In Chapter 2, we assumed for simplicity that CB, at each date $t$,
- directly controlled not only $i_t$, but also $\pi_t$ and $x_t$,
- observed the history of the exogenous shocks $(\bar{r}_{t-k}, u_{t-k})_{k \geq 0}$.

In this chapter, we more relevantly assume that CB, at each date $t$,
- directly controls only $i_t$,
- may not observe the history of the exogenous shocks,
and find that this affects the analysis in some important ways.

The equation describing how CB sets $i_t$ (as a function of some endogenous variables and/or exogenous shocks) is called “interest-rate rule.”

The main goal of the chapter is to address the question of what kind of interest-rate rule CB should follow, given its observation set.
We argue that the interest-rate rule should ensure local-equilibrium determinacy, i.e. be such that the log-linearized system made of the structural equations and this rule has a unique stationary solution.

We state the determinacy conditions in a general framework, i.e. for a broad class of linear rational-expectations systems.

We apply these general results to the log-linearized system made of

- the structural equations of the basic NK model,
- a simple interest-rate rule, like, e.g., Taylor’s (1993) rule,

and thus establish in particular the Taylor principle.

We apply the latter results to US monetary policy between 1960 and 1996.
We also argue that the interest-rate rule should be such that this unique stationary solution coincides with the **optimal feasible path**, i.e. the path that maximizes RH’s welfare subject to CB’s observation-set constraint.

We show that, in the basic NK model, for some reasonable observation sets of CB, there may exist no interest-rate rule consistent with CB’s observation set, consistent with the optimal feasible path, ensuring local-equilibrium determinacy, so that the **optimal feasible path may not be implementable**.

Lastly, we explore the related but distinct issue of the **multiplicity of determinate projections** conditional on a given interest-rate path.
Outline of the chapter

1. Introduction

2. Determinacy conditions (Blanchard and Kahn, 1980)

3. Taylor principle (e.g., Woodford, 2003, C4)


5. Implementability of optimal feasible MP (Loisel, 2016)

6. Multiplicity of determinate projections (Galí, 2011)
In the first section of this chapter, we are interested in the conditions under which linear rational-expectations systems have a unique stationary solution.

We consider the class of linear rational-expectations systems that can be written in Blanchard and Kahn’s (1980) form:

\[ E_t \{Z_{t+1}\} = AZ_t + \xi_t, \]

where

- \( Z_t \) is a vector of endogenous variables set at date \( t \) or earlier,
- \( \xi_t \) is a vector of exogenous disturbances realized at date \( t \) or earlier (often called “fundamental disturbances” as they appear in the system),
- \( A \) is a matrix with real-number elements.
Blanchard and Kahn’s (1980) conditions

- Let $m$ denote the number of non-predetermined variables of the system (loosely speaking, the degrees of freedom due to the presence of expected future variables).

- Let $n$ denote the number of eigenvalues of $A$ that are outside the unit circle.

- Blanchard and Kahn (1980) show that, provided that a certain rank condition is met (as is typically the case),
  - if $m < n$, then the system has no stationary solution,
  - if $m = n$, then the system has a unique stationary solution,
  - if $m > n$, then the system has an infinity of stationary solutions.

- Note that these conditions do not involve $\xi_t$, so that they are the same for the deterministic system $E_t \{Z_{t+1}\} = AZ_t$. 
A simple illustration

Consider the following one-equation one-variable system:

\[ \mathbb{E}_t \{ z_{t+1} \} = az_t + \xi_t, \]

where \( a > 0 \) and \( \xi_t \) is an i.i.d fundamental shock (i.e. shock that appears in the system).

- Blanchard and Kahn’s (1980) conditions say that this system has
  - a unique stationary solution if \( a > 1 \),
  - an infinity of stationary solutions if \( a < 1 \).

- When \( a > 1 \), the unique stationary solution is \( z_t = \frac{-\xi_t}{a} \).

- When \( a < 1 \), for any i.i.d. “sunspot shock” \( \zeta_t \) (i.e. shock that does not appear in the system), \( z_t = \frac{-\xi_t}{a} + \sum_{k=0}^{+\infty} a^k \zeta_{t-k} \) is a stationary solution.
Local-equilibrium determinacy

- When an economic system has an infinity of stationary solutions, sunspot shocks may make the economy more volatile, which typically decreases welfare.

- Therefore, an interest-rate rule should ensure **local-equilibrium determinacy**, i.e. be such that the log-linearized system made of the structural equations and this rule has a unique stationary solution.

- Bernanke and Woodford (1997) provide an easy-to-interpret example of multiple local equilibria.

- They consider an interest-rate rule prescribing to raise the short-term nominal interest rate $i_t$ in response to a rise in the long-term nominal interest rate $i_t^\ell$, interpreted rightly or wrongly as an “inflation scare.”

- Then, markets’ expectations of an increase in $i_t$ will entail an increase in $i_t^\ell$ and therefore an increase in $i_t$ that will validate these expectations.
Rules with an exogenous right-hand side I

- Consider any interest-rate rule setting the interest rate **exogenously** (i.e., as a linear function of only current and past exogenous shocks, with intercept \(\bar{i}\)), and note it \(R_1\).

- The deterministic version of the log-linearized system made of the IS equation, the Phillips curve, and \(R_1\) is

\[
\begin{align*}
\chi_t &= E_t \{\chi_{t+1}\} - \frac{1}{\sigma} \left( i_t - E_t \{\pi_{t+1}\} - \bar{i} \right), \\
\pi_t &= \beta E_t \{\pi_{t+1}\} + \kappa \chi_t, \\
i_t &= \bar{i},
\end{align*}
\]

and can be rewritten as \(i_t = \bar{i}\) and

\[
\begin{bmatrix}
\frac{1}{\sigma} & 1 \\
\beta & 0
\end{bmatrix}
E_t \left\{ \begin{bmatrix}
\pi_{t+1} \\
\chi_{t+1}
\end{bmatrix} \right\}
= \begin{bmatrix}
0 & 1 \\
1 & -\kappa
\end{bmatrix}
\begin{bmatrix}
\pi_t \\
\chi_t
\end{bmatrix}.
\]
The latter system can be rewritten in Blanchard and Kahn’s (1980) form

\[ \mathbb{E}_t \{ \mathbf{X}_{t+1} \} = A_1 \mathbf{X}_t, \]

where

\[
\mathbf{X}_t \equiv \begin{bmatrix} \pi_t \\ x_t \end{bmatrix} \quad \text{and} \quad A_1 \equiv \begin{bmatrix} \frac{1}{\beta} & \frac{-\kappa}{\beta} \\ \frac{-1}{\beta \sigma} & 1 + \frac{\kappa}{\beta \sigma} \end{bmatrix}.
\]

The eigenvalues of \( A_1 \),

\[
\delta \equiv \frac{(1 + \beta + \frac{\kappa}{\sigma}) - \sqrt{(1 + \beta + \frac{\kappa}{\sigma})^2 - 4\beta}}{2\beta},
\]

\[
\delta' \equiv \frac{(1 + \beta + \frac{\kappa}{\sigma}) + \sqrt{(1 + \beta + \frac{\kappa}{\sigma})^2 - 4\beta}}{2\beta},
\]

are such that \( 0 < \delta < 1 \) and \( \delta' > 1 \).
Rules with an exogenous right-hand side III

- So the system has two non-predetermined variables ($E_t\{\pi_{t+1}\}$ and $E_t\{x_{t+1}\}$), but only one eigenvalue outside the unit circle ($\delta' > 1$).

- Therefore, this system has multiple stationary solutions, i.e. there are multiple local equilibria.

- The result that setting the interest rate exogenously may lead to indeterminacy was first obtained by Sargent and Wallace (1975) in a non-micro-founded model.

- McCallum (1981) subsequently showed that interest-rate rules with an endogenous right-hand side could ensure determinacy, by making the interest rate react “out of equilibrium” to endogenous variables.
A Taylor rule I

- So consider, for instance, the following interest-rate rule, noted $R_2$:

$$i_t = \bar{i} + \phi_\pi \pi_t + \phi_x x_t,$$

where $\phi_\pi \geq 0$ and $\phi_x \geq 0$, which is similar to Taylor’s (1993) rule.

- Using this rule to replace $i_t$ in the IS equation, we obtain the deterministic system $\mathbb{E}_t \{X_{t+1}\} = A_2 X_t$, where

$$A_2 \equiv \begin{bmatrix} \frac{1}{\beta} & \frac{-\kappa}{\beta \phi_x + \kappa} \\ \frac{\beta \phi_\pi - 1}{\beta \sigma} & 1 + \frac{\beta \phi_x + \kappa}{\beta \sigma} \end{bmatrix},$$

so that $R_2$ ensures determinacy if and only if the two eigenvalues of $A_2$ are outside the unit circle (since the system still has two non-predet. variables).
As shown by, e.g., Woodford (2003, C4), this happens if and only if

\[ \phi_{\pi} + \frac{1 - \beta}{\kappa} \phi_x > 1. \]

The Phillips curve implies that a 1-unit permanent increase in the inflation rate leads to a \( \frac{1 - \beta}{\kappa} \)-unit permanent increase in the output gap.

So the left-hand side of the **determinacy condition** above represents the permanent increase in the interest rate prescribed by \( R_2 \) in response to a 1-unit permanent increase in the inflation rate.

Therefore, the determinacy condition corresponds to the **Taylor principle**: in the long term, the (nominal) interest rate should rise by more than the increase in the inflation rate in order to ensure determinacy.
A Taylor rule with inertia I

Now consider the following interest-rate rule, noted $R_3$:

$$i_t = (1 - \rho) \bar{i} + \rho i_{t-1} + \phi_\pi \pi_t + \phi_x x_t,$$

where $\rho \geq 0$, $\phi_\pi \geq 0$, and $\phi_x \geq 0$, which includes $R_2$ as a special case.

We then obtain the deterministic system $\mathbb{E}_t \{Y_{t+1}\} = A_3 Y_t$, where

$$Y_t \equiv \begin{bmatrix} \pi_t \\ x_t \\ i_{t-1} - \bar{i} \end{bmatrix} \quad \text{and} \quad A_3 \equiv \begin{bmatrix} 1 - \frac{k}{\beta} & \frac{-\kappa}{\beta} & 0 \\ \frac{1}{\beta} + \frac{k}{\beta} & \frac{-\kappa}{\beta} & \frac{\rho}{\sigma} \\ \phi_\pi & \phi_x & \rho \end{bmatrix}.$$
A Taylor rule with inertia II

- As shown by, e.g., Woodford (2003, C4), this happens if and only if
  \[ \phi_\pi + \frac{1 - \beta}{\kappa} \phi_x > 1 - \rho. \]

- When \( \rho < 1 \), this \textbf{determinacy condition} corresponds once again to the \textbf{Taylor principle}: in the long term, the (nominal) interest rate should rise by more than the increase in the inflation rate in order to ensure determinacy.

- When \( \rho \geq 1 \), this determinacy condition is necessarily satisfied, and so is the Taylor principle since the prescribed increase in the interest rate is infinite.

- So, whether \( \rho < 1 \) or \( \rho \geq 1 \), this determinacy condition corresponds to the Taylor principle.
Next, consider the following interest-rate rule, noted $R_4$:

$$i_t = \bar{i} + \phi_\pi \pi_t^{E_t} + \phi_x x_t,$$

where $\phi_\pi \geq 0$ and $\phi_x \geq 0$.

We then obtain the deterministic system $E_t \{X_{t+1}\} = A_4 X_t$, where

$$A_4 \equiv \begin{bmatrix}
\frac{1}{\beta} & -\frac{\kappa}{\beta} \\
\frac{\phi_\pi - 1}{\beta \sigma} & 1 + \frac{\beta \phi_x - \kappa (\phi_\pi - 1)}{\beta \sigma}
\end{bmatrix},$$

so that $R_4$ ensures determinacy if and only if the two eigenvalues of $A_4$ are outside the unit circle (since the system still has two non-predet. variables).
A forward-looking Taylor rule II

- As shown by, e.g., Woodford (2003, C4), this happens if and only if
  \[ \phi_\pi + \frac{1 - \beta}{\kappa} \phi_x > 1 \]
  and
  \[ \phi_\pi < 1 + \frac{1 + \beta}{\kappa} \left( \phi_x + \frac{2}{\sigma} \right). \]

- So, for this rule, the **Taylor principle** is necessary, but not sufficient for determinacy.
A forward-looking Taylor rule with inertia I

- Now turn to the following interest-rate rule, noted $R_5$:

$$i_t = (1 - \rho) \bar{i} + \rho i_{t-1} + \phi_\pi E_t \{ \pi_{t+1} \} + \phi_x x_t,$$

where $\rho \geq 0$, $\phi_\pi \geq 0$, and $\phi_x \geq 0$, which includes $R_4$ as a special case.

- We then obtain the deterministic system $E_t \{ Y_{t+1} \} = A_5 Y_t$, where

$$A_5 = \begin{bmatrix} \frac{1}{\bar{\beta}} & \frac{-\kappa}{\bar{\beta}} & 0 \\ \frac{\phi_\pi - 1}{\beta \sigma} & 1 + \frac{\beta \phi_x - \kappa (\phi_\pi - 1)}{\beta \sigma} & \frac{\rho}{\sigma} \\ \frac{\phi_\pi}{\bar{\beta}} & \frac{\kappa \phi_\pi}{\bar{\beta}} & \rho \end{bmatrix},$$

so that $R_5$ ensures determinacy if and only if $A_5$ has exactly two eigenvalues outside the unit circle (since the system still has two non-predet. variables).
As shown by, e.g., Woodford (2003, C4), this happens only if

\[
\phi_\pi + \frac{1 - \beta}{\kappa} \phi_x > 1 - \rho
\]

and

\[
\phi_\pi < 1 + \rho + \frac{1 + \beta}{\kappa} \left[ \phi_x + \frac{2(1 + \rho)}{\sigma} \right].
\]

So, for this rule too, the Taylor principle is necessary, but not sufficient for determinacy.
Lastly, consider the following interest-rate rule, noted $R_6$:

$$i_t = \bar{i} + \phi_p p_t + \phi_x x_t,$$

where $\phi_p > 0$ and $\phi_x \geq 0$, which includes Wicksell’s (1898) rule as a special case (namely the case in which $\phi_x = 0$).

We then obtain the deterministic system $\mathbb{E}_t \{Z_{t+1}\} = A_6 Z_t$, where

$$Z_t \equiv \begin{bmatrix} p_t \\ p_{t-1} \\ x_t \end{bmatrix} \quad \text{and} \quad A_6 \equiv \begin{bmatrix} \frac{1+\beta}{\beta} & -1 & -\kappa \\ 1 & 0 & 0 \\ \frac{\beta\phi_p-1}{\beta\sigma} & \frac{1}{\beta\sigma} & 1 + \frac{\beta\phi_x+\kappa}{\beta\sigma} \end{bmatrix},$$

so that $R_6$ ensures determinacy if and only if $A_6$ has exactly two eigenvalues outside the unit circle (since the system still has two non-predet. variables).
As shown by, e.g., Woodford (2003, C4), this happens whatever the values of $\phi_p > 0$ and $\phi_x \geq 0$.

This (absence of) determinacy condition corresponds once again to the Taylor principle.

Indeed, any permanent increase in the inflation rate eventually leads to an infinite increase in the price level, and therefore an infinite increase in the interest rate provided that $\phi_p > 0$. 
We now turn to an application of the previous determinacy results.

Clarida, Galí, and Gertler (2000), henceforth CGG, argue that the observed decrease in macroeconomic volatility in the US between the 1960-1979 and 1979-1996 periods may be due to a change in MP in 1979.

They estimate the following interest-rate rule on US data:

$$i_t = \rho(L)i_{t-1} + (1 - \rho) [i^* + \beta (\mathbb{E}_t \{\pi_{t+k}\} - \pi^*) + \gamma \mathbb{E}_t \{x_{t+q}\}] ,$$

where $L$ is the lag operator ($L^i_t \equiv i_{t-1}$), $\rho(L) \equiv \rho_1 + \rho_2 L + \ldots + \rho_n L^{n-1}$, $\rho \equiv \rho(1)$, and $\beta$ does not denote the discount factor (in this section).

The Taylor principle (neglecting the effect of a permanent increase in the inflation rate on the output gap) states that $\beta$ should be higher than one in order to ensure determinacy.
### TABLE I
**Aggregate Volatility Indicators**

<table>
<thead>
<tr>
<th></th>
<th>Inflation Level</th>
<th>Output Gap</th>
<th>Output hp</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Pre-Volcker</strong></td>
<td>2.77</td>
<td>2.71</td>
<td>1.83</td>
</tr>
<tr>
<td><strong>Volcker-Greenspan</strong></td>
<td>2.18</td>
<td>2.36</td>
<td>1.49</td>
</tr>
<tr>
<td><strong>post–82</strong></td>
<td>1.00</td>
<td>2.06</td>
<td>1.34</td>
</tr>
</tbody>
</table>

An application to US monetary policy II

- CGG find that the estimated value of $\beta$ is significantly
  - lower than one over the “pre-Volcker” period (1960-1979),
  - higher than one over the “Volcker-Greenspan” period (1979-1996).

- They conclude that MP
  - did not ensure determinacy during the pre-Volcker period,
  - did ensure determinacy during the Volcker-Greenspan period,
which could contribute to explain the decrease in macroeconomic volatility.

- In fact, the condition considered ($\beta > 1$) may be neither necessary nor sufficient for determinacy under such a rule, but Lubik and Schorfheide (2004) re-do the exercise using the necessary and sufficient condition, for $n = 1$ and $k = q = 0$, and reach similar conclusions.
CGG’s estimation method

- Let $\Xi \equiv [\rho_1 \cdots \rho_n \pi^* \beta \gamma]^\prime$ the vector of parameters to be estimated ($i^*$ being calibrated, as $\pi^*$ and $i^*$ are not separately identifiable).

- The rational-expectations assumption implies that
  \[
  \mathbb{E}\{Z_t (\pi_{t+k} - \mathbb{E}_t \{\pi_{t+k}\})\} = \mathbb{E}\{Z_t (x_{t+q} - \mathbb{E}_t \{x_{t+q}\})\} = 0
  \]
  for any vector $Z_t$ of variables (called “instruments”) observed by the private sector when it forms its expectations at date $t$.

- Using the interest-rate rule, we then get the orthogonality condition
  \[
  \mathbb{E}\{Z_t g_t(\Xi)\} = 0,
  \]
  where $g_t(\Xi) \equiv i_t - \rho(L)i_{t-1} - (1 - \rho) [i^* + \beta (\pi_{t+k} - \pi^*) + \gamma x_{t+q}]$.

- When $\text{dim}(Z_t) \geq \text{dim}(\Xi)$, this provides the basis for the estimation of $\Xi$ using Hansen’s (1982) generalized method of moments (GMM).
GMM estimator

- Noting $T$ the number of dates in the sample, we define
  - $\mathbf{m}(\Xi) \equiv \mathbb{E} \{ \mathbf{Z}_t g_t(\Xi) \}$ the vector of moments,
  - $\hat{\mathbf{m}}(\Xi) \equiv \frac{1}{T} \sum_{t=1}^{T} \mathbf{Z}_t g_t(\Xi)$ the vector of sample moments.

- For any symmetric, positive, and definite weight matrix $W$, the GMM estimator is defined as
  $$\hat{\Xi}_{GMM} \equiv \text{arg min}_{\Xi} \left[ \hat{\mathbf{m}}(\Xi)' W \hat{\mathbf{m}}(\Xi) \right].$$

- Whatever the weight matrix, the GMM estimator is consistent and asymptotically normal.

- When the weight matrix is the inverse of the variance-covariance matrix of the sample moments, the GMM estimator is also asymptotically efficient.
Hansen-Sargan test

- When \( \operatorname{dim}(Z_t) > \operatorname{dim}(\Xi) \), there are some overidentifying restrictions that can be tested with the **Hansen-Sargan test** (Hansen, 1982, Sargan, 1958).

- The null and alternative hypotheses of this test are
  - \( H_0 : \mathbf{m}(\Xi) = 0 \),
  - \( H_1 : \mathbf{m}(\Xi) \neq 0 \).

- The test statistic \( J_T \equiv T \hat{\mathbf{m}}(\hat{\Xi}_{GMM})' W_T \hat{\mathbf{m}}(\hat{\Xi}_{GMM}) \), where \( W_T \) converges in probability towards the efficient weight matrix, is asymptotically
  - chi-squared with \( \operatorname{dim}(Z_t) - \operatorname{dim}(\Xi) \) degrees of freedom under \( H_0 \),
  - unbounded under \( H_1 \).

- In Tables II, III, IV, and V, “\( p \)” denotes the \( p \)-value of \( J_T \) under \( H_0 \): for instance, \( H_0 \) is rejected at the 95% confidence level when \( p < 0.05 \).
For the **benchmark estimation** (whose results are presented in Table II),
- $\pi$ is measured by the GDP-deflator inflation rate and $x$ by the output gap constructed by the Congressional Budget Office,
- the expectation horizons are $k = 1$ and $q = 1$,
- the two periods considered are 1960Q1-1979Q2 and 1979Q3-1996Q4.

The benchmark-estimation results are shown to be **robust** to the consideration of
- alternative measures of $\pi$ and $x$ (in Table III),
- alternative values of $k$ and $q$ (in Table IV),
- different subperiods within each of the two periods (in Table V).
Benchmark-estimation results

### TABLE II

**Baseline Estimates**

<table>
<thead>
<tr>
<th></th>
<th>$\pi^*$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>$\rho$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-Volcker</td>
<td>4.24</td>
<td>0.83</td>
<td>0.27</td>
<td>0.68</td>
<td>0.834</td>
</tr>
<tr>
<td></td>
<td>(1.09)</td>
<td>(0.07)</td>
<td>(0.08)</td>
<td>(0.05)</td>
<td></td>
</tr>
<tr>
<td>Volcker-Greenspan</td>
<td>3.58</td>
<td>2.15</td>
<td>0.93</td>
<td>0.79</td>
<td>0.316</td>
</tr>
<tr>
<td></td>
<td>(0.50)</td>
<td>(0.40)</td>
<td>(0.42)</td>
<td>(0.04)</td>
<td></td>
</tr>
</tbody>
</table>

Standard errors are reported in parentheses. The set of instruments includes four lags of inflation: output gap, the federal funds rate, the short-long spread, and commodity price inflation.

Robustness of the results I

**TABLE III**
**ALTERNATIVE VARIABLES**

<table>
<thead>
<tr>
<th></th>
<th>(\pi^*)</th>
<th>(\beta)</th>
<th>(\gamma)</th>
<th>(\rho)</th>
<th>(p)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Detrended output</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pre-Volcker</td>
<td>4.17</td>
<td>0.75</td>
<td>0.29</td>
<td>0.67</td>
<td>0.801</td>
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<tr>
<td></td>
<td>(0.68)</td>
<td>(0.07)</td>
<td>(0.08)</td>
<td>(0.05)</td>
<td></td>
</tr>
<tr>
<td>Volcker-Greenspan</td>
<td>4.52</td>
<td>1.97</td>
<td>0.55</td>
<td>0.76</td>
<td>0.289</td>
</tr>
<tr>
<td></td>
<td>(0.58)</td>
<td>(0.32)</td>
<td>(0.30)</td>
<td>(0.05)</td>
<td></td>
</tr>
<tr>
<td><strong>Unemployment rate</strong></td>
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</tr>
<tr>
<td>Pre-Volcker</td>
<td>3.80</td>
<td>0.84</td>
<td>0.60</td>
<td>0.63</td>
<td>0.635</td>
</tr>
<tr>
<td></td>
<td>(0.87)</td>
<td>(0.05)</td>
<td>(0.11)</td>
<td>(0.04)</td>
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</tr>
<tr>
<td>Volcker-Greenspan</td>
<td>4.42</td>
<td>2.01</td>
<td>0.56</td>
<td>0.73</td>
<td>0.308</td>
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<tr>
<td></td>
<td>(0.44)</td>
<td>(0.28)</td>
<td>(0.41)</td>
<td>(0.05)</td>
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<td><strong>CPI</strong></td>
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<tr>
<td>Pre-Volcker</td>
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<td>0.431</td>
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<td>(0.06)</td>
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<td>Volcker-Greenspan</td>
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<td>0.88</td>
<td>0.138</td>
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<tr>
<td></td>
<td>(0.79)</td>
<td>(0.52)</td>
<td>(0.87)</td>
<td>(0.03)</td>
<td></td>
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</table>

Standard errors are reported in parentheses. The set of instruments includes four lags of inflation, output gap, the federal funds rate, the short-long spread, and commodity price inflation.

Robustness of the results II

### TABLE IV

#### ALTERNATIVE HORIZONS

<table>
<thead>
<tr>
<th></th>
<th>$\pi^*$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>$\rho$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k = 4, q = 1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pre-Volcker</td>
<td>3.58</td>
<td>0.86</td>
<td>0.34</td>
<td>0.73</td>
<td>0.835</td>
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<tr>
<td></td>
<td>(1.42)</td>
<td>(0.05)</td>
<td>(0.08)</td>
<td>(0.04)</td>
<td></td>
</tr>
<tr>
<td>Volcker-Greenspan</td>
<td>3.25</td>
<td>2.62</td>
<td>0.83</td>
<td>0.78</td>
<td>0.876</td>
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<td>(0.23)</td>
<td>(0.31)</td>
<td>(0.28)</td>
<td>(0.03)</td>
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<td>$k = 4, q = 2$</td>
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<tr>
<td>Pre-Volcker</td>
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<td>0.88</td>
<td>0.34</td>
<td>0.73</td>
<td>0.833</td>
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<tr>
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<td>(1.80)</td>
<td>(0.06)</td>
<td>(0.09)</td>
<td>(0.04)</td>
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<tr>
<td>Volcker-Greenspan</td>
<td>3.21</td>
<td>2.73</td>
<td>0.92</td>
<td>0.78</td>
<td>0.886</td>
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<td>(0.21)</td>
<td>(0.34)</td>
<td>(0.31)</td>
<td>(0.03)</td>
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</tbody>
</table>

Standard errors are reported in parentheses. The set of instruments includes four lags of inflation, output gap, the federal funds rate, the short-long spread, and commodity price inflation.

Consider first the pre-Volcker period. Interestingly, no significant differences arise across Chairmen in either the value of the inflation target $p^*$, or in the inflation coefficient $\beta$. The point estimates for the inflation target (in the 5–7 percent range) are somewhat above the baseline estimates for the full pre-Volcker period. Since we also dummy all the instruments, we only use only two instrument lags in our subsample stability analysis, thus keeping the total number of instruments and the degrees of freedom comparable to other specifications.

The estimated value for the inflation coefficient is below unity and in line with the baseline estimates. The only exception is Greenspan (1,1), where the coefficient is slightly above unity. The results for Greenspan suggest that the inflation coefficient is slightly more stable than in the other specifications, with standard errors reported in parentheses. The set of instruments includes two lags of inflation, output gap, the federal funds rate, the short-long spread, and commodity price inflation, as well as the same variables with a multiplicative subperiod dummy.


TABLE V
SUBSAMPLE STABILITY

<table>
<thead>
<tr>
<th></th>
<th>$\pi^*$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>$\rho$</th>
<th>$p$</th>
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<td>0.86</td>
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<td>0.524</td>
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<td>(0.16)</td>
<td>(0.06)</td>
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<td>0.63</td>
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<td>2.02</td>
<td>0.99</td>
<td>0.63</td>
<td>0.612</td>
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<td>(0.33)</td>
<td>(0.15)</td>
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</table>

Standard errors are reported in parentheses. The set of instruments includes two lags of inflation, output gap, the federal funds rate, the short-long spread, and commodity price inflation, as well as the same variables with a multiplicative subperiod dummy.
A caveat

- Cochrane (2011) questions the relevance of CGG’s results.

- He argues that parameter $\beta$ may simply not be identifiable.

- This is the case, for instance, when $\rho(L) = i^* = \pi^* = \gamma = 0$ and $k = 1$.

- Indeed, given data $(x_t, \pi_t, i_t)_{t \in \mathbb{Z}}$ generated by the three-equation system

$$
x_t = \delta_1 E_t \{x_{t+1}\} + \delta_2 (i_t - E_t \{\pi_{t+1}\}) + \nu_{1,t},
$$

$$
\pi_t = \delta_3 E_t \{\pi_{t+1}\} + \delta_4 x_t + \nu_{2,t},
$$

$$
i_t = \delta_5 E_t \{\pi_{t+1}\} + \nu_{3,t},
$$

where $\nu_{1,t}, \nu_{2,t},$ and $\nu_{3,t}$ are i.i.d. exogenous shocks, parameter $\delta_5$ is not identifiable since the stationary solution of this system, when it is unique, is

$$
x_t = \nu_{1,t} + \delta_2 \nu_{3,t},
$$

$$
\pi_t = \delta_4 \nu_{1,t} + \nu_{2,t} + \delta_2 \delta_4 \nu_{3,t},
$$

$$
i_t = \nu_{3,t}.$$
Structural equations and exogenous disturbances

- We now turn to the question of the **implementability of the optimal feasible path** in the basic NK model.

- Start with only two exogenous disturbances, affecting the discount factor and the elasticity of substitution between differentiated goods, both ARMA(1,1).

- The **structural equations** and **exogenous disturbances** are then:

\[
\begin{align*}
    c_t &= \mathbb{E}_t \{ c_{t+1} \} - \sigma^{-1} (i_t - \mathbb{E}_t \{ \pi_{t+1} \}) + \eta_t, \\
    \pi_t &= \beta \mathbb{E}_t \{ \pi_{t+1} \} + \kappa y_t + u_t, \\
    y_t &= c_t, \\
    n_t &= (1 - \alpha)^{-1} y_t, \\
    w_t &= \sigma c_t + \varphi n_t, \\
    \eta_t &= \rho \eta \eta_{t-1} + \varepsilon_t + \theta \eta \varepsilon_{t-1}, \\
    u_t &= \rho_u u_{t-1} + \varepsilon_t + \theta_u \varepsilon_{t-1}.
\end{align*}
\]
Consider the following date-\( t \) observation set for CB: \( O_t = \{c^{t-1}, \pi^{t-1}, y^{t-1}, n^{t-1}, w^{t-1}, i^{t-1}\} \), where, for any variable \( z \), \( z^t \equiv (z_{t-k})_{k \geq 0} \) denotes the history of \( z \) until date \( t \) included.

The optimal feasible path (noted \( P \)) is the path minimizing, from Woodford’s (1999) timeless perspective, \( L_0 = \mathbb{E}_0\{\sum_{t=0}^{\infty} \beta^t [(\pi_t)^2 + \lambda (y_t)^2]\} \), subject to the structural equations and the observation-set constraint.

One can show that there exists an interest-rate rule (noted \( R \)) of type

\[
i_t = \sum_{j=1}^{9} \left( f_j^\pi \pi_{t-j} + f_j^y y_{t-j} + g_j i_{t-j} \right),
\]

such that any interest-rate rule consistent with \( O_t \) and \( P \) can be written in a form of type \( \alpha(L)[i_t - \sum_{j=1}^{9} (f_j^\pi \pi_{t-j} + f_j^y y_{t-j} + g_j i_{t-j})] + \beta(L)[y_t - c_t] + \gamma(L)[n_t - (1 - \alpha)^{-1} y_t] + \delta(L)[w_t - \sigma c_t - \varphi n_t] = 0 \), where \( L \) denotes the lag operator, \( [\alpha(X), \beta(X), \gamma(X), \delta(X)] \in \mathbb{R}[X]^4 \), and \( \alpha(0) \neq 0 \).
Therefore, any rule consistent with $O_t$ and $P$ “robustly ensures determinacy” (i.e., ensures determinacy even when an exogenous MP shock is added to this rule) if only if $R$ does.

Therefore, $P$ is “implementable” (i.e., can be obtained as the robustly unique local equilibrium under a rule consistent with $O_t$) if and only if $R$ robustly ensures determinacy.

Consider two alternative calibrations for $(\beta, \sigma^{-1}, \kappa, \lambda)$:

- $(0.99, 1.00, 0.125, 0.021)$ as in Galí (2015),
- $(0.99, 6.25, 0.022, 0.003)$ as in Woodford (2003),

and focus on values of $(\rho_\eta, \rho_u, \theta_\eta, \theta_u)$ such that $\rho_\eta = \rho_u \equiv \rho$ and $\theta_\eta = \theta_u \equiv \theta$ (so that $\theta$ is not a measure of price stickiness in this section).

The next slide shows that $P$ is not implementable for many values of $\rho$ and $\theta$ (in particular for news shocks: $\theta \to +\infty$).
(Non-)implementability of the optimal feasible path II

Galí's calibration

\[ \theta(1 + |\theta|)^{-1} \]

Woodford's calibration

\[ \theta(1 + |\theta|)^{-1} \]

- Implementability
- Non-implementability: non-robustness
- Non-implementability: multiplicity
- Non-implementability: multiplicity and non-rob.
Implications for the conduct of MP

- One key lesson of the NK literature is the importance for CBs to track some key unobserved exogenous rates of interest (Galí, 2015, Woodford, 2003).

- From a normative perspective, the most important of these rates of interest is the exogenous value taken by $i_t$ on the optimal feasible path $P$.

- However, even when this value can be inferred in many alternative ways from $O_t$ on $P$, there may be no way of setting $i_t$ as a function of $O_t$ that implements $P$ as the robustly unique local equilibrium.

- In this case, any attempt to track this rate and implement $P$ will result in
  - local-equilibrium multiplicity,
  - non-existence of a local equilibrium.
Robustness analysis

- Now introduce three additional disturbances, affecting government purchases, technology, and consumption utility or labor disutility.

- The Euler equation is left unchanged, the other structural equations become

\[
\begin{align*}
\pi_t &= \beta \mathbb{E}_t \{\pi_{t+1}\} + \kappa (y_t - \phi_g g_t - \phi_a a_t - \phi_v v_t) + u_t, \\
y_t &= (1 - s) c_t + s g_t, \\
n_t &= (1 - \alpha)^{-1} y_t - (1 - \alpha)^{-1} a_t, \\
w_t &= \sigma c_t + \varphi n_t + v_t,
\end{align*}
\]

and \( L_0 \) becomes

\[
L_0 = \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[ (\pi_t)^2 + \lambda (y_t - \phi_g g_t - \phi_a a_t - \phi_v v_t)^2 \right] \right\}.
\]

- Assume that the additional disturbances follow ARMA processes and that \( O_t = \{ c^{t-1}, \pi^{t-1}, y^{t-1}, n^{t-1}, w^{t-1}, i^{t-1}, \varepsilon g, t-1 \} \).

- It is easy to show that we then get exactly the same results as previously.
In the last section of this chapter, we address the related but distinct issue of **multiple determinate projections** (Galí, 2011).

CBs typically do macroeconomic **projections**, i.e. macroeconomic **forecasts**, conditional on a given interest-rate path.

In practice, there are **three main alternative assumptions** about the interest-rate path over the projection period:

- the interest rate is constant,
- the interest rate evolves according to markets’ expectations,
- the interest rate evolves according to CB’s intentions.

In the first two assumptions, the interest-rate path is given **exogenously**.
If the interest-rate path were specified as exogenous in the projection exercise, then the projection would be \textit{indeterminate}.

So CBs usually consider an interest-rate rule ensuring determinacy and such that the path of the interest rate at the unique local equilibrium coincides with (or is close to) the exogenously given path.

However, even though such projections are \textit{determinate}, they are not \textit{uniquely defined}.

Indeed, for any exogenous interest-rate path, there exist several interest-rate rules ensuring determinacy and implementing

- the same path for the interest rate,
- different paths for the other endogenous variables,

so that there is a \textit{multiplicity of determinate projections}.
Consider an arbitrary exogenous path for the interest rate, noted \((i_t^*)_{t \in \mathbb{Z}}\).

Consider three alternative interest-rate rules:

- **Rule I**: \(i_t = \varphi \pi_t + \nu_t\), where \(\varphi > 1\) and \(\nu_t\) is an exogenous term,
- **Rule II**: \(i_t = i_t^* - \gamma i_{t-1}^* + \gamma (\pi_t + \sigma \Delta x_t + \bar{r}_{t-1}^n)\), where \(\gamma > 1\),
- **Rule III**: \(i_t = i_t^* - \gamma i_{t-1}^* + \gamma (\pi_t + r_{t-1})\), where \(\gamma > 1\) and \(r_t \equiv i_t - \mathbb{E}_t \{\pi_{t+1}\}\) is the ex ante real short-term interest rate.

All three rules ensure determinacy:
- we have already shown that Rule I ensures determinacy,
- it is easy to show, in a similar way, that so do Rules II and III.

All three rules are such that \(i_t = i_t^*\) at the unique local equilibrium:
- in Rule I, \(\nu_t\) is chosen such that this is indeed the case,
- Rule II (combined with the IS equation) and Rule III imply \(i_t - i_t^* = \frac{1}{\gamma} \mathbb{E}_t \{i_{t+1} - i_{t+1}^*\}\) and therefore \(i_t = i_t^*\).
Three different local equilibria

- However, the three rules do not lead to the same unique local equilibrium, as they implement different paths for the inflation rate and the output gap.

- Therefore, the projection made at date $t$ conditionally on $i_{t+k} = i^*_{t+k}$ for $k \geq 0$ is **not uniquely defined**.

- This **multiplicity of determinate projections** is illustrated on the next three slides, for the constant-interest-rate (CIR) assumption,
  - first using the basic New Keynesian model (calibrated),
  - then using Smets and Wouters’ (2007) DSGE model (estimated).

- As apparent on these slides, the difference between the projections can be quantitatively important.

- On all three slides, the “actual rule” denotes the rule $i_t = \phi \pi_t$ with $\phi > 1$. 
CIR-based responses to a cost-push shock in the NK model

Responses of the inflation rate

Responses of the output gap

Source: Galí (2011).
CIR-based responses to a cost-push shock in SW’s model

6. Multiple determinate projections: an illustration using the Smets–Wouters model

The present section reports the findings of an exercise similar to the one described above, but using a more realistic version of the New Keynesian model, namely, the estimated DSGE model of Smets and Wouters (2007). Relative to the basic New Keynesian model above, the Smets–Wouters model incorporates a number of features, including endogenous capital accumulation (subject to adjustment costs), habit formation in consumption, variable capital utilization, staggered wage and price setting with partial indexation, and as many as seven different structural shocks. That model, as well as the related models in Smets and Wouters (2003) and Christiano et al. (2005), can be viewed as the backbone of the estimated DSGE models developed at central banks in recent years and used for monetary policy analysis and forecasting. The reader is referred to the original Smets and Wouters (2007) paper (and its companion technical appendix) for details.

The construction of inflation and output projections under alternative rules using the Smets–Wouters model requires two main changes relative to the analysis above. First, and given the presence of endogenous state variables, the exogenous monetary policy shocks $f_{vt}$ that must be fed into rule I ("modest interventions") in order to keep the nominal interest rate on target.

Source: Galí (2011).
Responses of the inflation rate

Responses of the output gap

Source: Galí (2011).