Macroeconomics 1 (7/7)

Fiscal policy in the overlapping-generations model (Weil)

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Goal, motivation, and limitations of the chapter

- This chapter presents the overlapping-generations model and studies the effects of fiscal policy in this model.
- Unlike the Cass-Koopmans-Ramsey model, this model does not assume that generations are linked together by bequest and altruism: what are the consequences of fiscal policy?
- As in Chapter 6, we restrict the analysis to public expenditures that
  - do not affect the production function,
  - do not affect the private-consumption-utility function,
  - are financed with lump-sum taxes or debt emission (assuming no sovereign-default risk).
- As in Chapter 6, we focus on the positive – not normative – analysis of the effects of fiscal policy.

The overlapping-generations model in the literature

- First versions in **discrete time**: Allais (1947), Samuelson (1958), Diamond (1965).
- First version in **continuous time**: Blanchard (1985), who assumes that households have a **finite** lifespan.
- In this chapter, we consider the continuous-time version of Weil (1989), who assumes that households have an **infinite** lifespan, which makes his version simpler than Blanchard's (1985).
- When the population-growth rate is zero, this version coincides with the Cass-Koopmans-Ramsey model.

Allais, Blanchard

- **Maurice F. C. Allais**: French economist, born in 1911 in Paris, deceased in 2010 in Saint-Cloud, professor at the École Nationale Supérieure des Mines de Paris from 1944, laureate of the Sveriges Riksbank's prize in economic sciences in memory of Alfred Nobel in 1988 "*for his pioneering contributions to the theory of markets and efficient utilization of resources*".

- **Olivier J. Blanchard**: French economist, born in 1948 in Amiens, professor at MIT since 1983.

Diamond, Samuelson

- **Peter A. Diamond**: American economist, born in 1940 in New York, professor at MIT since 1966, co-laureate (with Dale T. Mortensen and Christopher A. Pissarides) of the Sveriges Riksbank's prize in economic sciences in memory of Alfred Nobel in 2010 "*for their analysis of markets with search frictions*".

- **Paul A. Samuelson**: American economist, born in 1915 in Gary, deceased in 2009 in Belmont, professor at MIT from 1940, laureate of the Sveriges Riksbank's prize in economic sciences in memory of Alfred Nobel in 1970 "*for the scientific work through which he has developed static and dynamic economic theory and actively contributed to raising the level of analysis in economic science*".

General overview of the model I *

- Firms rent capital (owned by households) and employ labor (supplied by households) to produce goods.

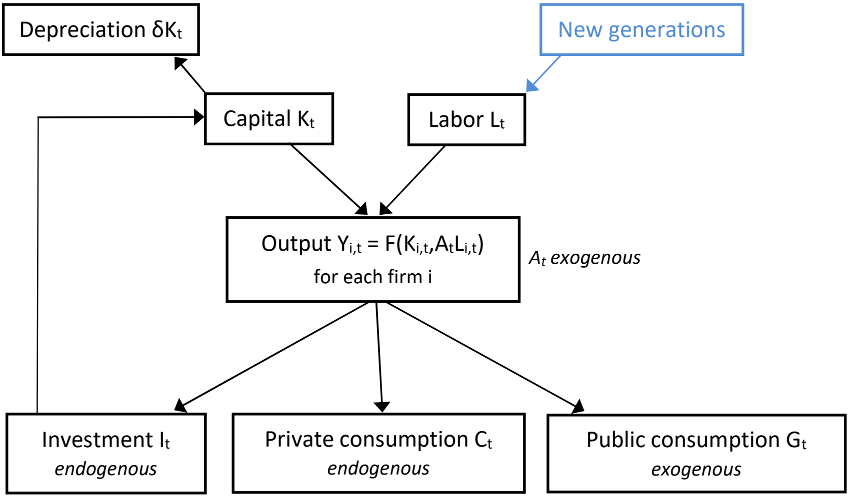
- The goods produced by firms are used for
  - households' consumption,
  - the government's consumption,
  - investment in new capital.

- The saving rate (quantity of goods saved−invested by households / quantity of goods consumed or saved−invested by households) is endogenous, optimally chosen by households.

- Capital evolves over time due to investment and capital depreciation.

(In the pages whose title is followed by an asterisk, compared to Chapter 6, in blue: additions; in red: replacements.)

General overview of the model II *



Good, private agents, markets *

- The good, private agents and markets are the same as in Chapter 6, with one exception.
- This exception is about households: in the overlapping-generations model, there exist different generations of households, who are not linked with each other by bequest and altruism.
- As in Chapter 6, there are therefore, in this model,
  - only one type of good,
  - two types of private agents (households, firms),
  - five markets (goods, labor, capital, loans, public debt).

Exogenous variables *

- **Neither flows nor stocks**:
  - continuous time, indexed by t,
  - price of goods $\equiv$ numéraire $= 1$,
  - (large) number of firms I.

- **Flows**:
  - labor supply $= 1$ per person,
  - real public expenditures G_t,
  - real lump-sum taxes T_t.

- **Stocks**:
  - agregate initial capital $K_0 > 0$,
  - population $L_t = L_0 e^{nt}$, where $L_0 > 0$ and $n \geq 0$,
  - productivity parameter $A_t = A_0 e^{gt}$, where $A_0 > 0$ and $g \geq 0$,
  - real initial total amount of assets $e_\nu^\nu = 0$ of a household born at time ν,
  - real initial public debt D_0.

Endogenous variables *

- **Prices**:
  - real usage cost of capital z_t,
  - real wage w_t,
  - real interest rate r_t.
- **Quantities – flows**:
  - aggregate output Y_t,
  - aggregate labor demand N_t,
  - consumption c_t^{ν} of a household born at time ν.
- **Quantities – stocks**:
  - aggregate capital K_t (except at $t = 0$),
  - real amount of private assets b_t^{ν} of a household born at time ν,
  - real total amount of assets e_t^{ν} of a h. born at time ν (except at $t = \nu$),
  - real public debt D_t (except at $t = 0$).

Aggregation of individual variables

- For any individual variable x_t^ν, we define the aggregate variable

$$\overline{X}_t \equiv x_t^0 L_0 + \int_0^t x_t^\nu \dot{L}_\nu d\nu$$

as well as the per-capita aggregate variable $\overline{x}_t \equiv \frac{\overline{X}_t}{L_t}$.

Chapter outline

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2. Equilibrium conditions
3. Equilibrium conditions on κ_t and γ_t
4. Equilibrium determination
5. Effects of fiscal policy
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Equilibrium conditions

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Households' intertemporal utility *

- At time ν, the **intertemporal utility** of households born at time ν is

$$U_\nu \equiv \int_\nu^{+\infty} e^{-\rho(t-\nu)}[\ln(c_t^\nu) + v(g_t)]dt$$

  where

  - $g_t \equiv \frac{G_t}{L_t}$ is, by assumption, independent of age,
  - ρ is the rate of time preference ($\rho > 0$),
  - v is the instantaneous-utility function for public expenditures.

- The instantaneous-utility function for private consumption u is thus such that the elasticity of intertemporal substitution is constant, equal to 1.

- The "actualization" of instantaneous utility at rate ρ, rather than $\rho - n$, and the assumption $e_\nu^\nu = 0$ capture the **absence of intergenerational links**.

Households' assets *

- As in Chapter 6, each household can hold three types of assets:
  - loans to other households,
  - capital ownership titles,
  - public debt.

- As in Chapter 6, households must in equilibrium be indifferent between these three asset types, so

$$\begin{aligned} r_t &\equiv \text{real interest rate on loans to households} \\ &= \text{real rate of return on ownership titles} \\ &= \text{real interest rate on public debt.} \end{aligned}$$

Households' budget constraint I

- The **instantaneous budget constraint** of households born at time ν is

$$\dot{e}_t^{\nu} = w_t - t_t + r_t e_t^{\nu} - c_t^{\nu},$$

where the wage w_t (by result) and the lump-sum tax $t_t \equiv \frac{T_t}{L_t}$ (by assumption) are independent of age.

- Re-arranging the terms and multiplying by the exponential, we get

$$\left[\dot{e}_t^{\nu} - r_t e_t^{\nu}\right] e^{-\int_{\nu}^{t} r_\tau d\tau} = (w_t - t_t - c_t^{\nu}) e^{-\int_{\nu}^{t} r_\tau d\tau}.$$

Households' budget constraint II

- Integrating from ν to T and using $e_\nu^\nu = 0$, we then get

$$e_T^\nu e^{-\int_\nu^T r_\tau d\tau} = \int_\nu^T (w_t - t_t - c_t^\nu) e^{-\int_\nu^t r_\tau d\tau} dt.$$

- Going to the limit $T \to +\infty$, we get the **intertemporal budget constraint** of households born at time ν

$$\int_\nu^{+\infty} c_t^\nu e^{-\int_\nu^t r_\tau d\tau} dt \leq \int_\nu^{+\infty} (w_t - t_t) e^{-\int_\nu^t r_\tau d\tau} dt$$

$$\text{if and only if} \qquad \lim_{T \to +\infty} \left(e_T^\nu e^{-\int_\nu^T r_\tau d\tau} \right) \geq 0$$

(**solvency constraint** of households born at ν).

Households' optimization problem

- The optimization problem of households born at time ν is thus the following: at time ν, for some given $(r_t, w_t, g_t, t_t)_{t \geq \nu}$ and $e_\nu^\nu = 0$,

$$\max_{(c_t^\nu)_{t \geq \nu}, (e_t^\nu)_{t > \nu}} \int_\nu^{+\infty} e^{-\rho(t-\nu)} [\ln(c_t^\nu) + v(g_t)] dt$$

subject to the constraints

1. $\forall t \geq \nu$, $c_t^\nu \geq 0$ (constraint of consumption non-negativity),
2. $\forall t \geq \nu$, $\dot{e}_t^\nu = w_t - t_t + r_t e_t^\nu - c_t^\nu$ (instantaneous budget constraint),
3. $\lim_{t \to +\infty} \left(e_t^\nu e^{-\int_\nu^t r_\tau d\tau} \right) \geq 0$ (solvency constraint).

Solving households' optimization problem I

- Solving this optimization problem leads, with the same kind of computations as in Chapter 2, to the following conditions on $(c_t^\nu)_{t\geq\nu}$ and $(e_t^\nu)_{t>\nu}$:

1. $\frac{\dot{c}_t^\nu}{c_t^\nu} = r_t - \rho$ (**Euler equation**),
2. $\dot{e}_t^\nu = w_t - t_t + r_t e_t^\nu - c_t^\nu$ (instantaneous budget constraint),
3. $\lim_{t\to+\infty}\left(e_t^\nu e^{-\int_\nu^t r_\tau d\tau}\right) = 0$ (**transversality condition**).

Solving households' optimization problem II

- Conditions 2 and 3 imply that the intertemporal budget constraint is binding at any time after birth: $\forall t \geq \nu$,

$$\int_t^{+\infty} c_s^\nu e^{-\int_t^s r_\tau d\tau} ds = e_t^\nu + h_t$$

where $h_t \equiv \int_t^{+\infty} (w_s - t_s) e^{-\int_t^s r_\tau d\tau} ds$ is the actualized value at time t of future after-tax incomes (independent of age).

- The Euler equation implies that $\forall s \geq t \geq \nu$, $c_s^\nu = c_t^\nu e^{\int_t^s (r_\tau - \rho) d\tau}$.

- From these two results, we deduce that individual consumption at any time is proportional to current individual intertemporal wealth (which is a consequence of the assumption $\theta = 1$):

$$c_t^\nu = \rho(e_t^\nu + h_t).$$

Government's budget constraint *

- As in Chapter 6, with $d_t \equiv \frac{D_t}{L_t}$, the government's **instantaneous budget constraint** is

$$\dot{d}_t = (r_t - n)d_t + g_t - t_t,$$

their **intertemporal budget constraint** (assumed to be binding) is

$$d_0 = \int_0^{+\infty} (t_t - g_t) e^{-\int_0^t (r_\tau - n) d\tau} dt$$

and their **solvency constraint** (binding as well) is

$$\lim_{t \to +\infty} \left[d_t e^{-\int_0^t (r_\tau - n) d\tau} \right] = 0.$$

Firms' behavior and market clearing *

- Firms are modeled as in Chapter 2, so their behavior is characterized by the same first-order conditions as in Chapter 2.
- The market-clearing conditions are:
    - for the goods market: $Y_t = \overline{C}_t + G_t + \dot{K}_t + \delta K_t$,
    - for the labor market: $N_t = L_t$,
    - for the capital and loans markets: $\overline{B}_t = K_t$,
    - for the public-debt market: $\overline{E}_t = \overline{B}_t + D_t$.

Equilibrium conditions on κ_t and γ_t

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General overview of the model resolution *

- As in Chapters 2, 4, 5 and 6, we first determine the equilibrium paths $(s_t)_{t\geq 0}$ and $(f_t)_{t\geq 0}$ of a key stock and a key flow, using
  - a differential equation in $\dot{s}_t$, obtained from households' instantaneous budget constraint or from the goods-market-clearing condition (the "or" being a consequence of Walras' law),
  - a differential equation in $\dot{f}_t$, obtained from the Euler equation,
  - an initial condition on s_0,
  - a terminal condition, obtained from the transversality condition.

- We then determine the equilibrium paths of the other variables, from $(s_t)_{t\geq 0}$ and $(f_t)_{t\geq 0}$, using the other equilibrium conditions.

- In this chapter, $s_t = \kappa_t \equiv \frac{K_t}{A_t L_t}$ and $f_t = \gamma_t \equiv \frac{\overline{c}_t}{A_t} = \frac{\overline{C}_t}{A_t L_t}$, as in Chapters 2, 4 and 6 (except for households' heterogeneity).

Obtaining the terminal condition I

- We admit that e_t^ν, where $\nu \leq t$, is a decreasing function of ν for a given t (each generation being born without assets and accumulating more and more assets over time).

- The transversality condition $\lim_{t \to +\infty} \left(e_t^\nu e^{-\int_\nu^t r_\tau d\tau} \right) = 0$ for $\nu = 0$ is

$$\lim_{t \to +\infty} \left(e_t^0 e^{-\int_0^t r_\tau d\tau} \right) = 0$$

and implies

$$\lim_{t \to +\infty} \left(\overline{e}_t e^{-\int_0^t r_\tau d\tau} \right) = 0$$

as $\overline{e}_t \leq e_t^0$.

Obtaining the terminal condition II

- Now, the government's solvency constraint $\lim_{t \to +\infty} \left[d_t e^{-\int_0^t (r_\tau - n) d\tau} \right] = 0$ implies
$$\lim_{t \to +\infty} \left(d_t e^{-\int_0^t r_\tau d\tau} \right) = 0.$$

- So, using the public-debt-market-clearing condition ($\overline{e}_t = \overline{b}_t + d_t$), we can rewrite the transversality condition as
$$\lim_{t \to +\infty} \left(\overline{b}_t e^{-\int_0^t r_\tau d\tau} \right) = 0.$$

- Using $\overline{b}_t = k_t \equiv \frac{K_t}{L_t}$, we then get, in the same way as in Chapter 2,
$$\lim_{t \to +\infty} \left\{ \kappa_t e^{-\int_0^t [f'(\kappa_\tau) - (g + \delta)] d\tau} \right\} = 0.$$

Obtaining the differential equation in $\dot{\kappa}_t$ I

- We show in the appendix that

$$\dot{\overline{e}}_t = \overline{\dot{e}}_t - n\overline{e}_t.$$

- On the right-hand side of this equality,
  - the first term represents the variation in $\overline{e}_t$ on the **intensive margin** (the stock of assets of each generation grows over time),
  - the second term represents the variation in $\overline{e}_t$ on the **extensive margin** (new generations are born, who have no assets).

- So, we can aggregate the instantaneous budget constraint $\dot{e}_t^{\nu} = w_t - t_t + r_t e_t^{\nu} - c_t^{\nu}$ of households born at ν as

$$\dot{\overline{e}}_t = w_t - t_t + (r_t - n)\overline{e}_t - \overline{c}_t.$$

Obtaining the differential equation in $\dot{\kappa}_t$ II

- Using $\overline{e}_t = \overline{b}_t + d_t$ and the government's instantaneous budget constraint $\dot{d}_t = (r_t - n)d_t + g_t - t_t$, we then get

$$\dot{\overline{b}}_t = w_t - g_t + (r_t - n)\overline{b}_t - \overline{c}_t.$$

- Then, using $\overline{b}_t = k_t$, in the same way as in Chapter 2,

$$\dot{\kappa}_t = f(\kappa_t) - \gamma_t - \chi_t - (n + g + \delta)\,\kappa_t$$

where $\chi_t \equiv \frac{g_t}{A_t} = \frac{G_t}{A_t L_t}$.

- This differential equation implies the goods-market-clearing condition: $\dot{K}_t = Y_t - \overline{C}_t - G_t - \delta K_t$ (consequence of Walras' law).

Obtaining the differential equation in $\dot{\gamma}_t$ I

- Aggregating $c_t^{\nu} = \rho(e_t^{\nu} + h_t)$, we get $\overline{c}_t = \rho(\overline{e}_t + h_t)$.

- Moreover, we show in the appendix that $\dot{h}_t = r_t h_t - (w_t - t_t)$.

- Using these last two results and $\dot{\overline{e}}_t = w_t - t_t + (r_t - n)\overline{e}_t - \overline{c}_t$, we get $\dot{\overline{c}}_t = \rho(\dot{\overline{e}}_t + \dot{h}_t) = \rho(r_t - n)\overline{e}_t - \rho\overline{c}_t + \rho r_t h_t = (r_t - \rho)\overline{c}_t - n\rho\overline{e}_t$.

- Hence the **aggregate Euler equation**

$$\frac{\dot{\overline{c}}_t}{\overline{c}_t} = r_t - \rho - n\rho\frac{\overline{e}_t}{\overline{c}_t}.$$

- The growth rate of per-capita aggregate consumption $\overline{c}_t$ is therefore lower than the growth rate $r_t - \rho$ of each existing generation's consumption.

Obtaining the differential equation in $\dot{\gamma}_t$ II

- The difference between these two growth rates,

$$n\rho\frac{\overline{e}_t}{\overline{c}_t} = n\rho\frac{\overline{e}_t - e_t^t}{\overline{c}_t} = n\frac{\overline{c}_t - c_t^t}{\overline{c}_t},$$

is due to the arrival of new generations who consume less than existing generations ($c_t^t < \overline{c}_t$) because they are born without assets ($e_t^t = 0$).

- Using $\overline{e}_t = \overline{b}_t + d_t$ and $\overline{b}_t = k_t$, we can, in the same way as in Chapter 2, rewrite the aggregate Euler equation as

$$\frac{\dot{\gamma}_t}{\gamma_t} = f'(\kappa_t) - (\rho + g + \delta) - n\rho\frac{\kappa_t + \phi_t}{\gamma_t}$$

where $\phi_t \equiv \frac{d_t}{A_t} = \frac{D_t}{A_t L_t}$.

Equilibrium conditions on κ_t and γ_t *

- The two differential equations, the initial condition and the terminal condition determining $(\kappa_t)_{t\geq 0}$ and $(\gamma_t)_{t\geq 0}$ are thus the following:

$$\dot{\kappa}_t = f(\kappa_t) - \gamma_t - \chi_t - (n+g+\delta)\kappa_t,$$

$$\frac{\dot{\gamma}_t}{\gamma_t} = f'(\kappa_t) - (\rho+g+\delta) - n\rho\frac{\kappa_t+\phi_t}{\gamma_t},$$

$$\kappa_0 = \frac{K_0}{A_0 L_0},$$

$$\lim_{t\to+\infty}\left\{\kappa_t e^{-\int_0^t [f'(\kappa_\tau)-(g+\delta)]d\tau}\right\} = 0,$$

where $\phi_t = \int_t^{+\infty}(\psi_\tau - \chi_\tau)e^{-\int_t^\tau [f'(\kappa_s)-(n+g+\delta)]d\tau}$, with $\psi_t \equiv \frac{t_t}{A_t} = \frac{T_t}{A_t L_t}$.

Equilibrium determination

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Steady state for χ_t constant I

- In the rest of this chapter, unless stated otherwise, we focus on the case in which public debt is constantly zero: $d_0 = 0$ and $\forall t \geq 0$, $t_t = g_t$.

- We temporarily assume that $\forall t \geq 0$, (i) $\chi_t = \chi > 0$ and (ii) households expect that $\forall \tau \geq t$, $\chi_\tau = \chi$.

- Let μ_κ and μ_γ denote the values of, respectively, $\frac{\dot{\kappa}_t}{\kappa_t}$ and $\frac{\dot{\gamma}_t}{\gamma_t}$ at the **steady state** ($\equiv$ situation in which κ_0 is such that, in equilibrium, all quantities are non-zero and grow at constant rates).

- At the steady state, we can rewrite the differential equation in $\dot{\gamma}_t$ as $f'(\kappa_t) - n\rho\frac{\kappa_t}{\gamma_t} = \mu_\gamma + \rho + g + \delta$, which implies that
  1. either $\mu_\gamma = \mu_\kappa = 0$,
  2. or $\mu_\gamma > \mu_\kappa > 0$,
  3. or $\mu_\gamma < \mu_\kappa < 0$.

Steady state for χ_t constant II

- At the steady state, we can rewrite the differential equation in $\dot{\kappa}_t$ as $\gamma_t = f(\kappa_t) - (n+g+\delta+\mu_\kappa)\kappa_t - \chi$ and then, differentiating this equality with respect to time, obtain $\mu_\kappa f'(\kappa_t) - \mu_\gamma \frac{\gamma_t}{\kappa_t} = \mu_\kappa(n+g+\delta+\mu_\kappa)$.

- This last equality is consistent with Case 1 but not with Cases 2 and 3.

- Therefore, $\mu_\kappa = \mu_\gamma = 0$: κ_t and γ_t are constant over time at the steady state.

- Replacing $\dot{\kappa}_t$ with 0 in the differential equation in $\dot{\kappa}_t$, we get

$$\gamma_t = f(\kappa_t) - (n+g+\delta)\kappa_t - \chi,$$

which corresponds to a **bell-shaped curve** in the plane (κ_t, γ_t).

Steady state for χ_t constant III

- Replacing $\dot{\gamma}_t$ with 0 in the differential equation in $\dot{\gamma}_t$, we get

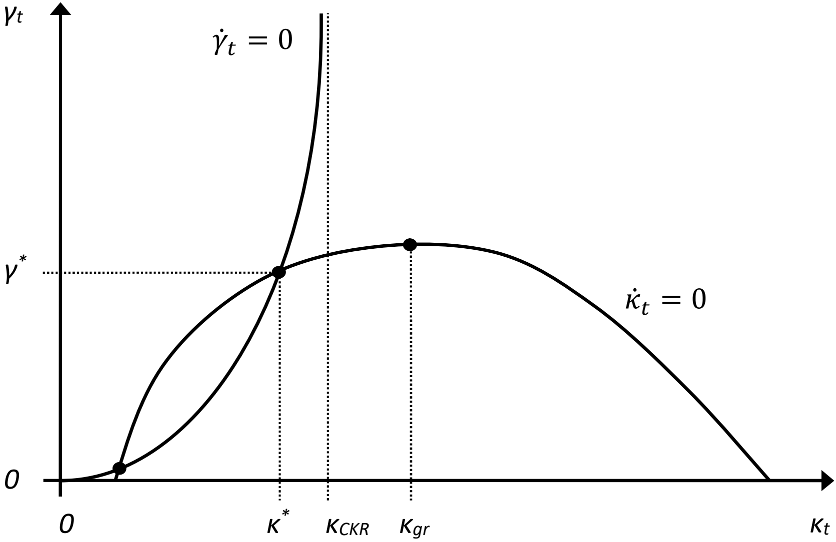
$$\gamma_t = \frac{n\rho\kappa_t}{f'(\kappa_t) - (\rho + g + \delta)},$$

  which corresponds, in the plane (κ_t, γ_t), to a **curve** that is **increasing** and convex, starts from the origin with a zero derivative, and has, as an asymptote, the vertical straight line whose equation is $f'(\kappa_t) = \rho + g + \delta$.

- For χ sufficiently small, these two curves have two **intersection points**, corresponding to the two possible steady-state values of (κ_t, γ_t).

- Among these two values, let (κ^*, γ^*) denote the value that corresponds to the more north-eastern intersection point.

Steady state for χ_t constant IV



Stylised facts of Kaldor (1961)

- At the steady state, like the Cass-Koopmans-Ramsey model, the overlapping-generations model of Weil (1989) therefore only accounts for the first five **stylised facts of Kaldor (1961)**:

  1. per-capita output grows: $\frac{\dot{y}_t}{y_t} = g \geq 0$,
  2. the per-capita capital stock grows: $\frac{\dot{k}_t}{k_t} = g \geq 0$,
  3. the rate of return of capital is constant: $r_t = f'(\kappa^*) - \delta$,
  4. the ratio capital / output is constant: $\frac{K_t}{Y_t} = \frac{\kappa^*}{f(\kappa^*)}$,
  5. the labor and capital shares of income are constant: $\frac{w_t L_t}{Y_t} = \frac{f(\kappa^*) - \kappa^* f'(\kappa^*)}{f(\kappa^*)}$ and $\frac{z_t K_t}{Y_t} = \frac{\kappa^* f'(\kappa^*)}{f(\kappa^*)}$,
  6. ~~the growth rate of per-capita output varies across countries.~~

Possibility of dynamic inefficiency for χ_t constant I

- The value $\kappa_{CKR} > \kappa^*$ of κ_t on the vertical straight line is defined by

$$f'(\kappa_{CKR}) = \rho + g + \delta$$

and corresponds to the steady-state value of κ_t in the Cass-Koopmans-Ramsey model with $\theta = 1$, i.e. with $u(c) = \ln(c)$.

- The value κ_{or} of κ_t maximizing the bell-shaped curve is defined by

$$f'(\kappa_{or}) = n + g + \delta$$

(**golden rule of capital accumulation** of the Solow-Swan model).

Possibility of dynamic inefficiency for χ_t constant II

- In the Cass-Koopmans-Ramsey model with $\theta = 1$, for U_0 to take a finite value, we must have $\rho > n$ and hence $\kappa_{CKR} < \kappa_{or}$.

- In the overlapping-generations model, $\rho > 0$ is enough for U_ν^ν to take a finite value, and we can therefore have $\rho < n$ and $\kappa_{CKR} > \kappa_{or}$.

- We admit that, when $\rho < n$, we can obtain $\kappa^* > \kappa_{or}$, i.e. there can be **dynamic inefficiency**, due to capital over-accumulation.

- In this last case, the competitive equilibrium is **not a Pareto optimum**.

- No matter whether there is dynamic inefficiency or not, the first welfare theorem does not apply, notably because of the infinite number of agent types (while this theorem applies in the Cass-Koopmans-Ramsey model, which is a representative-agent model).

Possibility of dynamic inefficiency for χ_t constant III

- Dynamic inefficiency, when it occurs, can be interpreted as the consequence of an **intergenerational pecuniary externality**:
  - the capital stock and hence the prices that future generations will face depend on the decisions of current generations,
  - current generations do not take into account the effect of their decisions on the welfare of future generations.

- Dynamic inefficiency gives a role to a **pay-as-you-go pension system** (transferring goods from younger to older generations at each time): by discouraging saving, such a system reduces dynamic inefficiency.

- On the contrary, a funded pension system (constraining young generations to save for their old days) cannot reduce dynamic inefficiency and can even amplify it.

Possibility of dynamic inefficiency for χ_t constant IV

- In this model, there is dynamic inefficiency when the "real rate of return", $f'(\kappa^*) - \delta$, is lower than the growth rate of GDP, $g + n$.

- When one uses the real interest rate on public debt as a proxy for the real rate of return, one concludes that the main industrialized economies are dynamically inefficient.

- However, using the ratio of capital income (less depreciation) to the value of the capital stock as a proxy leads to the opposite conclusion.

- The existence of different model-consistent empirical measures for the real rate of return is due in particular to the fact that the model does not take uncertainty into account.

- Abel, Mankiw, Summers and Zeckhauser (1989) determine the condition for dynamic inefficiency under uncertainty and conclude that the main industrialized economies are **dynamically efficient**.

Equilibrium path for χ_t constant I

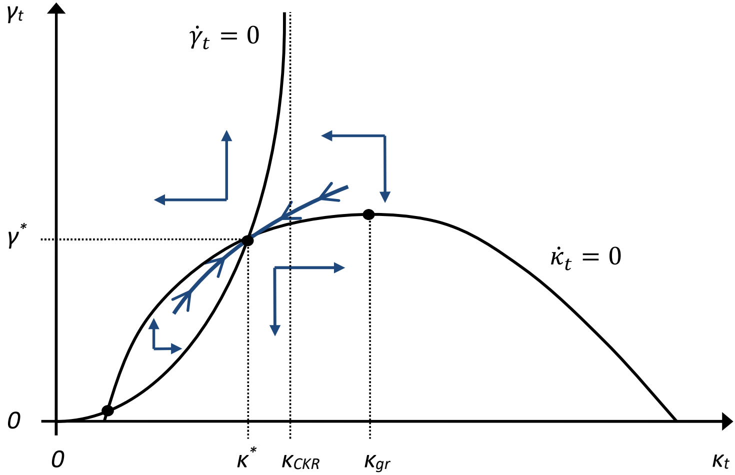
- In the rest of the chapter, as in the previous figure, we restrict the analysis to parameter values such that $\rho > n$, so that $\kappa_{CKR} < \kappa_{or}$ and hence $\kappa^* < \kappa_{or}$ (no dynamic inefficiency).

- The system of differential equations has
  - a stable arm and an unstable arm in the neighborhood of the point (κ^*, γ^*),
  - two unstable arms in the neighborhood of the other intersection point of the two curves.

- As in Chapters 2 and 6, there exists therefore a unique path, called "**saddle path**", along which (κ_t, γ_t) can converge to (κ^*, γ^*).

- We admit that the unique equilibrium path of (κ_t, γ_t) for some given κ_0 is this saddle path, provided that κ_0 is higher than the value of κ_t at the more south-western intersection point of the two curves.

Equilibrium path for χ_t constant II



Effects of fiscal policy

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No Ricardian equivalence when $n > 0$ I

- On this page and the next three pages, we relax the assumption that d_t is constantly zero.

- Given the equilibrium conditions on κ_t and γ_t, when $n > 0$, $(\kappa_t)_{t\geq 0}$ and $(\gamma_t)_{t\geq 0}$ depend in equilibrium not only on the path of public expenditures $(\chi_t)_{t\geq 0}$, but also on the path of lump-sum taxes $(\psi_t)_{t\geq 0}$.

- In other words, when $n > 0$, **the effect of public expenditures on the economy depends on the way they are financed** (current tax or current borrowing reimbursed with a future tax).

- The model therefore implies **no Ricardian equivalence** when $n > 0$.

- This result is due to the fact that, when $n > 0$, the per-capita aggregate intertemporal wealth of households depends not only on the path of public expenditures, but also on the path of lump-sum taxes, as we show on the next pages.

No Ricardian equivalence when $n > 0$ II

- Using the government's intertemporal budget constraint

$$d_t = \int_t^{+\infty} (t_s - g_s) e^{-\int_t^s (r_\tau - n) d\tau} ds,$$

we can rewrite the per-capita aggregate intertemporal wealth of households

$$\overline{e}_t + h_t = \overline{b}_t + d_t + \int_t^{+\infty} (w_s - t_s) e^{-\int_t^s r_\tau d\tau} ds$$

as

$$\overline{e}_t + h_t = \overline{b}_t + \int_t^{+\infty} (w_s - g_s) e^{-\int_t^s r_\tau d\tau} ds + \Omega_t$$

where $\Omega_t \equiv \int_t^{+\infty} (t_s - g_s) e^{-\int_t^s (r_\tau - n) d\tau} \left[1 - e^{-n(s-t)} \right] ds.$

No Ricardian equivalence when $n > 0$ III

- Denoting by $T_{s/t} \equiv T_s e^{-\int_t^s r_\tau d\tau} = L_t t_s e^{-\int_t^s (r_\tau - n) d\tau}$ the actualized value at time t of aggregate lump-sum taxes at time $s \geq t$, we get

$$\frac{\partial \Omega_t}{\partial T_{s/t}} = \frac{1 - e^{-n(s-t)}}{L_t}.$$

- When $n > 0$, $\frac{\partial \Omega_t}{\partial T_{s/t}}$ is strictly increasing in s: the more distant in the future the tax financing public expenditures, the less public expenditures reduce the intertemporal wealth of existing generations.

- The reason is that the more distant in the future this tax, the more this tax will be paid by new generations which existing generations are not linked to.

Ricardian equivalence when $n = 0$

- When $n = 0$, $(\kappa_t)_{t\geq 0}$ and $(\gamma_t)_{t\geq 0}$ depend in equilibrium on the path of public expenditures $(\chi_t)_{t\geq 0}$, but not on the path of lump-sum taxes $(\psi_t)_{t\geq 0}$.

- The model therefore implies the Ricardian equivalence in this particular case.

- The reason is that without the arrival of new generations, the intertemporal wealth of existing generations does not depend on the path of lump-sum taxes: $\frac{\partial \Omega_t}{\partial T_{s/t}} = 0$ for any $s \geq t$.

- The model then coincides with the Cass-Koopmans-Ramsey model in the case $u(c) = \ln(c)$ and $n = 0$.

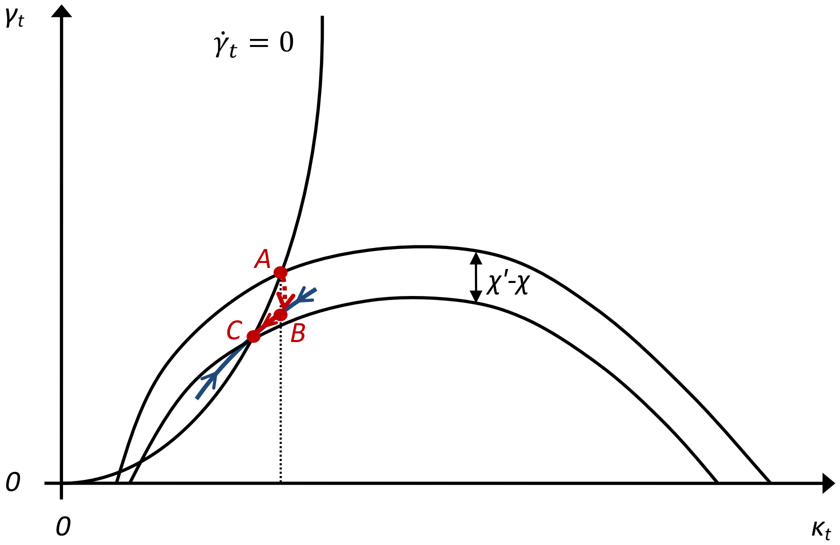
Variations in χ_t over time *

- As in Chapter 6, we assume in the rest of this chapter that
  - χ_t can vary over time,
  - these variations can be expected, or not expected, by households,
  - at each time, households are unaware that they may be surprised, at a later time, by the value of χ_t or by the announcement of its future path,
  - at each time at which they are surprised by the current value of χ_t or by the announcement of its future path, households solve their new optimization problem and change their current and expected future behavior accordingly.
- As in Chapter 6, κ_t is a continuous function of time, while γ_t and $\dot{\gamma}_t$ can be discontinuous only at the times at which households are surprised by the current value of χ_t or by the announcement of its future path.

Effect of an unexpected and permanent increase in χ_t I

- We now assume that there exists a time t_0 such that $t_0 > 0$ and
  - $\forall t \in [0; t_0[$, (i) $\chi_t = \chi$, (ii) households expect that $\forall \tau \geq t$, $\chi_\tau = \chi$, and (iii) the economy is at the corresponding steady state,
  - the government credibly announces at t_0 that $\forall t \geq t_0$, $\chi_t = \chi' > \chi$,
  - the government conducts, from t_0, the fiscal policy announced at t_0.
- From t_0, χ_t is constant over time and there are no more surprises, so the economy is on its new saddle path (lower than the old saddle path).
- The stock κ_t being a continuous function of time, it is the flow γ_t that adjusts at t_0 to put the economy on its new saddle path: therefore, the economy jumps from A to B at t_0 and then moves from B to C between t_0 and $+\infty$.
- Consumption γ_t falls by more than $\chi' - \chi$ in the long term because the decrease in the capital stock κ_t reduces the actualized value h_t of future after-tax incomes by increasing r_t and decreasing w_t.

Effect of an unexpected and permanent increase in χ_t II



Effect of an unexpected and temporary increase in χ_t

- Part 6 of the tutorials studies the effect of an unexpected and **temporary** increase in χ_t.

- The case of an unexpected and temporary increase in χ_t corresponds to the case in which there exist two dates t_0 and t_1 such that $0 < t_0 < t_1$ and
  - $\forall t \in [0; t_0[$, (i) $\chi_t = \chi$, (ii) households expect that $\forall \tau \geq t$, $\chi_\tau = \chi$, and (iii) the economy is at the corresponding steady state,
  - the government credibly announces at t_0 that (i) $\forall t \in [t_0; t_1[$, $\chi_t = \chi' > \chi$, and (ii) $\forall t \geq t_1$, $\chi_t = \chi$,
  - the government conducts, from t_0, the fiscal policy announced at t_0.

Conclusion

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Main predictions of the model

- In the long term, the first five stylised facts of Kaldor (1961) are obtained.
- There can be some dynamic inefficiency, due to capital over-accumulation, which a pay-as-you-go pension system can reduce.
- The effect of public expenditures on the economy depends on the way they are financed (no Ricardian equivalence).
- An unexpected and permanent increase in public expenditures reduces consumption and the capital stock permanently.

Appendix

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Differentiating $\overline{e}_t$ with respect to time

- We write $\overline{e}_t = \frac{1}{L_t}(e_t^0 L_0 + \mathcal{A}_t)$, where $\mathcal{A}_t \equiv \int_0^t e_t^\nu \dot{L}_\nu d\nu$.

- Writing $\mathcal{A}_t = g(t,t)$, where $g(u,v) \equiv \int_0^u e_v^\nu \dot{L}_\nu d\nu$, we get

$$\dot{\mathcal{A}}_t = \frac{\partial g}{\partial u}(t,t) + \frac{\partial g}{\partial v}(t,t) = e_t^t \dot{L}_t + \int_0^t \dot{e}_t^\nu \dot{L}_\nu d\nu = \int_0^t \dot{e}_t^\nu \dot{L}_\nu d\nu$$

because $e_t^t = 0$.

- We then get

$$\begin{aligned}
\dot{\overline{e}}_t &= \frac{-\dot{L}_t}{L_t^2}\left(e_t^0 L_0 + \mathcal{A}_t\right) + \frac{1}{L_t}\left(\dot{e}_t^0 L_0 + \dot{\mathcal{A}}_t\right) \\
&= \frac{-n}{L_t}\left(e_t^0 L_0 + \mathcal{A}_t\right) + \frac{1}{L_t}\left(\dot{e}_t^0 L_0 + \int_0^t \dot{e}_t^\nu \dot{L}_\nu d\nu\right) \\
&= \overline{\dot{e}}_t - n\overline{e}_t .
\end{aligned}$$

Differentiating h_t with respect to time

- We write $h_t = g(t,t)$, where $g(u,v) \equiv \int_u^{+\infty} (w_s - t_s) e^{-\int_v^s r_\tau d\tau} ds$.
- We then get

$$\begin{aligned} \dot{h}_t &= \frac{\partial g}{\partial u}(t,t) + \frac{\partial g}{\partial v}(t,t) \\ &= -(w_t - t_t) e^{-\int_t^t r_\tau d\tau} + r_t \int_t^{+\infty} (w_s - t_s) e^{-\int_t^s r_\tau d\tau} ds \\ &= r_t h_t - (w_t - t_t). \end{aligned}$$