g(X)$, and $S(X)$ have no root inside the unit circle, any rule of type (102) is equivalent to

$$\mathcal{R}_b(L)\,b_t + \mathcal{R}_\tau(L)\,\tau_t + \rho_a(L)^{-1}\rho_g(L)^{-1}S(L)^{-1}\left[\mathcal{R}_a(L)\,\varepsilon_t^a + \mathcal{R}_g(L)\,\varepsilon_t^g\right] = 0. \tag{103}$$

Under any rule of this type, (b_t, k_t, τ_t) is (uniquely or not uniquely) determined by the system made of (100)-(101) and (103), while $(y_t, x_t, h_t, c_t, r_t, w_t)$ is residually determined by (99). The system made of (100)-(101) and (103) has exactly one non-predetermined variable, corresponding to the term $\mathbb{E}_t\{P_k(0)\kappa_{t+1} + \mathbb{1}_{\ell=0}P_\tau(0)\tau_{t+1}\}$ in (100), where $\mathbb{1}_{\ell=0}$ is equal to 1 if $\ell = 0$ and to 0 if $\ell \geq 1$. The reciprocical characteristic polynomial of this system, $Z_b(X)X^\ell\mathcal{R}_b(X) + Z_\tau(X)\mathcal{R}_\tau(X)$, is the same as the reciprocical characteristic polynomial in Subsection B.2. Therefore, all rules of type (102) with $(\mathcal{R}_b(X), \mathcal{R}_\tau(X)) = (\widetilde{\mathcal{R}}_b^*(X), \widetilde{\mathcal{R}}_\tau^*(X))$, where $\widetilde{\mathcal{R}}_b^*(X)$ and $\widetilde{\mathcal{R}}_\tau^*(X)$ are arithmetically determined in Appendix B.3, robustly ensure local-equilibrium determinacy. Among them, the rule

$$\rho_a(L)\,\rho_g(L)\,S(L)\left[\widetilde{\mathcal{R}}_b^*(L)\,b_t + \widetilde{\mathcal{R}}_\tau^*(L)\,\tau_t\right] + \widehat{\mathcal{R}}_a^*(L)\,\varepsilon_t^a + \widehat{\mathcal{R}}_g^*(L)\,\varepsilon_t^g = 0, \tag{104}$$

where

$$\widehat{\mathcal{R}}_a^*(X) \equiv -\rho_a(X)\,\rho_g(X)\,S(X)\,\widetilde{\mathcal{R}}_b^*(X)\,\widehat{B}_a(X) - \widetilde{\mathcal{R}}_\tau^*(X)\,\widehat{T}_a(X),$$
$$\widehat{\mathcal{R}}_g^*(X) \equiv -\rho_a(X)\,\rho_g(X)\,S(X)\,\widetilde{\mathcal{R}}_b^*(X)\,\widehat{B}_g(X) - \widetilde{\mathcal{R}}_\tau^*(X)\,\widehat{T}_g(X),$$

is satisfied on the path $\widehat{P}$, since it is satisfied when both (95) and (98) hold. This rule, therefore, implements $\widehat{P}$ as the robustly unique local equilibrium.

In the fourth and last step, I transform the rule (104) into a rule consistent with O_t in a way that is neutral for robust local-equilibrium determinacy. Multiplying the left- and right-hand sides of (104) by $\theta_a(L)\theta_g(L)[1-(1-\delta)L]$, using (93) and (94) to replace $\theta_a(L)\varepsilon_t^a$ and $\theta_g(L)\varepsilon_t^g$, and then using (31) and (48) to replace g_t and $[1-(1-\delta)L]a_t$, I get

$$\begin{aligned}\theta_a(L)\,\theta_g(L)\,[1-(1-\delta)L]\,\rho_a(L)\,\rho_g(L)\,S(L)\left[\widetilde{\mathcal{R}}_b^*(L)\,b_t + \widetilde{\mathcal{R}}_\tau^*(L)\,\tau_t\right] \\ +\theta_g(L)\,\rho_a(L)\,\widehat{\mathcal{R}}_a^*(L)\,\{[1-(1-\delta)L]\,(y_t-\alpha h_t)-(1-\alpha)\,\delta x_{t-1}\} \\ +s_g^{-1}\theta_a(L)\,\rho_g(L)\,[1-(1-\delta)L]\,\widehat{\mathcal{R}}_g^*(L)\,[y_t - s_x x_t - (1-s_g-s_x)\,c_t] = 0.\end{aligned} \quad (105)$$

The rule (105) expresses τ_t as a function of only elements of O_t. Moreover, because the only equations used to transform (104) into (105) are the structural equations (29)-(31) (which imply (48)) and the stochastic processes (93)-(94), the system made of all structural equations and the rule (104) is equivalent to the system made of all structural equations and the rule (105). Since $\widehat{P}$ is the unique stationary solution of the former system, it is therefore also the unique stationary solution of the latter system. Finally, because $\theta_a(X)$, $\theta_g(X)$, and $1-(1-\delta)X$ have no root inside the unit circle, (105) still ensures local-equilibrium determinacy when added an exogenous tax-policy shock, like (104). So, to sum up, (105) is consistent with O_t and implements $\widehat{P}$ as the robustly unique local equilibrium. As a consequence, $\widehat{P}$ is implementable, and the rule (105) that I have arithmetically designed implements it.

Appendix C: Stabilization Policy in a General Framework

In this appendix, which complements Section 5, I prove Lemma 1, and I design the polynomials $\boldsymbol{\mathcal{F}}^*(X)$ and $\mathcal{G}^*(X)$ used to prove Proposition 1.

C.1 Proof of Lemma 1

Consider a rule of type (55) with $X^{\ell-z-1}\boldsymbol{\mathcal{F}}(X) \in \mathbb{R}^{1\times N}[X]$ and $\mathcal{F}_j(X) \neq 0$ for all $j \in \mathcal{J}^\star$. To lighten the exposition, remove the exogenous term $\boldsymbol{\mathcal{H}}(L)\boldsymbol{\varepsilon}_{t-\ell}$ from this rule, and similarly remove the exogenous term $\mathbf{C}(L)\boldsymbol{\xi}_t$ from the structural equations (52), as these exogenous terms do not matter for Lemma 1. Denote by S the system made of the structural equations and this rule, i.e. the system made of the following two equations:

$$\mathbb{E}_t\left\{\boldsymbol{\Delta}\left(L^{-1}\right)\left[\mathbf{A}\left(L\right)\mathbf{Y}_t + L^{-z}\mathbf{B}\left(L\right)i_t\right]\right\} = \mathbf{0}, \tag{106}$$

$$\boldsymbol{\mathcal{F}}\left(L\right)\mathbf{Y}_{t-\ell} + \mathcal{G}\left(L\right)i_t = 0. \tag{107}$$

Given the assumption $X^{\ell-z-1}\boldsymbol{\mathcal{F}}(X) \in \mathbb{R}^{1\times N}[X]$, I can use (107) to remove all the terms of type $\mathbb{E}_t\{i_{t+k}\}$ with $k \geq 0$ (if any) from (106), and thus rewrite (106) as

$$\mathbb{E}_t\left\{\boldsymbol{\Delta}\left(L^{-1}\right)\left[\mathbf{A}\left(0\right)\mathbf{Y}_t + \widetilde{\mathbf{A}}\left(L\right)\mathbf{Y}_{t-1}\right]\right\} + \widetilde{\mathbf{B}}\left(L\right)i_{t-1} = \mathbf{0},$$

where $\widetilde{\mathbf{A}}(X) \in \mathbb{R}^{N\times N}[X]$ and $\widetilde{\mathbf{B}}(X) \in \mathbb{R}^{N\times 1}[X]$. Next, given Assumption 1, I can re-express the system S in terms of $\widehat{\mathbf{Y}}_t \equiv \mathbf{A}(0)\mathbf{Y}_t$, instead of $\mathbf{Y}_t$:

$$\mathbb{E}_t\left\{\boldsymbol{\Delta}\left(L^{-1}\right)\left[\widehat{\mathbf{Y}}_t + \widehat{\mathbf{A}}\left(L\right)\widehat{\mathbf{Y}}_{t-1}\right]\right\} + \widetilde{\mathbf{B}}\left(L\right)i_{t-1} = \mathbf{0}, \tag{108}$$

$$\widehat{\boldsymbol{\mathcal{F}}}\left(L\right)\widehat{\mathbf{Y}}_{t-\ell} + \mathcal{G}\left(L\right)i_t = 0, \tag{109}$$

where $\widehat{\mathbf{A}}(X) \equiv \widetilde{\mathbf{A}}(X)[\mathbf{A}(0)]^{-1}$ and $\widehat{\boldsymbol{\mathcal{F}}}(X) \equiv \boldsymbol{\mathcal{F}}(X)[\mathbf{A}(0)]^{-1}$.

For each $j \in \{1, ..., N\}$, let $\widehat{Y}_t^j$ denote the j^{th} element of $\widehat{\mathbf{Y}}_t$, and γ_j the largest lag of $\widehat{Y}_t^j$ in the system (108)-(109), i.e. the integer such that $\widehat{Y}_{t-\gamma_j}^j$ appears in (108)-(109) and $\widehat{Y}_{t-\gamma_j-k}^j$ does not appear in (108)-(109) for any $k \geq 1$. The assumption $\mathcal{F}_j(X) \neq 0$ for all $j \in \mathcal{J}^\star$ implies that $\gamma_j \geq 0$ for all $j \in \{1, ..., N\}$. I focus on the case in which $(\gamma_j, \delta_j) \neq (0, 0)$ for each $j \in \{1, ..., N\}$. If this condition were not satisfied, i.e. if there existed $j \in \{1, ..., N\}$ such that $(\gamma_j, \delta_j) = (0, 0)$, then one could easily rewrite the system S in a block-recursive way, with N equations not featuring the variable $\widehat{Y}_t^j$ (at any past, current, or future date) and one equation residually determining $\widehat{Y}_t^j$. One could then focus on the N-equation subsystem $-$ and repeat the procedure until the subsystem obtained satisfies the condition (which will necessarily happen eventually, unless $(\gamma_j, \delta_j) = (0, 0)$ for all $j \in \{1, ..., N\}$, in which case Lemma 1 is trivial).

Similarly, let γ_{N+1} denote the largest lag of i_t in the system (108)-(109), i.e. the integer such that $i_{t-\gamma_{N+1}}$ appears in (108)-(109) and $i_{t-\gamma_{N+1}-k}$ does not appear in (108)-(109) for any $k \geq 1$. Since $\mathcal{G}(0) \neq 0$, we have $\gamma_{N+1} \geq 0$. I focus on the case in which $\gamma_{N+1} \geq 1$. In the alternative case in which $\gamma_{N+1} = 0$, one could easily rewrite the system S in a block-recursive way, with N equations not featuring the variable i_t (at any past, current, or future date) and one equation residually determining i_t. One could then focus on the N-equation subsystem without the variable i_t.

Since $\gamma_j \geq 0$, $\delta_j \geq 0$, and $(\gamma_j, \delta_j) \neq (0,0)$ for each $j \in \{1, ..., N\}$, and since $\gamma_{N+1} \geq 1$ and $\mathcal{G}(0) \neq 0$, I can easily (but tediously) rewrite the system (108)-(109) in the following Blanchard and Kahn's (1980) form:

$$\mathbb{E}_t \{\mathbf{Z}_{t+1}\} = \mathbf{M}\mathbf{Z}_t,$$

where

$$\mathbf{Z}_t \equiv \begin{bmatrix} \mathbf{Z}_t^1 \\ \vdots \\ \mathbf{Z}_t^{N+1} \end{bmatrix}, \mathbf{Z}_t^j \equiv \begin{bmatrix} \widehat{Y}^j_{t+\delta_j-1} \\ \vdots \\ \widehat{Y}^j_{t-\gamma_j} \end{bmatrix} \text{ for } j \in \{1, ..., N\}, \mathbf{Z}_t^{N+1} \equiv \begin{bmatrix} i_{t-1} \\ \vdots \\ i_{t-\gamma_{N+1}} \end{bmatrix},$$

and $\mathbf{M}$ is a square matrix with real-number elements. The system thus has $\delta \equiv \sum_{j=1}^{N} \delta_j$ non-predetermined variables, corresponding to the variables $\mathbb{E}_t\{\widehat{Y}^j_{t+k_j}\}$ for all $j \in \{1, ..., N\}$ such that $\delta_j \geq 1$ and all $k_j \in \{1, ..., \delta_j\}$.

The non-zero eigenvalues of the system S are those of the corresponding perfect-foresight deterministic system

$$\mathbf{\Upsilon}_1 (L) \begin{bmatrix} \mathbf{Y}_t \\ i_t \end{bmatrix} = \mathbf{0}, \tag{110}$$

where

$$\mathbf{\Upsilon}_1 (X) \equiv \begin{bmatrix} \mathbf{A}(X) & X^{-z}\mathbf{B}(X) \\ X^{\ell}\boldsymbol{\mathcal{F}}(X) & \mathcal{G}(X) \end{bmatrix}.$$

By assumption, we have $\mathbf{\Phi}_1(X) \equiv X^{\ell-z-1}\boldsymbol{\mathcal{F}}(X) \in \mathbb{R}^{1\times N}[X]$. Moreover, since $\mathcal{G}(0) \neq 0$, there exists $\mathbf{\Phi}_2(X) \in \mathbb{R}^{N\times 1}[X]$ such that $\mathbf{\Phi}_3(X) \equiv X^{-z-1}[\mathbf{B}(X) - \mathbf{\Phi}_2(X)\mathcal{G}(X)] \in \mathbb{R}^{N\times 1}[X]$. By multiplying the last line of (110) by $L^{-z}\mathbf{\Phi}_2(L)$ and subtracting the resulting N equations from the first N lines of (110), I can rewrite (110) as

$$\mathbf{\Upsilon}_2 (L) \begin{bmatrix} \mathbf{Y}_t \\ i_t \end{bmatrix} = \mathbf{0},$$

where

$$\boldsymbol{\Upsilon}_2(X) \equiv \begin{bmatrix} \mathbf{A}(X) - X\boldsymbol{\Phi}_2(X)\boldsymbol{\Phi}_1(X) & X\boldsymbol{\Phi}_3(X) \\ X^\ell \boldsymbol{\mathcal{F}}(X) & \mathcal{G}(X) \end{bmatrix}.$$

We have $\boldsymbol{\Upsilon}_2(X) \in \mathbb{R}^{(N+1)\times(N+1)}[X]$ and

$$\det[\boldsymbol{\Upsilon}_2(0)] = \det \begin{bmatrix} \mathbf{A}(0) & \mathbf{0} \\ 0^\ell \boldsymbol{\mathcal{F}}(0) & \mathcal{G}(0) \end{bmatrix} = \mathcal{G}(0)\det[\mathbf{A}(0)] \neq 0,$$

since $\mathcal{G}(0) \neq 0$ and $\det[\mathbf{A}(0)] \neq 0$ (given Assumption 1). Therefore, I can use a standard result in time-series analysis (see, e.g., Hamilton, 1994, Chapter 10, Proposition 10.1) and conclude that the non-zero eigenvalues of the system S are the inverses of the non-zero roots of $\det[\boldsymbol{\Upsilon}_2(X)]$. Now, adding a scalar multiple of one row to another row leaves the determinant of a matrix unchanged. Therefore, $\det[\boldsymbol{\Upsilon}_2(X)] = \det[\boldsymbol{\Upsilon}_1(X)]$. In turn, Laplace's expansion implies that $\det[\boldsymbol{\Upsilon}_1(X)]$ is equal to X^{-Nz^+} times the polynomial (56). Therefore, the non-zero eigenvalues of the system S are the inverses of the non-zero roots of (56). Lemma 1 follows.

C.2 Design of $\boldsymbol{\mathcal{F}}^*(X)$ and $\mathcal{G}^*(X)$

The greatest common divisor of $X^{1+\max(\ell,z)}\Psi_j(X)$ for $j \in \mathcal{J}$ and $\Psi_{N+1}(X)$, defined up to a multiplicative non-zero real-number scalar, is of type $X^m D_{\mathcal{J}}(X)$ with $m \in \mathbb{N}$. Therefore, Bézout's identity implies that there exist $U_j(X) \in \mathbb{R}[X] \smallsetminus \{0\}$ for $j \in \mathcal{J} \cup \{N+1\}$ such that

$$\sum_{j\in\mathcal{J}} U_j(X)\, X^{1+\max(\ell,z)}\Psi_j(X) + U_{N+1}(X)\,\Psi_{N+1}(X) = X^m D_{\mathcal{J}}(X). \tag{111}$$

We have $\Psi_{N+1}(X) = X^{Nz^+}\det[\mathbf{A}(X)]$, with $\det[\mathbf{A}(0)] \neq 0$ (given Assumption 1). Therefore, $\Psi_{N+1}(X)$ is a multiple of X^{Nz^+} but not of X^{1+Nz^+}. Moreover, for any $j \in \mathcal{J}$, $\Psi_j(X)$ is a multiple of $X^{(N-1)z^+}$, which implies that $X^{1+\max(\ell,z)}\Psi_j(X)$ is a multiple of X^{1+Nz^+}. Therefore, the greatest common divisor of all these polynomials, $X^m D_{\mathcal{J}}(X)$, is a multiple of X^{Nz^+} but not of X^{1+Nz^+}. Dividing the left- and right-hand sides of (111) by X^{Nz^+}, and replacing X by 0 in the resulting equation, I thus get $U_{N+1}(0) \neq 0$.

Let $Z(X) \in \mathbb{R}[X]$ be an arbitrarily given polynomial of degree δ such that: (i) if $\delta = 0$, then $Z(X) \neq 0$; and (ii) if $\delta \geq 1$, then all the δ roots of $Z(X)$ lie inside the unit circle and are non-zero. Given that Assumption 1 implies $\Psi_{N+1}(X) \neq 0$, let $Q(X) \in \mathbb{R}[X]$ and $R(X) \in \mathbb{R}[X]$ denote respectively the quotient and the remainder of the Euclidean division of $Z(X)$ by $X\Psi_{N+1}(X)$, i.e. the unique polynomials such that

$$Z(X) = X\Psi_{N+1}(X)Q(X) + R(X) \tag{112}$$

and $d(R) < 1 + d(\Psi_{N+1})$, where, for any polynomial $P(X)$, $d(P)$ denotes the degree of $P(X)$. Replacing X by 0 in (112) and using $Z(0) \neq 0$, I get $R(0) \neq 0$.

Multiplying the left- and right-hand sides of (111) by $R(X)$, and using (112), I obtain

$$\sum\nolimits_{j=1}^{N} \mathcal{R}_j^* (X) \Psi_j (X) + \mathcal{R}_{N+1}^* (X) \Psi_{N+1} (X) = X^m D_{\mathcal{J}} (X) Z (X) , \tag{113}$$

where

$$\begin{aligned} \mathcal{R}_j^* (X) &\equiv X^{1+\max(\ell,z)} R (X) U_j (X) \text{ for } j \in \mathcal{J}, \\ \mathcal{R}_j^* (X) &\equiv 0 \text{ for } j \in \{1, ..., N\} \setminus \mathcal{J}, \\ \mathcal{R}_{N+1}^* (X) &\equiv R (X) U_{N+1} (X) + X^{1+m} Q (X) D_{\mathcal{J}} (X) . \end{aligned}$$

Equation (113) can be rewritten as

$$\sum\nolimits_{j=1}^{N} (-1)^{N+1-j} X^{\ell} \mathcal{F}_j^* (X) \Psi_j (X) + \mathcal{G}^* (X) \Psi_{N+1} (X) = X^m \det[\mathbf{S}(X)] D_{\mathcal{J}} (X) Z (X) , \tag{114}$$

where

$$\begin{aligned} \mathcal{F}_j^*(X) &\equiv (-1)^{N+1-j} X^{-\ell} \det[\mathbf{S}(X)] \mathcal{R}_j^*(X) \text{ for } j \in \{1, ..., N\}, \\ \mathcal{G}^*(X) &\equiv \det[\mathbf{S}(X)] \mathcal{R}_{N+1}^*(X). \end{aligned}$$

We have $\boldsymbol{\mathcal{F}}^*(X) \equiv [\, \mathcal{F}_1^*(X) \quad \cdots \quad \mathcal{F}_N^*(X) \,] \in \mathbb{R}^{1 \times N}[X]$, $\mathcal{G}^*(X) \in \mathbb{R}[X]$, and $\mathcal{G}^*(0) \neq 0$ since $\det[\mathbf{S}(0)] \neq 0$, $R(0) \neq 0$, and $U_{N+1}(0) \neq 0$. Moreover, $\boldsymbol{\mathcal{F}}^*(X)$ and $\mathcal{G}^*(X)$ have the five properties mentioned in the main text (which I rewrite here in italics):

(i) $X^{\ell-z-1}\boldsymbol{\mathcal{F}}^*(X) \in \mathbb{R}^{1\times N}[X]$ *and* $\mathcal{F}_j^*(X) \neq 0$ *for all* $j \in \mathcal{J}^\star$, since $\det[\mathbf{S}(X)] \neq 0$, $R(X) \neq 0$, $U_j(X) \neq 0$ for $j \in \mathcal{J}$, and $\mathcal{J}^\star \subseteq \mathcal{J}$;

(ii) *the polynomial (56) with* $\boldsymbol{\mathcal{F}}(X) = \boldsymbol{\mathcal{F}}^*(X)$ *and* $\mathcal{G}(X) = \mathcal{G}^*(X)$ *has exactly* δ *non-zero roots inside the unit circle*, since (114) makes this polynomial equal to $X^m \det[\mathbf{S}(X)]$ $D_{\mathcal{J}}(X)Z(X)$, where $\det[\mathbf{S}(X)]$ and $D_{\mathcal{J}}(X)$ have no root inside the unit circle, and $Z(X)$ has exactly δ non-zero roots inside the unit circle;

(iii) $\boldsymbol{\mathcal{F}}^*(X)$ *and* $\mathcal{G}^*(X)$ *have no common root inside the unit circle*, since $\det[\mathbf{S}(X)]$ has no root inside the unit circle, and since $\mathcal{R}_j^*(X)$ for $j \in \mathcal{J}$ and $\mathcal{R}_{N+1}^*(X)$ have no common root inside the unit circle, except potentially in zero-measure cases (i.e. for a zero-measure subset of the set of possible choices for $Z(X)$);

(iv) $\{\det[\mathbf{S}(X)]\}^{-1}\boldsymbol{\mathcal{F}}^*(X) \in \mathbb{R}^{1\times N}[X]$ *and* $\{\det[\mathbf{S}(X)]\}^{-1}\mathcal{G}^*(X) \in \mathbb{R}[X]$;

(v) $\mathcal{F}_j^*(X) = 0$ *for all* $j \in \{1, ..., N\} \setminus \mathcal{J}$.

Finally, note that all the steps in the design of $\boldsymbol{\mathcal{F}}^*(X)$ and $\mathcal{G}^*(X)$ involve a finite number of arithmetic operations, in particular the computation of the quotient $Q(X)$ and remainder $R(X)$ of an Euclidean division of polynomials, and the computation of the Bézout polynomials $U_j(X)$ for $j \in \mathcal{J} \cup \{N+1\}$ (using the Euclidean algorithm, which essentially rests on the Euclidean division). Therefore, $\boldsymbol{\mathcal{F}}^*(X)$ and $\mathcal{G}^*(X)$ have been designed arithmetically.

Supplement to
"The Implementability and Implementation of Feasible Paths by Stabilization Policy"

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This supplementary appendix contains the (standard) algebraic derivations of the two optimal feasible paths considered in Section 3, which are used in the numerical simulations (to produce Figures 1 and 2). It also provides the expression of some reduced-form parameters introduced in Section 4, as functions of the structural parameters, so as to make the results ready-to-use for any numerical application.

S.1 Optimal Feasible Path in the Basic NK Model

In this subsection, I derive the timeless-perspective optimal feasible path (17). Given that the timeless-perspective optimal feasible path is defined as the limit of the date-t_0 Ramsey-optimal feasible path as $t_0 \to -\infty$, I start by deriving the date-t_0 Ramsey-optimal feasible path. To do so, I follow the undetermined-coefficients method. More specifically, I write the inflation rate and output as $\pi_{t_0+k} = \sum_{j=0}^{k} a_j^\pi \varepsilon_{t_0+k-j}^\eta + \sum_{j=0}^{k} b_j^\pi \varepsilon_{t_0+k-j}^u$ and $y_{t_0+k} = \sum_{j=0}^{k} a_j^y \varepsilon_{t_0+k-j}^\eta + \sum_{j=0}^{k} b_j^y \varepsilon_{t_0+k-j}^u$ for $k \geq 0$.[1] I look for the values of the coefficients a_j^π, b_j^π, a_j^y, and b_j^y for $j \geq 0$ that minimize

$$L_{t_0} = \mathbb{E}_{t_0} \left\{ \sum_{k=0}^{+\infty} \beta^k \left[\left(\sum_{j=0}^{k} a_j^\pi \varepsilon_{t_0+k-j}^\eta + \sum_{j=0}^{k} b_j^\pi \varepsilon_{t_0+k-j}^u \right)^2 + \lambda \left(\sum_{j=0}^{k} a_j^y \varepsilon_{t_0+k-j}^\eta + \sum_{j=0}^{k} b_j^y \varepsilon_{t_0+k-j}^u \right)^2 \right] \right\} \tag{S.1}$$

[1] To lighten the exposition, I do not consider a deterministic term c_k^π (respectively c_k^y) in the expression of π_{t_0+k} (respectively y_{t_0+k}), because this term is clearly zero on the timeless-perspective optimal feasible path ($\lim_{t_0 \to -\infty} c_k^\pi = \lim_{t_0 \to -\infty} c_k^y = 0$). Thus, the "date-$t_0$ Ramsey-optimal feasible path" that I consider is, in fact, the date-t_0 Ramsey-optimal feasible path up to a deterministic term.

subject to the constraints

$$a_0^y - a_1^y - \sigma^{-1} a_1^\pi - 1 = 0, \tag{S.2}$$

$$b_0^y - b_1^y - \sigma^{-1} b_1^\pi = 0, \tag{S.3}$$

$$a_j^\pi - \beta a_{j+1}^\pi - \kappa a_j^y = 0 \text{ for } j \geq 0, \tag{S.4}$$

$$b_0^\pi - \beta b_1^\pi - \kappa b_0^y - 1 = 0, \tag{S.5}$$

$$b_j^\pi - \beta b_{j+1}^\pi - \kappa b_j^y - (\rho_u + \theta_u)\rho_u^{j-1} = 0 \text{ for } j \geq 1. \tag{S.6}$$

The constraints (S.2)-(S.3) come from the IS equation (13) and the observation set $\widetilde{O}_t \equiv \{\varepsilon^{\eta,t-1}, \varepsilon^{u,t-1}\}$, which implies that i_t cannot depend on $(\varepsilon_t^\eta, \varepsilon_t^u)$. The constraints (S.4)-(S.6) come from the Phillips curve (14). I denote respectively by γ_a, γ_b, $(\delta_j^a)_{j\geq 0}$, δ_0^b, and $(\delta_j^b)_{j\geq 1}$ the Lagrange multipliers associated with these constraints.

I start with the determination of the coefficients a_j^π and a_j^y for $j \geq 0$. The first-order conditions of the Lagrangian minimization with respect to these coefficients are

$$2V_\eta \beta^k a_0^\pi - \delta_0^a = 0,$$

$$2V_\eta \beta^k a_1^\pi + \sigma^{-1}\gamma_a - \delta_1^a + \beta\delta_0^a = 0,$$

$$2V_\eta \beta^k a_j^\pi - \delta_j^a + \beta\delta_{j-1}^a = 0 \text{ for } j \geq 2,$$

$$2V_\eta \beta^k \lambda a_0^y - \gamma_a + \kappa\delta_0^a = 0,$$

$$2V_\eta \beta^k \lambda a_1^y + \gamma_a + \kappa\delta_1^a = 0,$$

$$2V_\eta \beta^k \lambda a_j^y + \kappa\delta_j^a = 0 \text{ for } j \geq 2,$$

where V_η denotes the variance of ε_t^η. By getting rid of the Lagrange multipliers, I can rewrite these first-order conditions as

$$\beta\lambda a_1^y + (1 + \kappa\sigma^{-1})\lambda a_0^y + \beta\kappa a_1^\pi + (1 + \beta + \kappa\sigma^{-1})\kappa a_0^\pi = 0, \tag{S.7}$$

$$\beta\kappa a_2^\pi + \beta\lambda a_2^y + \beta\kappa a_1^\pi + \kappa\lambda\sigma^{-1} a_0^y + (\beta + \kappa\sigma^{-1})\kappa a_0^\pi = 0, \tag{S.8}$$

$$\kappa a_j^\pi + \lambda a_j^y - \lambda a_{j-1}^y = 0 \text{ for } j \geq 3. \tag{S.9}$$

The equations (S.4) and (S.9) imply the recurrence equation $\beta\lambda a_{j+2}^\pi - (\beta\lambda + \kappa^2 + \lambda) a_{j+1}^\pi + \lambda a_j^\pi = 0$ for $j \geq 2$. The roots of the corresponding characteristic polynomial are μ (defined in the main text) and $\mu' \equiv (2\beta\lambda)^{-1}[\lambda + \beta\lambda + \kappa^2 + \sqrt{(\lambda + \beta\lambda + \kappa^2)^2 - 4\beta\lambda^2}]$. Since $0 < \mu < 1$ and $\beta\mu'^2 \geq 1$, as can be readily checked, the solution of the recurrence equation that minimizes L_{t_0} is of the form $a_j^\pi = a_2^\pi \mu^{j-2}$ for $j \geq 2$. The equation (S.4) then implies that $a_j^y = (1-\beta\mu)\kappa^{-1} a_2^\pi \mu^{j-2}$ for $j \geq 2$. Coefficients a_0^π, a_1^π, a_2^π, a_0^y, and a_1^y are then determined by the linear system made of (S.2), (S.4) for $j \in \{0, 1\}$, (S.7), (S.8), and $a_2^y = (1-\beta\mu)\kappa^{-1} a_2^\pi$. I thus eventually obtain $a_0^\pi = a_0$, $a_1^\pi = a_1$, $a_j^\pi = a_2 \mu^{j-2}$ for $j \geq 2$, $a_0^y = \kappa^{-1}(a_0 - \beta a_1)$, $a_1^y = \kappa^{-1}(a_1 - \beta a_2)$, and $a_j^y = (1-\beta\mu)\kappa^{-1} a_2 \mu^{j-2}$ for $j \geq 2$, where $[\,a_0 \quad a_1 \quad a_2\,]^T \equiv \mathbf{M}^{-1}[\,0 \quad 0 \quad \kappa\,]^T$ and

$$\mathbf{M} \equiv \begin{bmatrix} (\beta\kappa^2 + \kappa\lambda\sigma^{-1} + \kappa^2 + \kappa^3\sigma^{-1} + \lambda) & \beta\kappa\,(\kappa - \lambda\sigma^{-1}) & -\beta^2\lambda \\ (\beta\kappa + \kappa^2\sigma^{-1} + \lambda\sigma^{-1})\,\kappa & \beta\kappa\,(\kappa - \lambda\sigma^{-1}) & \beta\,(-\beta\lambda\mu + \kappa^2 + \lambda) \\ 1 & -\,(1 + \beta + \kappa\sigma^{-1}) & \beta \end{bmatrix}.$$

I now turn to the determination of the coefficients b_j^π and b_j^y for $j \geq 0$. The first-order conditions of the Lagrangian minimization with respect to these coefficients are the same as those with respect to the coefficients a_j^π and a_j^y for $j \geq 0$, except that a_j^π, a_j^y, γ_a, δ_j^a, and V_η should be respectively replaced by b_j^π, b_j^y, γ_b, δ_j^b, and V_u, where V_u denotes the variance of ε_t^u. Therefore, by getting rid of the Lagrange multipliers, I can rewrite these first-order conditions as

$$\beta\lambda b_1^y + (1 + \kappa\sigma^{-1})\lambda b_0^y + \beta\kappa b_1^\pi + (1 + \beta + \kappa\sigma^{-1})\kappa b_0^\pi = 0, \tag{S.10}$$

$$\beta\kappa b_2^\pi + \beta\lambda b_2^y + \beta\kappa b_1^\pi + \kappa\lambda\sigma^{-1} b_0^y + (\beta + \kappa\sigma^{-1})\kappa b_0^\pi = 0, \tag{S.11}$$

$$\kappa a_j^\pi + \lambda a_j^y - \lambda a_{j-1}^y = 0 \quad \text{for} \quad j \geq 3, \tag{S.12}$$

which correspond to (S.7), (S.8), and (S.9) in which a_j^π and a_j^y have been respectively replaced by b_j^π and b_j^y for all $j \geq 0$. The equations (S.6) and (S.12) imply the recurrence equation $\beta\lambda b_{j+2}^\pi - (\beta\lambda + \kappa^2 + \lambda) b_{j+1}^\pi + \lambda b_j^\pi = \lambda(1 - \rho_u)(\rho_u + \theta_u)\rho_u^{j-1}$ for $j \geq 2$, which is identical to the recurrence equation obtained above for $(a_j^\pi)_{j\geq 2}$ except for the term on the right-hand side. Therefore, the roots of the corresponding characteristic polynomial are μ, μ', and ρ_u. Given that $0 < \mu < 1$ and $\beta\mu'^2 \geq 1$, and given that I focus on the generic case $\rho_u \neq \mu$, the solution of the recurrence equation that minimizes L_{t_0} is of the form $b_j^\pi = (b_2^\pi - b)\mu^{j-2} + b\rho_u^{j-2}$ for $j \geq 2$ with $b \in \mathbb{R}$. The recurrence equation for $j = 2$ implies that $b = [\beta\lambda\rho_u^2 - (\beta\lambda + \kappa^2 + \lambda)\rho_u + \lambda]^{-1}\lambda(1 - \rho_u)(\rho_u + \theta_u)\rho_u$. The equation (S.6) then implies that $b_j^y = (1 - \beta\mu)\kappa^{-1}(b_2^\pi - b)\mu^{j-2} + [(1 - \beta\rho_u)b - (\rho_u + \theta_u)\rho_u]\kappa^{-1}\rho_u^{j-2}$ for $j \geq 2$. The coefficients b_0^π, b_1^π, b_2^π, b_0^y, and b_1^y are then determined by the linear system made of (S.3), (S.5), (S.6) for $j = 1$, (S.10), (S.11), and $b_2^y = \kappa^{-1}[(1 - \beta\mu)b_2^\pi + \beta(\mu - \rho_u)b - (\rho_u + \theta_u)\rho_u]$. I thus eventually obtain $b_0^\pi = b_0$, $b_1^\pi = b_1$, $b_j^\pi = (b_2 - b)\mu^{j-2} + b\rho_u^{j-2}$

for $j \geq 2$, $b_0^y = \kappa^{-1}(b_0 - \beta b_1 - 1)$, $b_1^y = \kappa^{-1}[b_1 - \beta b_2 - (\rho_u + \theta_u)]$, and $b_j^y = (1 - \beta\mu)\kappa^{-1}(b_2 - b)\mu^{j-2} + [(1-\beta\rho_u)b - (\rho_u + \theta_u)\rho_u]\kappa^{-1}\rho_u^{j-2}$ for $j \geq 2$, where $[\,b_0 \quad b_1 \quad b_2\,]^T \equiv \mathbf{M}^{-1}[\,\lambda\,[1 + \beta(\rho_u + \theta_u) + \kappa\sigma^{-1}] \quad \lambda\,[\beta(\rho_u + \theta_u)\rho_u - \beta^2(\mu - \rho_u)b + \kappa\sigma^{-1}] \quad 1-(\rho_u+\theta_u)\,]^T$.

The coefficients a_j^π, b_j^π, a_j^y, and b_j^y for $j \geq 0$ that I have obtained give me the inflation rate and output on the date-t_0 Ramsey-optimal feasible path as functions of shocks having occurred since date t_0. By making t_0 tend towards $-\infty$, I straightforwardly get these two variables on the timeless-perspective optimal feasible path as functions of all current and past shocks:

$$\mathbf{Y}_t = \mathbf{T}_0^Y \boldsymbol{\varepsilon}_t + \mathbf{T}_1^Y \boldsymbol{\varepsilon}_{t-1} + \sum\nolimits_{j=2}^{+\infty} \left(\rho_u^{j-2}\mathbf{T}_u^Y + \mu^{j-2}\mathbf{T}_\mu^Y\right)\boldsymbol{\varepsilon}_{t-j}, \tag{S.13}$$

where $\mathbf{T}_0^Y \equiv \begin{bmatrix} a_0 & b_0 \\ \frac{a_0 - \beta a_1}{\kappa} & \frac{b_0 - \beta b_1 - 1}{\kappa} \end{bmatrix}$, $\mathbf{T}_1^Y \equiv \begin{bmatrix} a_1 & b_1 \\ \frac{a_1 - \beta a_2}{\kappa} & \frac{b_1 - \beta b_2 - (\rho_u + \theta_u)}{\kappa} \end{bmatrix}$,

$\mathbf{T}_u^Y \equiv \begin{bmatrix} 0 & b \\ 0 & \frac{(1-\beta\rho_u)b - (\rho_u+\theta_u)\rho_u}{\kappa} \end{bmatrix}$, and $\mathbf{T}_\mu^Y \equiv \begin{bmatrix} a_2 & b_2 - b \\ \frac{(1-\beta\mu)a_2}{\kappa} & \frac{(1-\beta\mu)(b_2 - b)}{\kappa} \end{bmatrix}$.

Multiplying the left- and right-hand sides of (S.13) by $(1-\rho_u L)(1-\mu L)$ leads to the first two lines of (17) with

$$\begin{aligned}\mathbf{T}_Y(X) \;\equiv\;& \mathbf{T}_0^Y + \left[-\left(\rho_u + \mu\right)\mathbf{T}_0^Y + \mathbf{T}_1^Y\right]X + \left[\rho_u\mu\mathbf{T}_0^Y - (\rho_u + \mu)\mathbf{T}_1^Y + \mathbf{T}_u^Y + \mathbf{T}_\mu^Y\right]X^2 \\ &+ \left[\rho_u\mu\mathbf{T}_1^Y - \mu\mathbf{T}_u^Y - \rho_u\mathbf{T}_\mu^Y\right]X^3.\end{aligned}$$

Moreover, $\text{rank}[\mathbf{T}_Y(0)] = 2$, since $\text{rank}(\mathbf{T}_0^Y) = 2$.

Then, using the IS equation (13) and (S.13), I residually obtain the interest rate on the timeless-perspective optimal feasible path as a function of all past shocks:

$$i_t = \mathbf{T}_1^i \boldsymbol{\varepsilon}_{t-1} + \sum\nolimits_{j=2}^{+\infty}\left(\rho_\eta^{j-2}\mathbf{T}_\eta^i + \rho_u^{j-2}\mathbf{T}_u^i + \mu^{j-2}\mathbf{T}_\mu^i\right)\boldsymbol{\varepsilon}_{t-j}, \tag{S.14}$$

where $\mathbf{T}_1^i \equiv \begin{bmatrix} \frac{\left(1+\beta-\beta\mu+\kappa\sigma^{-1}\right)a_2 - a_1 + \kappa(\rho_\eta + \theta_\eta)}{\kappa\sigma^{-1}} & \frac{\left(1+\beta-\beta\mu+\kappa\sigma^{-1}\right)b_2 - b_1 + \beta(\mu - \rho_u)b + (\rho_u+\theta_u)(1-\rho_u)}{\kappa\sigma^{-1}} \end{bmatrix}$,

$\mathbf{T}_\eta^i \equiv \begin{bmatrix} (\rho_\eta + \theta_\eta)\rho_\eta\sigma & 0 \end{bmatrix}$, $\mathbf{T}_u^i \equiv \begin{bmatrix} 0 & b\rho_u - \frac{[(1-\beta\rho_u)b - (\rho_u+\theta_u)\rho_u](1-\rho_u)\sigma}{\kappa} \end{bmatrix}$,

and $\mathbf{T}_\mu^i \equiv \begin{bmatrix} \frac{-(\kappa\sigma - \lambda)a_2\mu}{\lambda} & \frac{-(\kappa\sigma-\lambda)(b_2 - b)\mu}{\lambda} \end{bmatrix}$.

Multiplying the left- and right-hand sides of (S.14) by $(1-\rho_\eta L)(1-\rho_u L)(1-\mu L)$ leads to the last line of (17) with

$$\begin{aligned}\mathbf{T}_i(X) \;\equiv\;& \mathbf{T}_1^i + \left[-(\rho_\eta + \rho_u + \mu)\mathbf{T}_1^i + \mathbf{T}_\eta^i + \mathbf{T}_u^i + \mathbf{T}_\mu^i\right]X \\ &+ \left[(\rho_\eta\rho_u + \rho_\eta\mu + \rho_u\mu)\mathbf{T}_1^i - (\rho_u + \mu)\mathbf{T}_\eta^i - (\rho_\eta + \mu)\mathbf{T}_u^i - (\rho_\eta + \rho_u)\mathbf{T}_\mu^i\right]X^2 \\ &+ \left[-\rho_\eta\rho_u\mu\mathbf{T}_1^i + \rho_u\mu\mathbf{T}_\eta^i + \rho_\eta\mu\mathbf{T}_u^i + \rho_\eta\rho_u\mathbf{T}_\mu^i\right]X^3.\end{aligned}$$

Finally, it can be checked that $[1 \quad 0]\mathbf{T}_Y(\rho_u^{-1}) \neq \mathbf{0}$, $[1 \quad 0]\mathbf{T}_Y(\mu^{-1}) \neq \mathbf{0}$, $[0 \quad 1]\mathbf{T}_Y(\rho_u^{-1}) \neq \mathbf{0}$, $[0 \quad 1]\mathbf{T}_Y(\mu^{-1}) \neq \mathbf{0}$, $\mathbf{T}_i(\rho_\eta^{-1}) \neq \mathbf{0}$, $\mathbf{T}_i(\rho_u^{-1}) \neq \mathbf{0}$, and $\mathbf{T}_i(\mu^{-1}) \neq \mathbf{0}$, except possibly in zero-measure cases. Therefore, the ARMA(p,q) representation (17) of the path $\widetilde{P}$ is generically of *minimal* orders p and q.

S.2 Optimal Feasible Path in Svensson and Woodford's Model

Svensson and Woodford (2005) compute the timeless-perspective optimal feasible path when $\mathcal{CB}$'s observation set is $\{\varepsilon^{\eta,t-1}, \varepsilon^{u,t-1}\}$. They provide the following expressions for $\mathbb{E}_t\{\pi_{t+1}\}$, $\mathbb{E}_t\{y_{t+1}\}$, and i_t on this path, as functions of η_{t-1} and u^t:[2]

$$\mathbb{E}_t\{\pi_{t+1}\} = \frac{\rho_u\mu}{1-\beta\rho_u\mu}u_t - \frac{(1-\mu)\rho_u\mu}{1-\beta\rho_u\mu}\sum_{j=1}^{+\infty}\mu^{j-1}u_{t-j}, \tag{S.15}$$

$$\mathbb{E}_t\{y_{t+1}\} = \frac{-\kappa\rho_u\mu}{\lambda(1-\beta\rho_u\mu)}\sum_{j=0}^{+\infty}\mu^j u_{t-j}, \tag{S.16}$$

$$i_t = \sigma\rho_\eta\eta_{t-1} + \frac{(\lambda-\kappa\sigma)\rho_u^2\mu}{\lambda(1-\beta\rho_u\mu)}u_{t-1} - \frac{(\lambda-\kappa\sigma)(1-\mu)\rho_u\mu}{\lambda(1-\beta\rho_u\mu)}\sum_{j=1}^{+\infty}\mu^{j-1}u_{t-j}. \tag{S.17}$$

Using these expressions, the IS equation (25), the Phillips curve (26), and the definition of μ, I easily get π_t and y_t on this path as functions of ε_t^η and u^t:

$$\pi_t = u_t + \left(\frac{\rho_u\mu}{1-\beta\rho_u\mu} - \rho_u\right)u_{t-1} - \frac{(1-\mu)\rho_u\mu}{1-\beta\rho_u\mu}\sum_{j=2}^{+\infty}\mu^{j-2}u_{t-j}, \tag{S.18}$$

$$y_t = \varepsilon_t^\eta - \frac{\kappa\rho_u\mu}{\lambda(1-\beta\rho_u\mu)}\sum_{j=1}^{+\infty}\mu^{j-1}u_{t-j}. \tag{S.19}$$

Multiplying the left- and right-hand sides of (S.18) and (S.19) by $(1-\rho_u L)(1-\mu L)$ leads to the first two lines of (27), with

$$\mathbf{T}_Y^{SW}(X) \equiv \begin{bmatrix} 0 & 1 + \left(\frac{\rho_u\mu}{1-\beta\rho_u\mu} - \rho_u - \mu\right)X - \frac{\beta\rho_u^2\mu^2}{1-\beta\rho_u\mu}X^2 \\ (1-\rho_u X)(1-\mu X) & \frac{-\kappa\rho_u\mu}{\lambda(1-\beta\rho_u\mu)}X \end{bmatrix}.$$

Multiplying the left- and right-hand sides of (S.17) by $(1-\rho_\eta L)(1-\rho_u L)(1-\mu L)$ leads to the last line of (27), with

$$\mathbf{T}_i^{SW}(X) \equiv \begin{bmatrix} \sigma\rho_\eta(1-\rho_u X)(1-\mu X) & \frac{-(\lambda-\kappa\sigma)\rho_u\mu}{\lambda(1-\beta\rho_u\mu)}(1-\rho_\eta X)(1-\rho_u-\mu+\rho_u\mu X) \end{bmatrix}.$$

It is easy to check that $[1 \quad 0]\mathbf{T}_Y^{SW}(\rho_u^{-1}) \neq \mathbf{0}$, $[1 \quad 0]\mathbf{T}_Y^{SW}(\mu^{-1}) \neq \mathbf{0}$, $[0 \quad 1]\mathbf{T}_Y^{SW}(\rho_u^{-1}) \neq \mathbf{0}$, $[0 \quad 1]\mathbf{T}_Y^{SW}(\mu^{-1}) \neq \mathbf{0}$, $\mathbf{T}_i^{SW}(\rho_\eta^{-1}) \neq \mathbf{0}$, $\mathbf{T}_i^{SW}(\rho_u^{-1}) \neq \mathbf{0}$, and $\mathbf{T}_i^{SW}(\mu^{-1}) \neq \mathbf{0}$, except possibly in zero-measure cases. Therefore, the ARMA(p,q) representation (27) is

[2] There are two differences between Equations (S.15), (S.16), (S.17) in this supplementary appendix, and Equations (26), (27), (32) in Svensson and Woodford (2005). First, as mentioned in the main text, I have set the mean of η_t to zero for simplicity. Second, I have corrected a typo in their Equation (27); more specifically, I have removed the negative sign just after the equality sign.

generically of *minimal* orders p and q. Finally, it is also easy to check that the polynomials $(1-\rho_\eta X)\det[\mathbf{T}_Y^{SW}(X)]$ and $\mathbf{T}_i^{SW}(X)adj[\mathbf{T}_Y^{SW}(X)]$ are divisible by $D(X) \equiv (1-\rho_u X)(1-\mu X)$, but not by any scalar polynomial of higher degree. Therefore, $D(X)$ is the greatest common divisor (defined up to a multiplicative non-zero real-number scalar) of these polynomials.

S.3 Some Reduced-Form Parameters

$$\mathbf{A} \equiv \frac{1}{\alpha+\left(1+\chi-\frac{\alpha}{1-\tau}\right)\frac{s_c}{\sigma}}\left[\mathbf{A}_1 \quad \mathbf{A}_2\right],$$

where

$$\mathbf{A}_1 \equiv \begin{bmatrix} \frac{\alpha s_x}{\delta} & \frac{-(1-\delta)\alpha s_x}{\delta}+(1-\alpha)(1+\chi)\frac{s_c}{\sigma} \\ \frac{\alpha}{\delta}+\left(1+\chi-\frac{\alpha}{1-\tau}\right)\frac{s_c}{\delta\sigma} & \frac{-(1-\delta)}{\delta}\left[\alpha+\left(1+\chi-\frac{\alpha}{1-\tau}\right)\frac{s_c}{\sigma}\right] \\ \frac{s_x}{\delta} & \frac{-(1-\delta)s_x}{\delta}-(1-\alpha)\left[1-\frac{s_c}{(1-\tau)\sigma}\right] \\ -\left(1+\chi-\frac{\alpha}{1-\tau}\right)\frac{s_x}{\delta} & \left(1+\chi-\frac{\alpha}{1-\tau}\right)\frac{(1-\delta)s_x}{\delta}+(1-\alpha)(1+\chi) \\ \left(1+\frac{\omega\tau}{1-\tau}\right)\frac{\alpha s_x}{\delta} & \frac{(1-\alpha)(1+\chi)\omega\tau s_c}{(1-\tau)\sigma}-\left(1+\frac{\omega\tau}{1-\tau}\right)\frac{(1-\delta)\alpha s_x}{\delta}-\alpha\left[1+\left(\chi-\frac{\tau}{1-\tau}\right)\frac{s_c}{\sigma}\right] \\ -\left(1-\frac{\alpha}{1-\tau}\right)\frac{s_x}{\delta} & \left(1-\frac{\alpha}{1-\tau}\right)\frac{(1-\delta)s_x}{\delta}+(1-\alpha)\left[1+\frac{\chi s_c}{(1-\tau)\sigma}\right] \\ \frac{-\alpha s_x}{\delta} & \frac{(1-\delta)\alpha s_x}{\delta}-(1-\alpha)(1+\chi)\frac{s_c}{\sigma} \end{bmatrix},$$

$$\mathbf{A}_2 \equiv \begin{bmatrix} (1+\chi)\frac{s_c}{\sigma} & \left(1-\frac{s_c}{\varphi\sigma}\right)\alpha s_g \\ 0 & 0 \\ -\left[1-\frac{s_c}{(1-\tau)\sigma}\right] & \left(1-\frac{s_c}{\varphi\sigma}\right)s_g \\ 1+\chi & -\left(1+\chi+\frac{\alpha}{\varphi}-\frac{\alpha}{1-\tau}\right)s_g \\ (1+\chi)\left(1+\frac{\omega\tau}{1-\tau}\right)\frac{s_c}{\sigma} & \left(1-\frac{s_c}{\varphi\sigma}\right)\left(1+\frac{\omega\tau}{1-\tau}\right)\alpha s_g-\left[\alpha+\left(1+\chi-\frac{\alpha}{1-\tau}\right)\frac{s_c}{\sigma}\right]\frac{s_g\omega}{\varphi} \\ \left[1+\frac{\chi s_c}{(1-\tau)\sigma}\right] & -\left(1+\frac{\alpha\sigma+\chi s_c}{\varphi\sigma}-\frac{\alpha}{1-\tau}\right)s_g \\ -(1+\chi)\frac{s_c}{\sigma} & \left[\alpha+\left(1+\chi-\frac{\alpha}{1-\tau}\right)\frac{s_c}{\sigma}\right](1-\tau)\frac{s_g}{\varphi\tau}-\left(1-\frac{s_c}{\varphi\sigma}\right)\alpha s_g \end{bmatrix},$$

with $s_c \equiv 1-s_g-s_x$ and $\varphi \equiv (1-\tau)\left[\alpha+(1-\alpha)\omega\right]$;

$$\mathbf{B} \equiv \frac{s_c}{(1+\chi)s_c+\alpha(\sigma-s_c)}\left[\mathbf{B}_1 \quad \mathbf{B}_2\right],$$

where

$$\mathbf{B}_1 \equiv \begin{bmatrix} \frac{\alpha\sigma s_x}{\delta s_c} & (1-\alpha)(1+\chi)-\frac{(1-\delta)\alpha\sigma s_x}{\delta s_c} \\ \frac{1+\chi}{\delta}+\frac{\alpha(\sigma-s_c)}{\delta s_c} & \frac{-(1-\delta)}{\delta}\left[1+\chi+\frac{\alpha(\sigma-s_c)}{s_c}\right] \\ \frac{\sigma s_x}{\delta s_c} & \frac{-(1-\alpha)(\sigma-s_c)}{s_c}-\frac{(1-\delta)\sigma s_x}{\delta s_c} \\ \frac{(\alpha\sigma-s_c)\sigma s_x}{\delta s_c^2} & \frac{(1-\alpha)(1+\chi)\sigma}{s_c}-\frac{(1-\delta)(\alpha\sigma-s_c)\sigma s_x}{\delta s_c^2} \\ \frac{\alpha\sigma s_x}{\delta s_c} & -\alpha\left(\chi+\frac{\sigma}{s_c}\right)-\frac{\alpha(1-\delta)\sigma s_x}{\delta s_c} \\ \left(\chi+\frac{\alpha\sigma-s_c}{s_c}\right)\frac{\sigma s_x}{\delta s_c} & (1-\alpha)\left(\chi+\frac{\sigma}{s_c}\right)-\frac{(1-\delta)\sigma s_x}{\delta s_c}\left(\chi+\frac{\alpha\sigma-s_c}{s_c}\right) \end{bmatrix},$$

$$\mathbf{B}_2 \equiv \begin{bmatrix} \frac{-\alpha\tau}{1-\tau} & 1+\chi & \frac{\alpha\sigma s_g}{s_c} \\ 0 & 0 & 0 \\ \frac{-\tau}{1-\tau} & \frac{-(\sigma-s_c)}{s_c} & \frac{\sigma s_g}{s_c} \\ \frac{-\alpha\sigma\tau}{(1-\tau)s_c} & \frac{(1+\chi)\sigma}{s_c} & \frac{(\alpha\sigma-s_c)\sigma s_g}{s_c^2} \\ \frac{-\alpha\tau}{1-\tau} - \left[1+\chi+\frac{\alpha(\sigma-s_c)}{s_c}\right]\frac{\omega\tau}{1-\tau} & 1+\chi & \frac{\alpha\sigma s_g}{s_c} \\ -\left(\chi+\frac{\alpha\sigma}{s_c}\right)\frac{\tau}{1-\tau} & \chi+\frac{\sigma}{s_c} & \left(\chi+\frac{\alpha\sigma-s_c}{s_c}\right)\frac{\sigma s_g}{s_c} \end{bmatrix};$$

and

$$P_k(X) \equiv \left[B_{41} - \frac{1-\beta(1-\delta)}{\sigma}B_{51}\right] + \left[B_{42} - B_{41} - \frac{1-\beta(1-\delta)}{\sigma}B_{52}\right]X - B_{42}X^2,$$

$$P_\tau(X) \equiv \left[B_{43} - \frac{1-\beta(1-\delta)}{\sigma}B_{53}\right] - B_{43}X,$$

$$Q_b(X) \equiv 1 - \beta^{-1}X,$$

$$Q_k(X) \equiv [\alpha + (1-\alpha)\omega]\,\tau\,(B_{11} + B_{12}X),$$

$$P_a \equiv -B_{44},$$

$$P_g \equiv -B_{45},$$

$$Q_\tau \equiv [\alpha + (1-\alpha)\omega]\,\tau\,(B_{13} + 1),$$

$$Q_a \equiv [\alpha + (1-\alpha)\omega]\,\tau B_{14},$$

$$Q_g \equiv [\alpha + (1-\alpha)\omega]\,\tau B_{15} - s_g.$$